The Distribution of the Equilibrium Quantity Traded*

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Abstract

We show that in a large family of models the equilibrium quantity traded has a general non-central hypergeometric distribution. This family includes mechanism design models of two-sided markets, k-double auctions, and any standard auction, provided traders' types are independent and, on each side of the market, identically distributed. Our results exploit the facts that the efficient quantity traded has a non-central hypergeometric distribution and that the equilibrium quantity traded in these models is *quasi-Walrasian*. That is, it is determined by the intersection of the demand and supply schedules constructed from monotone transformations of values and costs. Asymptotically, the equilibrium quantity traded is normally distributed.

Keywords: Efficient quantity, general non-central hypergeometric distribution, two-sided market design, k-double auctions, Bayesian optimal mechanism, multi-unit auctions.
JEL-Classification: C72, D44, D82

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1 Introduction

In a large family of economic models, the equilibrium quantity traded is a non-degenerate random variable. Examples include Bayesian mechanism design models of the market, k-double auctions, two-sided markets with per unit taxes or subsidies, any standard auction and clock auctions with one-sided private information when agents' types are independent and, on each side of the market, identically distributed. In Loertscher, Muir, and Taylor (2017) we show that under this assumption the equilibrium quantity traded in a Walrasian market, that is, the ex post efficient quantity traded, has a non-central hypergeometric distribution. This distribution models biased sampling without replacement and is a generalisation of the standard hypergeometric distribution (which arises when sampling is unbiased) and Wallenius' non-central hypergeometric distribution (which arises when the sampling bias simply depends on some constant weight (see Wallenius, 1963)).

In this paper, we show that this result extends well beyond efficiency. Indeed, it holds for any of the aforementioned models because in all these models the quantity traded is *quasi-Walrasian* in the sense that it is determined by the intersection of a monotone transformation of buyers' values and sellers' costs. To be precise, letting $V_{[i]}$ be the i-th highest buyer value and $C_(j)$ be the j-th lowest seller cost and letting B and S be monotone transformations of the schedule of values $\{V_1, \dots, V_n\}$ and and costs $\{C_1, \dots, C_m\}$ respectively, the equilibrium quantity traded in these models is given by the largest index Q such that

$$B(V_{[Q]}) - S(C_{(Q)}) \geq 0.$$

For any model in which the equilibrium quantity traded is quasi-Walrasian, the result then follows from the result for the ex post efficient quantity by replacing the true types V_i and C_j with the transformed types $B(V_i)$ and $S(C_j)$ and by using the distributions of the transformed types. Under mild regularity conditions, we show that in the limit, as the market becomes large, the quantity traded in any quasi-Walrasian equilibrium converges in distribution to a normal random variable.

We hope that these results prove useful for future research on k-double-auctions and on the design of two-sided market mechanisms and help advance the research agenda outlined by Satterthwaite and Williams (2002, p.1841):

> Currently, economic theory provides little guidance to financial exchanges in the selection of computer algorithms and floor procedures for trading. A theory of

market mechanisms would provide such guidance and also complement the rich literature on markets in experimental economics, which is currently the main source of guidance in the design of market mechanisms.

The remainder of this note is organised as follows. The general market setup is introduced and the main result is described in Section 2. In Sections 3.1 and 3.2 we discuss these results and show that our market setup subsumes mechanism design models of two-sided markets, k-double auctions, two-sided market with per unit taxes or subsidies, any standard auction and clock auctions with one-sided private information, provided traders' types are independent and, on each side of the market, identically distributed. Here, we also determine the symmetric equilibrium bidding function for a multi-unit standard auction with reserves. Finally, large market asymptotics are discussed in Section 3.3 and Section 4 concludes.

2 Setup

Consider a market in which n buyers and m sellers with unitary demand and supply trade units of a homogeneous indivisible good. Each buyer $i \in \{1, \ldots, n\}$ has a valuation $V_i \in [\underline{v}, \overline{v}]$ independently drawn from an absolutely continuous distribution function F. Each seller $j \in \{1, \ldots, m\}$ has a cost $C_j \in [\underline{c}, \overline{c}]$ drawn independently drawn from an absolutely continuous distribution function G. To obtain an interesting setup we require that $\overline{v} > \underline{c}$ and $\underline{v} < \overline{c}$. We assume that agents are risk neutral and payoffs are quasilinear. That is, if trade occurs at price p the utility of a buyer with valuation v who receives a unit of the good with probability r is $rv - p$ and the profit of a seller with cost c who is paid the price p and asked to produce a unit with probability r is $p - rc$. We assume that the value of the outside option of not trading is 0 for every agent. We denote this economy by $\langle n, m, F, G \rangle$.

We assume there exists an intermediary who accepts bids from buyers and sellers and facilitates trade. Let $X_{(i)}$ and $X_{[i]}$ be, receptively, the i-th lowest and i-th highest element in $\boldsymbol{X} = (X_1, \ldots, X_l)$ for $i \in \{1, \ldots, l\}$. We adhere to the convention of setting $V_{[l]} = \min\{\underline{v}, \underline{c}\}$ for $l > n$ and $C_{(l)} = \max\{\overline{v}, \overline{c}\}$ for $l > m$. Similarly, we set $V_{[0]} = \overline{v}$ and $C_{(0)} = \underline{c}$.

Definition 1 *We say that a market mechanism is* quasi-Walrasian *if the equilibrium quantity traded Q is the largest index such that*

$$B\left(V_{[Q]}\right) - S\left(C_{(Q)}\right) \geq 0, \tag{1}$$

where B and S are non-decreasing functions with $B(\overline{v}) > S(\underline{c})$ and $B(\underline{v}) < S(\overline{c})$, and the types of trading agents are given by the sets $\{V_{[1]}, \ldots, V_{[Q]}\}$ and $\{C_{(1)}, \ldots, C_{(Q)}\}$.

We assume the intermediary selects a quasi-Walrasian mechanism. That is, we are interested in mechanisms that induce the ex post efficient quantity traded with respect to non-decreasing transformations of the true demand and supply schedules. The functions B and S depend on the objective of the intermediary and the equilibrium bidding strategies of buyers and sellers. This setup subsumes mechanism design models of two-sided markets, k-double auctions, and any standard auction, provided traders' types are independent and, on each side of the market, identically distributed.

Consider the set $\mathcal{X} = \{B(V_1), \ldots, B(V_n), S(C_1), \ldots, S(C_m)\}$ of transformed types. Observe that if $Q = q$, (1) is equivalent to requiring that precisely q of the n largest transformed types in the set $\mathcal{X}$ are transformed valuations. Intuitively, for every buyer among the n highest transformed types, there is a seller they can trade with among the m lowest transformed types. Thus, the equilibrium quantity traded can be considered as a random variable which counts the number of transformed valuations contained in a sample of n transformed types drawn without replacement from a population of $n + m$ transformed types. Since this sampling scheme is biased if $F \neq G$ or $B \neq S$, the equilibrium quantity traded has a non-central hypergeometric distribution. Letting $F_B := F \circ B^{-1}$ and $G_S := G \circ S^{-1}$, we have the following proposition, which follows directly from the results of Loertscher, Muir, and Taylor (2017).

Proposition 1 *When the equilibrium quantity traded in the economy $\langle n, m, F, G\rangle$ is characterised by (1), we have*

$$Q \stackrel{d}{=} \mathrm{HgG}(n, n, n + m, G_S \circ F_B^{-1}),$$

where HgG *denotes the general non-central hypergeometric distribution of Loertscher, Muir, and Taylor (2017).*

Interesting special cases arise if B and S are the identity function, so that the equilibrium quantity traded is ex post efficient. First, if $G = F$ the sampling method is unbiased and the distribution simplifies to the standard hypergeometric distribution. Second, if $G = F^{\omega}$ for some positive constant ω, the distribution simplifies to Wallenius' non-central hypergeometric distribution with weight ω.

3 Applications

We now illustrate briefly how this result applies to a variety of canonical models.

3.1 Two-Sided Private Information

We first illustrate how Proposition 1 applies to a large class of market models with unit traders when private information is held on both sides on the market. In what follows, we assume that F and G have identical, compact supports.

Bayesian mechanism design models The assumption of independently distributed types is standard in Bayesian mechanism design.[1] Bayesian mechanism design models with unit traders are considered, for example, by Myerson and Satterthwaite (1983), Gresik and Satterthwaite (1989), Williams (1999), Baliga and Vohra (2003) and Loertscher and Marx (2017). Under the ex post efficient mechanism, the equilibrium quantity traded is characterised by (1) when the functions B and S are the identity.

For a profit-maximising market maker (or designer), the equilibrium quantity traded under the optimal incentive compatible and individually rational mechanism is characterised by (1) with

$$B(v) = \Phi(v) \coloneqq v - \frac{1 - F(v)}{f(v)} \quad \text{and} \quad S(c) = \Gamma(c) \coloneqq c + \frac{G(c)}{g(c)}$$

where Φ and Γ are the virtual valuation and cost functions, respectively, which are assumed to be monotone. More generally, any designer who aims at reaching a point on the efficient frontier between expected revenue and expected social surplus under incentive compatible and individually rational mechanism uses an allocation rule that trades the largest quantity Q such that

$$\Phi_\alpha\left(V_{[Q]}\right) - \Gamma_\alpha\left(C_{(Q)}\right) \geq 0, \tag{2}$$

where, for $\alpha \in [0, 1]$, Φ_α and Γ_α are the weighted virtual types:[2]

$$\Phi_\alpha(v) \coloneqq (1 - \alpha)v + \alpha\Phi(v) \quad \text{and} \quad \Gamma_\alpha(c) \coloneqq (1 - \alpha)c + \alpha\Gamma(c). \tag{3}$$

[1] For discussions of its relaxation, see, for examples, Myerson (1981), Crémer and McLean (1985, 1988) and McAfee and Reny (1992). Kosmopoulou and Williams (1998) provide an argument for why models with independent type distributions are "robust".

[2] That points on the efficient frontier are characterised by allocation rules that depend on weighted virtual types is well known; see, for example, Norman (2004). For the specific trading setup we study here, see, for examples, Tatur (2005) or Loertscher and Marx (2017).

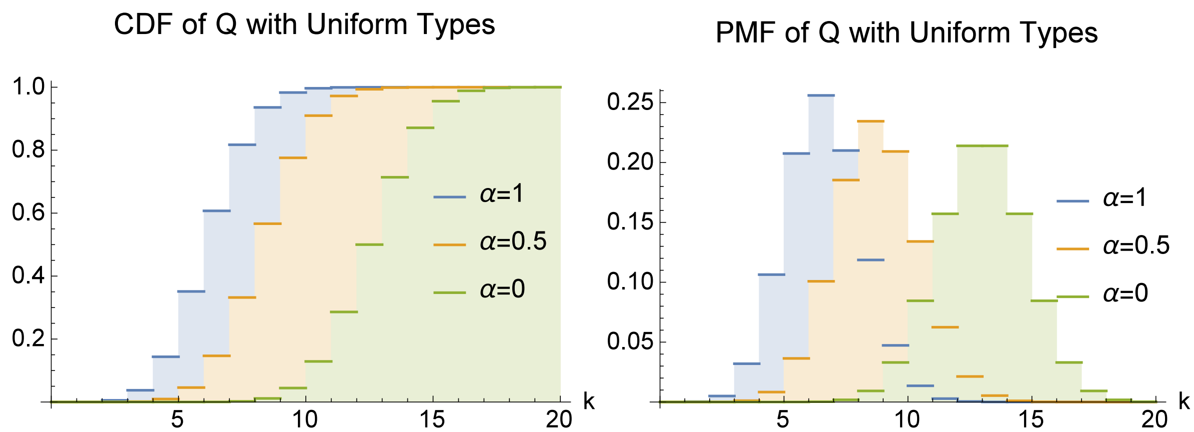

Figure 1: The effect of varying α on the distribution of the equilibrium quantity traded under the Bayesian optimal mechanism with uniform types and $n = m = 25$.

Figure 1 shows the distribution function and probability mass function of the equilibrium quantity traded for $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$, when agent types are uniformly distributed on $[0, 1]$ and $n = m = 25$. Similarly, Figure 2 illustrates the mean, variance and skewness of the equilibrium quantity traded as a function of α, when agent types are uniformly distributed on $[0, 1]$ and $n = m = 25$.

Double auctions The assumption of independent and identically distributed types with unit demands and supplies is also maintained in the vast literature on double auctions, including Chatterjee and Samuelson (1983), Satterthwaite and Williams (1989), Rustichini, Satterthwaite, and Williams (1994), Tatur (2005) and Yoon (2001).[3,4] In the case of k-double auctions, the auctioneer sets a market-clearing price $p(\mathcal{X})$, which lies between the n-th and the $(n+1)$-th highest bid. Therefore, Proposition 1 applies and the equilibrium quantity traded has a general non-central hypergeometric distribution. The functions B and S are the respective symmetric equilibrium bidding functions of buyers and sellers. For example, under the buyer's bid double auction with types uniformly distributed on $[0, 1]$, Williams (1988) showed that $B(v) = nv/(n + 1)$ and $S(c) = c$.

[3] See Cripps and Swinkels (2006) for a noteable exception.

[4] Interestingly, the equilibrium quantity traded does not have the distribution in Proposition 1 in the double auction of McAfee (1992) because the equilibrium in McAfee's double auction is not quasi-Walrasian. More generally, the equilibrium quantity traded in models with unit traders and two-sided private information will not be quasi-Walrasian in any (double) clock auction; see Loertscher and Marx (2017) for details and a formal definition of clock auctions. However, since the aforementioned mechanisms are quasi-Walrasian in the large market limit, the asymptotic results of Section 3.3 still apply.

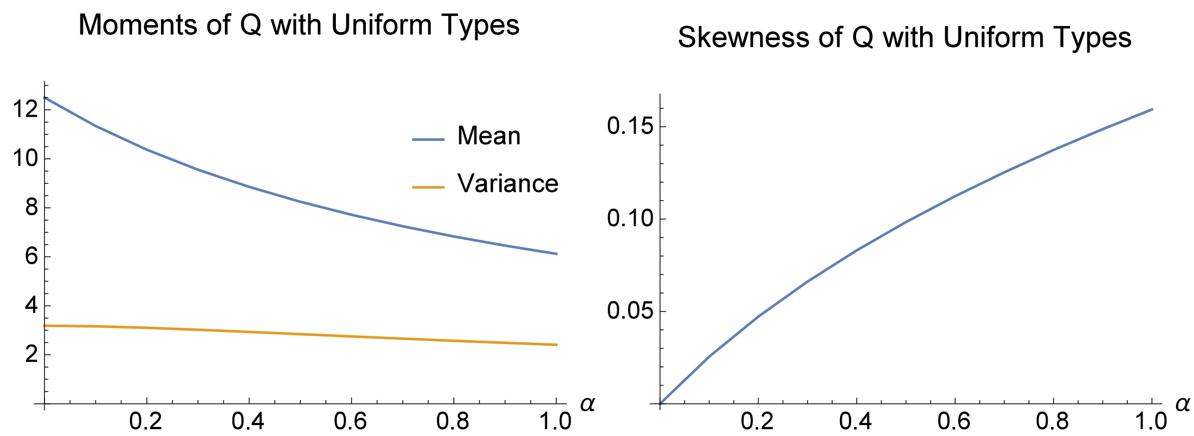

Figure 2: The effect of varying α on the moments of the equilibrium quantity traded under the Bayesian optimal mechanism with uniform types and $n = m = 25$.

Specific Taxes Now suppose that a participation fee of τ is charged to buyers (see Yoon, 2001) or that buyers are charged a transaction fee of τ (see Tatur, 2005). Such methods have been used to overcome the deficit problem.[5] Taking $B(v) = v - \tau$, the equilibrium quantity traded again has a general non-central hypergeometric distribution. Note that this result does not extend to the case in which the specific tax τ is not constant, because here announcing B does not provide a buyer with information sufficient for computing their transformed type $B(v)$ at the interim stage.

3.2 Optimal Selling Mechanism

The equilibrium quantity traded is also distributed according to the general non-central hypergeometric distribution in optimal selling mechanisms if we assume that the seller draws $m < n$ unit costs independently from the distribution G, after which the realisation $\mathbf{c}$ becomes public. We may write $c_1 < \cdots < c_m$ since the realised costs are almost surely unique. The demand side is as before, that is, we assume that there are n buyers who draw their values independently from the distribution F whose support is the same as that of G and F exhibits a monotone virtual value function $\Phi(v)$. For the sake of generality, we assume that the seller is interested in maximising a weighted sum of expected social surplus and his expected profit. This means that the equilibrium quantity traded will be quasi-Walrasian with $B(v) = \Phi_\alpha(v)$ and $S(c) = c$.

[5]See Chu (2009) for a multi-unit example.

Standard Auctions This is a standard auction setup.[6] Accordingly, the optimal mechanisms can be implemented via any standard auction format. Consider first implementation via a generalised *second-price auction*. In this auction, if the equilibrium quantity traded is q, the buyers with the q highest values obtain an item. Every trading buyer pays the maximum of $v_{[q+1]}$ and $\Phi_\alpha^{-1}(c_q)$, where $\Phi_\alpha^{-1}(c_j)$ is the reserve the seller sets for the j-th unit. For example, for $\alpha = 0$ we have ex post efficiency and thus $\Phi_\alpha^{-1}(c_j) = c_j$. For $\alpha = 1$, we obtain Myerson's optimal reserve when the cost is c_j, which is $\Phi^{-1}(c_j)$ (see Myerson, 1981).

If the units are sold using a standard auction format, the equilibrium quantity traded has a non-central hypergeometric distribution. Under a standard auction format, such as a first-price, second-price or all-pay auction, we can make use of the revenue equivalence theorem[7] to compute the equilibrium bidding function of buyers. For example, let v denote the realised value of a given buyer and set $\Phi_\alpha^{-1}(c_{m+1}) = \overline{v}$ for convenience. Under any standard auction, the buyer's interim probability of trade is zero if $v < \Phi_\alpha^{-1}(c_1)$ and we assume that such buyers do not submit bids. Otherwise, letting $j \in \{1, \dots, m\}$ be such that $\Phi_\alpha^{-1}(c_j) \le v < \Phi_\alpha^{-1}(c_{j+1})$, the symmetric equilibrium bidding function under a first-price auction is given by

$$\beta_1(v) = v - \frac{\int_{\Phi_\alpha^{-1}(c_j)}^{v} F_{[j:n-1]}(t)\, dt}{F_{[j:n-1]}(v)},$$

where

$$F_{[i:n-1]}(x) = \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{\underline{v}}^{x} F^{n-i-1}(t)(1-F(t))^{i-1} f(t)\, dt.$$

Similarly, under an all-pay auction we have

$$\beta_{\text{APA}}(v) = vF_{[j:n-1]}(v) - \int_{\Phi_\alpha^{-1}(c_j)}^{v} F_{[j:n-1]}(t)\, dt.$$

The details of calculating β_1 and β_{APA} are deferred to Appendix A. In each of these cases, the equilibrium is quasi-Walrasian with $B(v) = \Phi_\alpha(v)$ and $S(c) = c$.

Given a deterministic $\mathbf{c}$ (as in Segal, 2003) the probability that the equilibrium quantity traded is $q \in \{0, 1, \dots, m\}$ is

$$\mathbb{P}(Q = q) = \frac{n!}{q!(n-q)!} F^{n-q}\left(\Phi_\alpha^{-1}(c_{q+1})\right)\left(1 - F(\Phi_\alpha^{-1}(c_q))\right)^q, \tag{4}$$

In this case, the equilibrium quantity traded does not have the general non-central hypergeometric distribution because the assumption that costs are independent and identically distributed

[6] For example, Segal (2003) provides an analysis in which $\mathbf{c}$ is deterministic.

[7] See, for example, Vickrey (1961), Myerson (1981), Riley and Samuelson (1981), Harris and Raviv (1981), Milgrom (2004), Krishna (2010) and Börgers (2015).

no longer holds (it is as if there are m sellers who draw their costs from non-identical degenerate distributions). For the special case of constant marginal costs, that is, if $C_j = c$ for all j, the distribution of the equilibrium quantity traded specialises to a binomial distribution. In this case we again have a general non-central hypergeometric distribution where G is the degenerate distribution with a point mass at c.

Clock Auctions and Obviously Strategy-Proof Mechanisms Clock auctions, that is, open ascending (or descending) auctions, have a number of advantages over sealed bid formats including privacy preservation, which protects bidders from hold-up by the seller and the seller from the (often political) risk of regret (see, for example, Ausubel, 2004; Lucking-Reiley, 2000; Milgrom, 2004), limiting the need for information acquisition by bidders (Milgrom and Segal, 2015), and endowing bidders with obviously dominant strategies (Li, 2017). Because with one-sided private information the optimal mechanism is clock-implementable (Loertscher and Marx, 2017), it follows that the equilibrium quantity traded will also have a non-central hypergeometric distribution when the optimal mechanism is implemented via a clock auction. Moreover, because by Theorem 3 in Li (2017), any obviously strategy-proof mechanism has a clock auction implementation, it also follows that, under the statistical assumptions we impose, the equilibrium quantity traded has a non-central hypergeometric distribution for any envy-free, obviously strategy-proof mechanism.[8]

3.3 Asymptotics

Convergence properties of mechanisms in large markets have been of great and long interest to economists, particularly in the vast literature on double auctions and in the burgeoning computer science literature on prior-free mechanisms. Based on the analysis of Borovkov and Muir (2017), we are also able to characterise the distribution of the equilibrium quantity traded in such large markets under any quasi-Walrasian mechanism, provided the monotone transformations that characterise the mechanism, B_n and S_m, are asymptotically independent of the numbers of traders (or units) n and m.

For ease of exposition we assume that $m = n$.[9] Let a quasi-Walrasian mechanism char-

[8]The qualification "envy-free" is required because Li's Theorem 3 permits individualised clocks and hence individualised prices. These are ruled out by envy-freeness, which is a property of the quasi-Walrasian equilibrium and dominant strategies.

[9]The results of this section can easily be generalised in order to allow $m = \lambda n$ for any fixed $\lambda > 0$. Such an assumption, which enables one to consider sequences of markets rather than nets of markets, is standard in the literature on k-double auctions (see, for example, Gresik and Satterthwaite, 1989; Rustichini, Satterthwaite, and

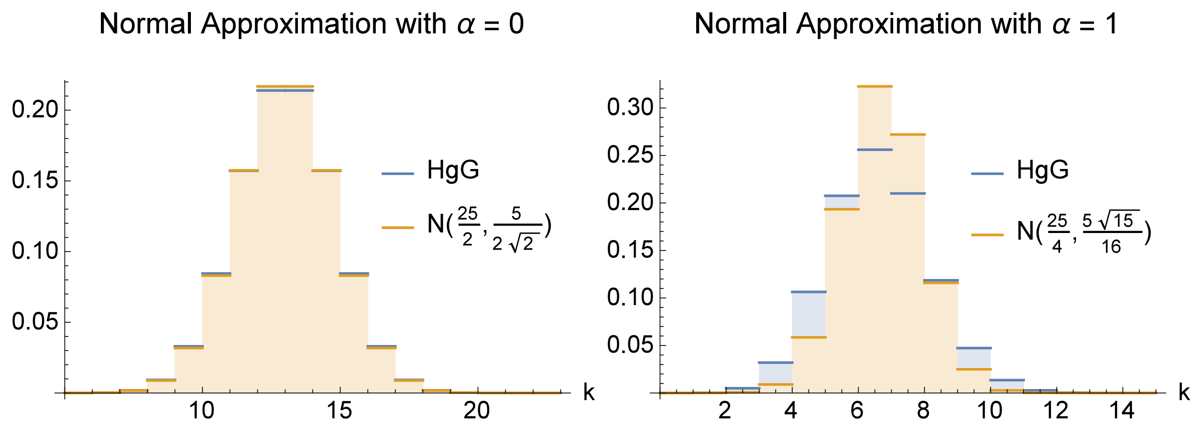

Figure 3: The normal approximation of distribution of the equilibrium quantity traded and the non-central hypergeometric distribution under the Bayesian optimal mechanisms with $\alpha = 0$ and $\alpha = 1$ for types uniformly distributed on $[0, 1]$ and $n = m = 25$.

acterised by the functions B_n and S_n be given and assume that, as $n \to \infty$, these functions converge pointwise to B and S, respectively. We need to impose another, mild regularity condition, namely that the distribution functions F_B and G_S are absolutely continuous with respective densities f_B and g_S that are bounded above and bounded away from zero on their respective supports. Moreover, the densities f_B and g_S are differentiable and the functions $\frac{d}{dt}\left(\frac{1}{f_B \circ F_B^{-1}(t)}\right)$ and $\frac{d}{dt}\left(\frac{1}{g_S \circ G_S^{-1}(t)}\right)$ are bounded on $(0, 1)$.

Denote by $E(t) = F_B^{-1}(1-t) - G_S^{-1}(t)$ the difference between the buyers' transformed type schedule and the sellers' transformed cost schedule in the limit economy when the per capita quantity traded is t. Furthermore, let t^* denote the per capita equilibrium quantity traded in the limit economy, i.e. t^* is such that $E(t^*) = 0$. Finally, let

$$\sigma^2 = \frac{1}{(E'(t^*))^2}\left[\frac{t^*(1-t^*)}{f_B^2 \circ F_B^{-1}(1-t^*)} + \frac{t^*(1-t^*)}{g_S^2 \circ G_S^{-1}(t^*)}\right].$$

The following result, where Q_n is the equilibrium quantity traded in the economy parameterised by n, is then a corollary to the main result of Borovkov and Muir (2017).[10]

Proposition 2 *As $n \to \infty$, we have $\frac{Q_n - nt^*}{n^{1/2}} \xrightarrow{d} \mathrm{N}\left(0, \sigma^2\right)$.*

Thus, the distribution of the equilibrium quantity traded is asymptotically normal and the parameters of the approximating normal distribution can be expressed in terms of the primitives

Williams, 1994). A more general treatment in which the ratio m/n is allowed to vary can be found in Borovkov and Muir (2017).

[10]Borovkov and Muir (2017) provide a stronger approximation result (in terms of almost sure convergence) and bound the approximation rate.

of the economy. Figure 3 provides an illustration for the two extreme cases that we used in Figure 1, i.e. for $\alpha = 0$ and $\alpha = 1$, assuming types are uniformly distributed on $[0, 1]$ and $n = m = 25$.[11] As becomes clear from Figure 3, the approximation performs better for $\alpha = 0$. In this case, B and S are the identity function and we have $F = G$ and a skewness of 0 (see Figure 2). Of course, Proposition 2 also applies to market mechanisms that are only quasi-Walrasian in the limit as $n \to \infty$, such as the double-auction mechanism of McAfee (1992) or the prior-free optimal clock auction of Loertscher and Marx (2017) when private information pertains to both sides of the market.

4 Conclusion

In this paper we show that in a large family of economic models the equilibrium quantity traded has a general non-central hypergeometric distribution, including mechanism design models of two-sided markets, k-double auctions, two-sided markets with per unit taxes or subsidies, any standard auction and clock auctions with one-sided private information, provided traders' types are independent and, on each side of the market, identically distributed. Our results depend on two insights. The first is that the ex post efficient quantity traded has a general non-central hypergeometric distribution. The second is that in each of the aforementioned models, the equilibrium is quasi-Walrasian. That is, the equilibrium is ex post efficient with respect to monotone transformations of the schedule of buyer valuations and the schedule of seller costs.

When agents jointly own an asset (see Cramton, Gibbons, and Klemperer, 1987), our analysis applies in the extreme case in which one agent initially owns the good. Our model also applies in a somewhat parsimonious way to the provision of public goods and club goods. In these cases, one can regard the buyers' side of the market as consisting of a single buyer whose type is drawn from the appropriate convolution.

[11] The density of the approximating normal distribution has been integrated over appropriate unit intervals in order to compare it to the probability mass function of the relevant general non-central hypergeometric distribution.

References

Ausubel, L. M. (2004): "An Efficient Ascending-Bid Auction for Multiple Objects," *American Economic Review*, 94(5), 1452–1475.

Baliga, S., and R. Vohra (2003): "Market Research and Market Design," *Advances in Theoretical Economics*, 3(1).

Börgers, T. (2015): *An Introduction to the Theory of Mechanism Design*. Oxford University Press, New York.

Borovkov, K. A., and E. V. Muir (2017): "Approximating the Equilibrium Quantity Traded and Welfare in Large Markets," *Stochastic Models*, Forthcoming.

Chatterjee, K., and W. Samuelson (1983): "Bargaining Under Incomplete Information," *Operations Research*, 31(1), 835–851.

Chu, L. Y. (2009): "Truthful Bundle/Multiunit Double Auctions," *Management Science*, 55(7), 1184–1198.

Cramton, P., R. Gibbons, and P. Klemperer (1987): "Dissolving a Partnership Efficiently," *Econometrica*, 55(3), 615–632.

Crémer, J., and R. P. McLean (1985): "Optimal Selling Strategies Under Uncertainty for a Discriminating Monopolist when Demands are Interdependent," *Econometrica*, 53(2), 345–362.

——— (1988): "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56(6), 1247–1257.

Cripps, M. W., and J. M. Swinkels (2006): "Efficiency of Large Double Auctions," *Econometrica*, 74(1), 47–92.

Gresik, T., and M. Satterthwaite (1989): "The Rate at which a Simple Market Converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms," *Journal of Economic Theory*, 48(1), 304–332.

Harris, M., and A. Raviv (1981): "Allocation Mechanisms and the Design of Auctions," *Econometrica*, 49(6), 1477–1499.

Kosmopoulou, G., and S. R. Williams (1998): "The Robustness of the Independent Private Value Model in Bayesian Mechanism Design," *Economic Theory*, 12(2), 393–421.

Krishna, V. (2010): *Auction Theory*. Academic Press, Burlington, Massachusetts, 2 edn.

Li, S. (2017): "Obviously Strategy-Proof Mechanisms," Working Paper.

Loertscher, S., and L. Marx (2017): "Optimal Clock Auctions," Working Paper.

Loertscher, S., E. V. Muir, and P. G. Taylor (2017): "A general non-central hypergeometric distribution," *Communications in Statistics – Theory and Methods*, 46(9), 4579–4598.

Lucking-Reiley, D. (2000): "Vickrey Auctions in Practice: From Nineteenth-Century Philately to Twenty-First-Century E-Commerce.," *Journal of Economic Perspectives*, 14(3), 183 – 192.

McAfee, R. P. (1992): "A Dominant Strategy Double Auction," *Journal of Economic Theory*, 56(2), 434–450.

McAfee, R. P. (1992): "A Dominant Strategy Double Auction," *Journal of Economic Theory*, 56(2), 434–450.

McAfee, R. P., and P. J. Reny (1992): “Correlated Information and Mechanism Design,” *Econometrica*, 60(2), 395–421.

Milgrom, P., and I. Segal (2015): “Deferred-Acceptance Auctions and Radio Spectrum Reallocation,” Working Paper, Stanford University.

Milgrom, P. R. (2004): *Putting Auction Theory to Work*. Cambridge University Press, Cambridge.

Myerson, R. (1981): “Optimal Auction Design,” *Mathematics of Operations Research*, 6(1), 58–78.

Myerson, R., and M. Satterthwaite (1983): “Efficient Mechanisms for Bilateral Trading,” *Journal of Economic Theory*, 29(2), 265–281.

Norman, P. (2004): “Efficient Mechanisms for Public Goods with Use Exclusions.,” *Review of Economic Studies*, 71(4), 1163 – 1188.

Riley, J. G., and W. F. Samuelson (1981): “Optimal Auctions,” *American Economic Review*, 71(3), 381–392.

Rustichini, A., M. A. Satterthwaite, and S. R. Williams (1994): “Convergence to Efficiency in a Simple Market with Incomplete Information,” *Econometrica*, 62(5), 1041–1063.

Satterthwaite, M., and S. R. Williams (1989): “Bilateral Trade with the Sealed k-Double Auction: Existence and Efficiency,” *Journal of Economic Theory*, 48(1), 107–133.

——— (2002): “The Optimality of a Simple Market Mechanism,” *Econometrica*, 70(5), 1841–1863.

Segal, I. (2003): “Optimal Pricing Mechanisms with Unknown Demand,” *American Economic Review*, 93(3), 509–529.

Tatur, T. (2005): “On the Tradeoff Between Deficit and Inefficiency and the Double-Auction with a Fixed Transation Fee,” *Econometrica*, 73(2), 517–570.

Vickrey, W. (1961): “Counterspeculation, Auction, and Competitive Sealed Tenders,” *Journal of Finance*, 16(1), 8–37.

Wallenius, K. T. (1963): “Biased Sampling: The Non-central Hypergeometric Probability Distribution,” Ph.D. thesis, Department of Statistics, Stanford University, Also published with the same title as Technical Report no. 70.

Williams, S. R. (1988): “The Nature of Equilibria in the Buyer’s Bid Double Auction,” Working Paper.

——— (1999): “A Characterization of Efficient, Bayesian Incentive Compartible Mechanisms,” *Economic Theory*, 14(1), 155–180.

Yoon, K. (2001): “The Modified Vickrey Double Auction.,” *Journal of Economic Theory*, 101(2), 572 – 584.

A Symmetric Equilibrium Bidding Function under a Standard Multi-Unit Auction

Let v denote the valuation of a given buyer, with $v \geq \Phi_\alpha^{-1}(c_1)$. Let $j \in \{1, \ldots, m\}$ be such that $\Phi_\alpha^{-1}(c_j) \leq v < \Phi_\alpha^{-1}(c_{j+1})$ and let $V_{[j:n-1]}$ denote the j-th highest valuation among the $n-1$ competing buyers. The buyer's probability of trade is

$$r(v) = \mathbb{P}(v \geq V_{[j:n-1]}) = F_{[j:n-1]}(v)$$

and the interim expected payment under a second price auction is

$$\begin{aligned}
m_2(v) &= \mathbb{E}[\max\{V_{[j:n-1]}, \Phi_\alpha^{-1}(c_j)\} \mid v \geq V_{[j:n-1]}]\mathbb{P}(v \geq V_{[j:n-1]}) \\
&= \mathbb{E}[V_{[j:n-1]} \mid v \geq V_{[j:n-1]} \geq \Phi_\alpha^{-1}(c_j)]\mathbb{P}(v \geq V_{[j:n-1]} \geq \Phi_\alpha^{-1}(c_j)) \\
&+ \mathbb{E}[\Phi_\alpha^{-1}(c_j) \mid \Phi_\alpha^{-1}(c_j) \geq V_{[j:n-1]}]\mathbb{P}(\Phi_\alpha^{-1}(c_j) \geq V_{[j:n-1]}) \\
&= \int_{\Phi_\alpha^{-1}(c_j)}^{v} t\, dF_{[j:n-1]}(t) + \Phi_\alpha^{-1}(c_j) \int_{\underline{v}}^{\Phi_\alpha^{-1}(c_j)} dF_{[j:n-1]}(t).
\end{aligned}$$

Integrating by parts we have

$$\begin{aligned}
m_2(v) &= \left[tF_{[j:n-1]}(t)\right]_{\Phi_\alpha^{-1}(c_j)}^{v} - \int_{\Phi_\alpha^{-1}(c_j)}^{v} F_{[j:n-1]}(t)\, dt + \Phi_\alpha^{-1}(c_j) \left[F_{[j:n-1]}(t)\right]_{\underline{v}}^{\Phi_\alpha^{-1}(c_j)} \\
&= vF_{[j:n-1]}(v) - \int_{\Phi_\alpha^{-1}(c_j)}^{v} F_{[j:n-1]}(t)\, dt.
\end{aligned}$$

We may now use the revenue equivalence theorem to determine the symmetric equilibrium bidding function under any standard auction. For example, under a first-price auction, the symmetric equilibrium bidding function is given by

$$\beta_1(v) = \frac{m_2(v)}{r(v)} = v - \frac{\int_{\Phi_\alpha^{-1}(c_j)}^{v} F_{[j:n-1]}(t)\, dt}{F_{[j:n-1]}(v)}.$$

Similarly, under an all-pay auction we have $\beta_{\text{APA}}(v) = m_2(v)$.