

Optimal labor procurement under minimum wages and monopsony power*

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Abstract

We characterize when involuntary unemployment arises under optimal monopsony procurement—with and without minimum wage constraints—providing a price-theoretic foundation for efficiency wages. Involuntary unemployment caused by market power can be distinguished from that caused by a minimum wage: only the former is accompanied by wage dispersion. If wage dispersion and involuntary unemployment arise absent regulation, a minimum wage equal to the highest laissez-faire wage increases employment and workers’ pay and decreases involuntary unemployment. Although minimum wages can induce discontinuous increases in wage dispersion and involuntary unemployment, social and worker surplus increase continuously in the minimum wage up to the competitive wage.

Keywords: Market power, efficiency wage, price regulation, involuntary unemployment, wage dispersion

JEL-Classification: C72, D47, D82

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1 Introduction

Minimum wage legislation has been around for over a century and continues to play a prominent role in controversial policy debates.¹ Yet how minimum wages affect workers' pay, total employment, and involuntary unemployment remains an open question, even in theory. This paper answers that question for a model in which a monopsony firm uses an optimal procurement mechanism, with and without minimum wage constraints. There is a continuum of workers of equal productivity who are privately informed about their opportunity costs of working.² In this otherwise standard labor market setting, we allow the firm to use any incentive-compatible and individually rational procurement mechanism satisfying, where applicable, a minimum wage constraint. The mechanism design approach yields a price-theoretic foundation for the efficiency wage theory of unemployment: the model explains why, and when, a firm optimally uses non-market-clearing wages that induce involuntary unemployment. By construction, the firm cannot do better by using more elaborate contracts, which is a problem that has plagued earlier theories of efficiency wages and involuntary unemployment.

We show that regardless of whether or not the firm faces a binding minimum wage, at most two wages are paid in equilibrium under a separation condition on the firm's procurement cost function.³ Cost-minimizing procurement under *laissez-faire* involves setting two wages (i.e., using a two-wage mechanism) and inducing involuntary unemployment if and only if the procurement cost function under optimal uniform wage setting lies above its convexification at the optimal employment level. If the firm optimally induces involuntary unemployment, it does so by hiring some workers at an efficiency wage that is above the market-clearing wage. Setting a minimum wage equal to the efficiency wage then reduces—and possibly eliminates—involuntary unemployment and increases total employment. A minimum wage can make a two-wage mechanism and involuntary unemployment optimal even if setting a market-clearing wage is always optimal under *laissez-faire*.⁴

Notwithstanding this richness, the analysis offers clear predictions regarding the effects

¹While precursors to minimum wage legislation date back to the Hammurabi Code (c. 1755–1750 BC), New Zealand became the first country to implement a minimum wage in 1894, followed by the Australian state of Victoria in 1896, and the United Kingdom in 1909 (Starr, 1981).

²See, for example, Lee and Saez (2012).

³The firm can pay two wages in equilibrium by rationing some workers at the high wage or by hiring both full-time and part-time workers and paying part-time workers a higher hourly wage. For empirical evidence that part-time workers earn higher hourly wages than full-time workers, see Booth and Wood (2008) and Day and Rodgers (2015) and the discussion in Section 5.2.

⁴For example, if the firm's marginal revenue product of labor is constant, then setting a market-clearing wage under *laissez-faire* is always optimal. However, using a two-wage mechanism may become optimal under a binding minimum wage.

of marginally increasing a prevailing minimum wage on the basis of whether or not there is wage dispersion and/or involuntary unemployment. We show that exactly three cases are possible. First, if there is both involuntary unemployment and wage dispersion, then a marginal increase in the minimum wage increases employment and decreases involuntary unemployment. Second, if there is no involuntary unemployment and, consequently, no wage dispersion at a given minimum wage, then a marginal increase in the minimum wage increases employment.⁵ Third, if there is involuntary unemployment and no wage dispersion, then any increase in the minimum wage decreases employment and increases involuntary unemployment.⁶ Thus, our analysis shows that a regulator observing involuntary unemployment can identify whether it is caused by market power or by a minimum wage on the basis of whether or not there is also wage dispersion. Intuitively, wage dispersion indicates that the firm is exerting market power by engaging in a form of wage discrimination, and an appropriately set minimum wage curbs that power.

The welfare effects of minimum wages are similarly clear-cut. For all minimum wages up to the competitive wage, employment, workers' pay, social surplus, and worker surplus increase continuously in the minimum wage. This is notable because when an increase in the minimum wage induces the firm to switch from a uniform wage to a two-wage mechanism, wage dispersion and involuntary unemployment increase discontinuously. Social surplus and worker surplus nevertheless increase continuously at such transitions because the firm initially hires workers at the efficiency wage with vanishingly small probability.

Deriving these results requires characterizing the monopsony's optimal procurement mechanism in the face of an arbitrary minimum wage constraint. If—as in almost all of the prior literature—the firm is restricted to setting a uniform wage, this is straightforward: it simply cannot screen workers whose opportunity cost of supplying labor lies below the minimum wage. We instead place no restriction on the contracts the firm can offer. It can then screen such workers by coupling wages with appropriately chosen employment lotteries⁷ and do strictly better as a result. Solving the firm's problem therefore amounts to determining how it *optimally* screens workers under a constraint that makes separating certain worker types more costly. That the optimal mechanism nevertheless reduces to no more than three parameters—the at most two wages and the probability of being rationed at the high wage—is what makes the comparative statics tractable, despite the countervailing effects that arise.

As we discuss in Section 5, many of our conclusions do not depend on the particular

⁵As we will see, in such cases the firm optimally sets a uniform wage. Because this effect was first observed by Robinson (1933), who assumed uniform wages, we collectively refer to these cases as the Robinson region.

⁶As we will see, in such cases the firm is a price taker and does not exert market power under the prevailing minimum wage. For this reason we collectively refer to these cases as the neoclassical region.

⁷Or, equivalently, by coupling each hourly wage with an appropriate level of part-time employment.

modeling assumptions we have made. The superiority of two-wage over uniform-wage mechanisms is robust to risk-averse workers, to small errors in setting the two wages, and to rationing schemes other than uniform random rationing. Our results also don't hinge on the firm being a monopsonist: replacing its objective with a Ramsey objective that parameterizes the degree of market power leaves the comparative statics intact. One feature of our model does play a central role in the analysis, but it is related to a standard assumption that we drop rather than impose. Specifically, convexity of the procurement cost function is customarily assumed in models with uniform wage setting, typically as a matter of analytical convenience rather than on the basis of empirical evidence. Section 5.1 discusses where and why non-convex costs can be expected to arise, and the evidence bearing on this. The separation condition that we do impose is a further restriction on the shape of the cost function, and Section 5.1 also shows what survives without it: three wages suffice, and the classification of minimum wage effects on employment by observables described above continues to hold. In particular, involuntary unemployment accompanied by wage dispersion still indicates market power, and a marginal increase in the minimum wage increases employment.

Related literature The idea that involuntary unemployment is beneficial for businesses and detrimental for workers is an old one, dating at least as far back as Friedrich Engels's and Karl Marx's notion of a *reserve army of labor*.⁸ Our paper provides an optimal pricing theory of involuntary unemployment. As we have already discussed, because the firm is choosing among all incentive-compatible mechanisms, it cannot do better by using more elaborate contracts, which is the difficulty that has plagued earlier theories of efficiency wages and involuntary unemployment.⁹

More generally, this paper relates to the literature on monopoly and monopsony pricing, including Mussa and Rosen (1978), Bulow and Roberts (1989) and Loertscher and Muir (2022), by studying a setting without aggregate uncertainty and without assuming regularity in the sense of Myerson (1981).¹⁰ Among these papers, it is closest to Mussa and

⁸See Engels (1845) and Marx (1867).

⁹For an overview of the early literature on the efficiency wage theory of involuntary unemployment, see the collection of essays in Akerlof and Yellen (1986). This literature formalized the idea that firms deliberately offer wages exceeding the market-clearing level so that the resulting excess supply of labor (and corresponding level of involuntary unemployment) can be used to discipline their workforce. For evidence on the inter-industry wage differentials that motivated much of this literature, see Krueger and Summers (1988). As Yellen (1984, p. 202) put it: "All these models suffer from a similar theoretical difficulty—that employment contracts more ingenious than the simple wage schemes considered, can reduce or eliminate involuntary unemployment."

¹⁰There has been a recent upsurge of interest in mechanism design problems without regularity; see, for example, Condorelli (2012), Dworzak, Kominers, and Akbarpour (2021), Kleiner et al. (2021) and

Rosen (1978) and Loertscher and Muir (2022), which likewise employ mechanism design techniques to solve the firm’s problem. Absent a binding minimum wage constraint, our problem is the monopsony version of the baseline monopoly pricing problem studied in Loertscher and Muir (2022).¹¹ The procurement setting itself builds on Lee and Saez (2012). Since it concerns minimum wages and labor procurement, this paper also relates to the labor literature, in which there is growing recognition that employers exert market power; see, for example, Card (2022a,b). Robinson (1933) was the first to recognize that in the presence of monopsony power, minimum wages can increase employment without causing involuntary unemployment.¹² Our analysis generalizes hers by allowing the firm to offer arbitrary contracts rather than restricting it to a uniform wage. From a methodological perspective, minimum wages directly constrain the transfer rule and thus differ from the constraints typically analyzed in the mechanism design literature, which can be imposed directly on the allocation rule. A notable exception is Gomes and Pavan (2024), who show how uniform-price constraints on one side of the market can be handled in a two-sided matching problem in which non-linear prices are optimal under *laissez-faire*.

While we do not derive optimally regulated prices, our methodology may prove useful for analyses of optimal regulation in the tradition of Ramsey (1927) and Boiteux (1956) according to which the regulator’s objective consists of a convex combination of social surplus and the firm’s profit, without restricting the firm’s contracting space; see also Wilson (1993). In light of the recent upsurge of interest in regulation, particularly in the context of “big tech” (see, for example, Australian Competition and Consumer Commission, 2019; Cr  mer et al., 2019; Furman et al., 2019; Stigler Center, 2019), there is potentially a wide range of applications for this methodology. Notable prior work along these lines includes Armstrong et al. (1995), who adopt a mechanism design approach to study price regulation of a monopoly seller that optimally engages in non-linear pricing. In contrast to our approach, these authors consider a regular setting in which the seller faces a concave revenue function. Consequently, applied to a labor market setting in which individual labor supply is price-inelastic, their analysis does not give rise to efficiency wages and involuntary unemployment and the corresponding comparative statics that are the focus of our paper.

Akbarpour, Dworzak, and Kominers (2024). However, none of these papers consider pricing constraints. Loertscher and Muir (2024) illustrate the optimal mechanisms in the presence of price caps and minimum wages for piecewise linear demand and supply.

¹¹Like Mussa and Rosen (1978), Loertscher and Muir (2022) allow for vertically differentiated products. They analyze resale that arises due to randomization and study optimal selling mechanisms with resale.

¹²See also Stigler (1946). As shown by Bhaskar, Manning, and To (2002), this basic logic also extends to imperfectly competitive markets. There is empirical evidence consistent with these effects, the classic reference being Card and Krueger (1994). More recently, Wiltshire (2021) provides an analysis of the labor market effects of Walmart supercenters and the effects of minimum wages in the presence of monopsony power, as well as a comprehensive overview of this strand of literature.

The remainder of this paper is structured as follows. Section 2 introduces the baseline procurement setup and revisits Robinson’s (1933) analysis of minimum wage effects in the presence of monopsony power, which assumes uniform pricing. In Section 3, we derive the optimal procurement mechanism with and without a minimum wage constraint. Section 4 derives the corresponding comparative statics and provides policy guidance. Section 5 discusses modeling assumptions, robustness and extensions. The paper concludes with Section 6.

2 Setup

We consider the procurement problem of a monopsony firm (hereafter, the firm) that faces a unit mass of workers. We let $V(Q)$ denote the firm’s marginal revenue product of labor when it hires $Q \in [0, 1]$ workers. Equivalently, $V(Q)$ is the firm’s willingness to pay for the Q -th unit of labor. We assume that V is continuous and (weakly) decreasing.¹³ If V is constant on some intervals and the firm’s demand for workers is therefore not uniquely determined, we break ties by assuming that it employs the maximum number of workers. Accordingly, the firm’s labor demand at any wage $w \in \mathbb{R}$ is $D(w) := \max\{Q \in [0, 1] : V(Q) \geq w\}$ if $w \leq V(0)$ and $D(w) := 0$ if $w > V(0)$.

Let W denote the inverse labor supply schedule faced by the firm, so that $W(Q)$ is the *market-clearing wage* required to induce $Q \in [0, 1]$ units of labor supply. We assume that W is strictly increasing and absolutely continuous, with a finite right derivative $W'_+ > 0$ at every $Q \in [0, 1)$ and a finite left derivative $W'_- > 0$ at every $Q \in (0, 1]$, and that $W(0) \geq 0$. The corresponding labor supply schedule is $S := W^{-1}$. For expositional convenience, we also assume that $V(0) > W(0)$ and $V(1) < W(1)$.¹⁴ The firm’s cost function

$$C(Q) := W(Q)Q \tag{1}$$

specifies the cost of procuring $Q \in [0, 1]$ units at the market-clearing wage $W(Q)$. It follows that the cost function C is also strictly increasing and absolutely continuous, with a right derivative C'_+ at every $Q \in [0, 1)$ and a left derivative C'_- at every $Q \in (0, 1]$.

¹³If the firm uses Q units of input to generate downstream profit $\Pi(Q)$, where Π is concave and differentiable with $\Pi'(0) > 0$, then the firm’s willingness to pay for the Q -th unit of input is $V(Q) = \Pi'(Q)$. For example, the firm could be a monopoly in the downstream market with access to a production technology that transforms labor into output. Because we allow V to be constant on some subintervals of $[0, 1]$, this specification also accommodates regions in which marginal revenue is constant due to ironing (see, for example, Loertscher and Muir, 2022). Alternatively, the firm could be a downstream price taker with access to a production technology exhibiting a constant or diminishing marginal product of labor.

¹⁴As we will see, these assumptions ensure that there are strictly positive masses of both employed and unemployed workers under optimal procurement.

The competitive level of employment, denoted Q^p , satisfies $V(Q^p) = W(Q^p)$. This is the employment level that would emerge under price-taking behavior by the firm.¹⁵ We refer to $W(Q^p)$ as the *competitive wage*.

Following Lee and Saez (2012), we microfound the inverse labor supply schedule W using an extensive-margin formulation. Lee and Saez argue that this is the empirically relevant margin, particularly in low-wage industries which are most affected by minimum wages. Specifically, we assume that the firm faces a unit mass of equally productive workers, each of whom supplies a single unit of labor inelastically. Each worker has a private opportunity cost $c \in [\underline{c}, \bar{c}] := [W(0), W(1)]$ of supplying labor. The cumulative distribution function of worker costs is denoted by G , with density $g > 0$ on $[\underline{c}, \bar{c}]$, implying that $G = S$, and $G^{-1} = W$. We also assume that workers are risk neutral and have quasi-linear utility. The interim expected payoff of a worker with cost $c \in [\underline{c}, \bar{c}]$ who is hired at wage $w \in \mathbb{R}$ with probability $x \in [0, 1]$ is then $x(w - c)$. We maintain this Lee-Saez microfoundation of W and the corresponding cost function C as defined in (1) throughout the paper.

Mechanisms We consider direct procurement mechanisms $\langle x, w \rangle$ consisting of an allocation rule $x : [\underline{c}, \bar{c}] \rightarrow [0, 1]$ that maps worker reports to their probability of employment and a wage schedule $w : [\underline{c}, \bar{c}] \rightarrow \mathbb{R}$ that maps worker reports to a wage that is paid conditional on employment. For all $c, \hat{c} \in [\underline{c}, \bar{c}]$, incentive compatibility requires

$$x(c)(w(c) - c) \geq x(\hat{c})(w(\hat{c}) - c). \quad (\text{IC})$$

An immediate and well-known implication is that a mechanism $\langle x, w \rangle$ satisfies (IC) only if x is decreasing. Assuming that each worker receives the same payoff from not working and normalizing this payoff to 0, individual rationality requires that, for all $c \in [\underline{c}, \bar{c}]$,

$$x(c)(w(c) - c) \geq 0. \quad (\text{IR})$$

By the revelation principle, focusing on direct mechanisms is without loss of generality. Since there is no aggregate uncertainty, it is further without loss of generality to restrict attention to direct mechanisms that determine the employment probability and wage of a given worker independently of the reports of other workers. Finally, we are also restricting attention to direct mechanisms in which $w(c)$ is a deterministic wage paid to a worker who reports cost c , conditional on employment. We refer to this as *ex post implementation*. Without loss of generality, we adopt the convention that $w(c) = 0$ for all $c \in [\underline{c}, \bar{c}]$ with $x(c) = 0$. A minimum

¹⁵Since V is decreasing and W is strictly increasing and these functions satisfy $V(0) > W(0)$ and $V(1) < W(1)$, $Q^p \in (0, 1)$ exists and is unique.

wage $\underline{w} \in [W(0), W(1)]$ can then be incorporated simply by requiring that, for all $c \in [\underline{c}, \bar{c}]$ with $x(c) > 0$,

$$w(c) \geq \underline{w}. \quad (\text{MW})$$

Beyond matching real-world practice, we will show that ex post implementation is without loss of generality, both under laissez-faire and under a minimum wage constraint.¹⁶

We say that an incentive compatible and individual rational direct mechanism $\langle x, w \rangle$ with total employment Q involves an *efficiency wage* w_e if $w_e > W(Q)$ and the set $\{c \in [\underline{c}, \bar{c}] : w(c) \geq w_e, x(c) > 0\}$ has positive measure. Any mechanism involving an efficiency wage necessarily induces *involuntary unemployment* since there is a positive mass of workers who are willing to supply labor at the efficiency wage but are nevertheless unemployed.¹⁷ The size of involuntary unemployment is the mass of such workers. A mechanism involves *wage dispersion* if it pays different wages to different workers, and we measure wage dispersion by the difference between the highest and lowest wages paid.

An incentive compatible and individually rational direct mechanism $\langle x, w \rangle$ is called a *uniform-wage mechanism* with wage w_u if there exists $\rho \in (0, 1]$ such that

$$x(c) = \begin{cases} \rho, & c \leq w_u, \\ 0, & c > w_u. \end{cases}$$

Total employment is then $\rho S(w_u)$, and involuntary unemployment is $(1 - \rho)S(w_u)$. If $\rho = 1$, the uniform-wage mechanism is a *market-clearing mechanism* with wage w_u and total employment $S(w_u)$. Uniform-wage mechanisms play a prominent role in the analysis of minimum wage effects including in the neoclassical model and in the analysis of Robinson (1933), both of which we revisit at the end of this section. A more general family of mechanisms, which we call *two-wage mechanisms*, is parameterized by three quantities Q , q_1 , and q_2 satisfying $0 \leq q_1 < Q \leq q_2 \leq 1$. Under the two-wage mechanism parameterized by (Q, q_1, q_2) , the firm hires the q_1 workers with costs below $W(q_1)$ with certainty at a low wage w_1 , and rations the $q_2 - q_1$ workers with costs in $[W(q_1), W(q_2))$ uniformly at random into $Q - q_1$ openings at a high wage w_2 . The individual rationality constraint of the marginal workers that participate

¹⁶In a general direct mechanism, a worker's transfer, conditional on employment status, may be random. We interpret the minimum wage as requiring that each worker's realized transfer is at least $\underline{w}$ times the amount of labor the worker supplies, which is one unit if employed and zero otherwise. Under ex post implementation this reduces to (MW). Theorem 1 shows that the minimal cost of procurement over general direct mechanisms is attained under ex post implementation.

¹⁷Defined this way, involuntary unemployment is observable, given the appropriate data; see, for example, Breza et al. (2021).

in the mechanism implies that $w_2 = W(q_2)$. Moreover, incentive compatibility implies that workers with cost $W(q_1)$ must be indifferent between working at the low wage w_1 and being hired at the high wage w_2 with probability $\beta(Q, q_1, q_2) := (Q - q_1)/(q_2 - q_1)$.¹⁸ That is, $w_1 - W(q_1) = \beta(Q, q_1, q_2)(W(q_2) - W(q_1))$ must hold, which is equivalent to

$$w_1(Q, q_1, q_2) = \beta(Q, q_1, q_2)W(q_2) + (1 - \beta(Q, q_1, q_2))W(q_1). \quad (2)$$

The firm's corresponding procurement cost is

$$q_1 w_1(Q, q_1, q_2) + (Q - q_1)w_2(q_2) = \beta(Q, q_1, q_2)C(q_2) + (1 - \beta(Q, q_1, q_2))C(q_1).$$

When $q_2 = Q$, the two-wage mechanism collapses to a market-clearing mechanism and when $q_1 = 0$ it reduces to the uniform-wage mechanism with $w_u = W(q_2)$ and $\rho = Q/q_2$. We call a two-wage mechanism with $0 < q_1 < Q < q_2$ *non-degenerate*. A non-degenerate two-wage mechanism pays the efficiency wage $W(q_2) > W(Q)$ to the $Q - q_1$ employed workers with costs above $W(q_1)$, and thus involves involuntary unemployment $q_2 - Q$ and wage dispersion $W(q_2) - w_1 > 0$.

Under an incentive compatible and individual rational direct mechanism $\langle x, w \rangle$ with total employment $Q = \int_{\underline{c}}^{\bar{c}} x(c) dG(c)$, worker surplus is $\int_{\underline{c}}^{\bar{c}} x(c)(w(c) - c) dG(c)$, social surplus is $\int_0^Q V(y) dy - \int_{\underline{c}}^{\bar{c}} x(c)c dG(c)$, and the firm's profit is equal to $\int_0^Q V(y) dy - \int_{\underline{c}}^{\bar{c}} x(c)w(c) dG(c)$ (i.e., the difference between social and worker surplus). It is straightforward to show that social surplus is maximized by the market-clearing mechanism with $w_u = W(Q^p)$.

Before studying the firm's optimal procurement problem in Section 3, we briefly revisit the *neoclassical model*, which assumes price-taking behavior by the firm, and Robinson's (1933) analysis of monopsony pricing, which assumes uniform wage setting.

Neoclassical model In what we refer to as the *neoclassical model*, the firm is assumed to be a price taker. This means that the laissez-faire quantity is equal to Q^p and introducing a minimum wage below $W(Q^p)$ has no effect. In contrast, introducing a minimum wage $\underline{w} > W(Q^p)$ induces the firm to hire $D(\underline{w})$ workers, leading to involuntary unemployment of size $S(\underline{w}) - D(\underline{w})$. Consequently, for $\underline{w} \geq W(Q^p)$, employment decreases in $\underline{w}$ and involuntary unemployment increases in $\underline{w}$.

The neoclassical model thus captures situations in which minimum wages are either ineffective or reduce social surplus. This contrasts with settings involving market power,

¹⁸It is straightforward to verify that the setup satisfies a single-crossing condition which ensures that all workers with $c < W(q_1)$ strictly prefer working with certainty at the low wage w_1 , all workers with $c \in (W(q_1), W(q_2))$ strictly prefer entering the lottery to work at the high wage w_2 , and workers with $c > W(q_2)$ strictly prefer not to participate in the mechanism.

where a minimum wage can increase employment and social surplus, and need not induce involuntary unemployment.

Robinson (1933) revisited We now consider Robinson’s (1933) analysis of monopsony power under uniform wage setting. Her analysis provides an explanation for the empirical findings of Card and Krueger (1994) and constitutes the first study of labor market power (see, for example, Card, 2022b).

We begin with the case in which C is strictly convex. The optimal level of employment under laissez-faire and uniform wage setting, denoted Q_U^ℓ , satisfies $C'_-(Q_U^\ell) \leq V(Q_U^\ell) \leq C'_+(Q_U^\ell)$. Observe that $Q_U^\ell \in (0, Q^p)$ and that the equilibrium wage $W(Q_U^\ell)$ is lower than the competitive wage $W(Q^p)$. Panel (a) of Figure 1 illustrates this. The gap between the laissez-faire equilibrium wage and the competitive wage creates scope for employment-increasing minimum wage regulation.

Under uniform wage setting, the cost $C_U(Q, \underline{w})$ of procuring the quantity Q given a minimum wage $\underline{w}$ is

$$C_U(Q, \underline{w}) = \begin{cases} \underline{w}Q, & Q \in [0, S(\underline{w})], \\ C(Q), & Q > S(\underline{w}). \end{cases} \quad (3)$$

For any minimum wage $\underline{w} \in (W(Q_U^\ell), W(Q^p))$, the firm’s marginal cost of hiring Q workers is $\underline{w}$ for $Q \leq S(\underline{w})$ and $C'_-(Q)$ for $Q > S(\underline{w})$. Since $\underline{w} < V(Q)$ for $Q \leq S(\underline{w})$, whereas $C'_-(Q) > V(Q)$ for $Q > S(\underline{w})$, the firm optimally chooses to employ $S(\underline{w})$ workers. Thus, a minimum wage in the interval $(W(Q_U^\ell), W(Q^p))$ increases employment without creating any involuntary unemployment. This is illustrated in panel (b) of Figure 1.

Because $C'_+(S(\underline{w})) > \underline{w}$, the marginal cost $C'_U(\cdot, \underline{w})$ of procurement under the minimum wage $\underline{w}$ is discontinuous at $Q = S(\underline{w})$.¹⁹ The employment- and social-surplus-increasing effects of minimum wages arise because $V(S(\underline{w})) \in [\underline{w}, C'_+(S(\underline{w}))]$. That is, $V(S(\underline{w}))$ lies within the discontinuity in the marginal cost function induced by the minimum wage $\underline{w}$. At the boundary $\underline{w} = W(Q^p)$, the firm still employs $S(\underline{w}) = Q^p$ workers, but the marginal cost function $C'_U(\cdot, \underline{w})$ is continuous (i.e., there is no discontinuity at $Q = S(\underline{w})$). Increasing the minimum wage beyond $W(Q^p)$ causes the firm to hire $D(\underline{w})$ workers, inducing involuntary unemployment of size $S(\underline{w}) - D(\underline{w})$.

In sum, when C is convex, there are three regions associated with the introduction of a minimum wage $\underline{w}$ under monopsony power and uniform wage setting. First, if $\underline{w} \leq W(Q_U^\ell)$, then the minimum wage does not bind. Second, there is what we call the *Robinson region*, characterized by $\underline{w} \in (W(Q_U^\ell), W(Q^p)]$, in which employment is $S(\underline{w})$ and the equilibrium

¹⁹Formally, we define the marginal cost envelope $C'_U(Q, \underline{w})$ to be the left derivative of C_U with respect to Q .

wage is $\underline{w}$. Finally, there is a *neoclassical region* characterized by $\underline{w} > W(Q^p)$ and involuntary unemployment of the size $S(\underline{w}) - D(\underline{w})$.

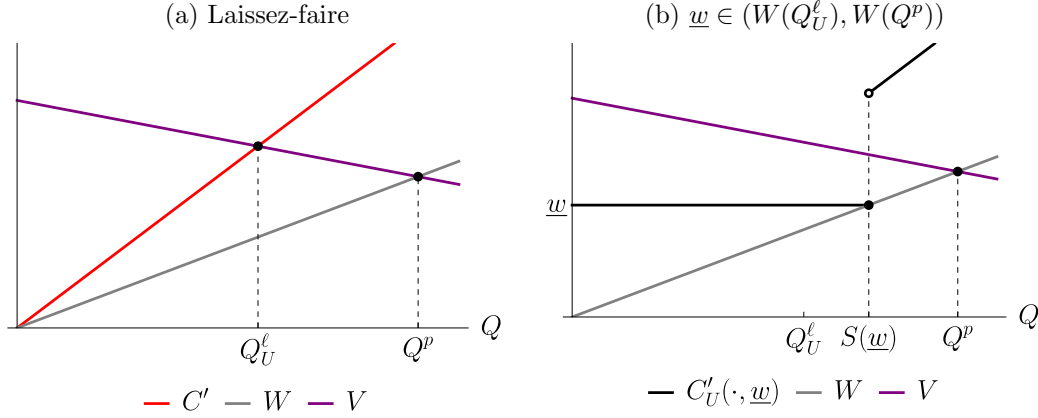


Figure 1: This figure illustrates Robinson's (1933) analysis when W is linear. Panel (a) illustrates the optimal level of employment Q_U^ℓ under laissez-faire and uniform wage setting, as well as the perfectly competitive level of employment Q^p . Panel (b) illustrates the introduction of a minimum wage $\underline{w} \in (W(Q_U^\ell), W(Q^p))$ under uniform wage setting, and shows that this induces the firm to hire $S(\underline{w})$ workers. Observe that $V(S(\underline{w}))$ falls within the discontinuity in the firm's marginal cost of procurement $C'_U(\cdot, \underline{w})$.

Under laissez-faire and uniform wages, dropping the assumption that C is convex mainly introduces the possibility of multiple local maxima of the firm's profit function, as illustrated in panel (a) of Figure 2. This complicates the analysis without adding substantive economic insights, which may explain why convexity is such a common assumption in the literature.²⁰

In contrast, introducing a minimum wage when C is not convex can have rich and, arguably, surprising effects even under uniform wage setting. The minimum wage effects are effectively the same as in the convex case whenever the global maximum of the firm's profit function coincides with the local maximum associated with the largest quantity. In this case, there is a non-binding region, a Robinson region in which employment is $S(\underline{w})$ and the equilibrium wage is $\underline{w}$, and a neoclassical region in which the firm hires $D(\underline{w})$ workers and involuntary unemployment is $S(\underline{w}) - D(\underline{w})$. Panel (b) of Figure 2 provides an illustration.

However, when the global profit maximum is not the local maximum with the largest quantity, a minimum wage $\underline{w}$ can affect outcomes in interesting and, to our knowledge, novel ways. In particular, $\underline{w}$ can increase equilibrium employment and the equilibrium wage above $S(\underline{w})$ and $\underline{w}$, respectively, whereas absent the minimum wage these would be $Q_U^\ell < S(\underline{w})$

²⁰Loertscher and Muir (2022, p. 574) make a similar observation in the context of monopoly pricing problems with respect to uniform pricing and the assumption that the revenue function is concave.

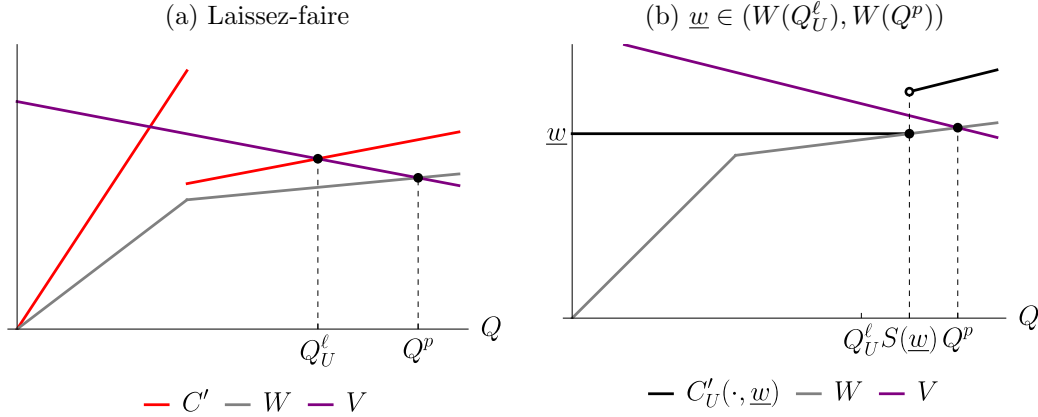


Figure 2: This figure illustrates Robinson's (1933) analysis for an example in which W is piecewise linear, and C is not convex. Panel (a) shows that under laissez-faire and uniform wages there are now multiple local maxima of the firm's profit function. Here, the global maximum of the firm's profit function coincides with the local maximum associated with the largest quantity. Panel (b) illustrates how the effects of introducing a minimum wage $\underline{w} \in (W(Q_U^\ell), W(Q^p))$ are then the same as when C is convex.

and $W(Q_U^\ell) < \underline{w}$.²¹ Although there is always a region in which the minimum wage does not bind and a neoclassical region as defined above, there may now also be subintervals within the Robinson region over which the minimum wage binds but the firm nevertheless optimally sets a wage above $\underline{w}$ and employment exceeds $S(\underline{w})$. In such cases, there is no involuntary unemployment, and transitions into these subintervals are associated with discontinuous increases in employment and the equilibrium wage. Moreover, the equilibrium wage and employment do not vary with marginal changes in $\underline{w}$ once they exceed $\underline{w}$ and $S(\underline{w})$, respectively.²² Thus, under uniform wage setting, non-convexity can make minimum wage effects substantially richer.

In the next section, we show that once the firm is allowed to use an optimal procurement mechanism, the same non-convexities become central even under laissez-faire because they determine when departing from a single wage is optimal.

²¹The effect arises when $\underline{w}$ is high enough that hiring $S(\underline{w})$ workers is less profitable than hiring the quantity associated with the global maximum of the laissez-faire profit function restricted to $Q \in [S(\underline{w}), 1]$.

²²This suggests additional challenges for analyzing minimum wage effects empirically, as the minimum wage is then locally non-binding—marginal changes in $\underline{w}$ have no effect—even though removing it would induce discontinuous changes in employment and the equilibrium wage.

3 Optimal procurement

We now turn to the firm's optimal procurement problem. Absent a minimum wage, the firm optimally departs from uniform wage setting when hiring Q workers if the cost function C is not convex at Q , in a sense made precise below. It then sets two wages rather than one, the higher of which is an efficiency wage at which labor is rationed uniformly at random, so that the resulting two-wage mechanism induces involuntary unemployment. Under a minimum wage, the firm continues to use at most two wages, provided the cost function satisfies a separation condition that we introduce in Section 3.2.

We first describe optimal procurement under *laissez-faire*, where a two-wage mechanism attains the minimal cost of procurement without further assumptions. We then turn to the problem with a minimum wage, where the analysis restricts attention to two-wage mechanisms and the separation condition makes this restriction without loss of generality.

3.1 Optimal procurement under *laissez-faire*

A useful object for characterizing optimal procurement under *laissez-faire* is the *convexification* of the cost function C . Denoted by $\underline{C}$, this is the largest convex function that is everywhere weakly below C . If C is convex, then $\underline{C} = C$. Otherwise, the set of quantities at which $\underline{C}$ lies strictly below C is a union of countably many disjoint open intervals, which we write as $\{(Q_1(m), Q_2(m))\}_{m \in \mathcal{M}}$, where $\mathcal{M}$ is a countable index set. On each of these intervals, $\underline{C}$ is the chord of C . That is,

$$\underline{C}(Q) = \begin{cases} C(Q_1(m)) + \frac{(Q - Q_1(m))(C(Q_2(m)) - C(Q_1(m)))}{Q_2(m) - Q_1(m)}, & \exists m \in \mathcal{M} \text{ s.t. } Q \in (Q_1(m), Q_2(m)), \\ C(Q), & \text{otherwise.} \end{cases}$$

We refer to the intervals $(Q_1(m), Q_2(m))_{m \in \mathcal{M}}$ as *ironing intervals* of C and say that C is *not convex at Q* if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Since W is strictly increasing, no ironing interval has left endpoint 0, that is, $Q_1(m) > 0$ for all $m \in \mathcal{M}$.²³ For all Q such that $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, introducing $\alpha(Q) := (Q - Q_1(m))/(Q_2(m) - Q_1(m))$ allows us to write $\underline{C}(Q)$ as the convex combination

$$\underline{C}(Q) = (1 - \alpha(Q))C(Q_1(m)) + \alpha(Q)C(Q_2(m)).$$

²³To see this, assume to the contrary that $(0, Q_2(m))$ is an ironing interval for some $m \in \mathcal{M}$. Then on $[0, Q_2(m)]$, $\underline{C}$ is the chord of C over this interval, which is $QW(Q_2(m))$. Because $\underline{C} \leq C$, this implies $W(Q_2(m)) \leq W(Q)$ for all $Q \in (0, Q_2(m))$, contradicting that W is strictly increasing.

For each $m \in \mathcal{M}$ and $i \in \{1, 2\}$, $Q_i(m)$ satisfies the first-order condition

$$C'_-(Q_i(m)) \leq \frac{C(Q_2(m)) - C(Q_1(m))}{Q_2(m) - Q_1(m)} \leq C'_+(Q_i(m)), \quad (4)$$

because $\underline{C}$ lies weakly below C and coincides with it at $Q_i(m)$. If $Q_2(m) = 1$ for some $m \in \mathcal{M}$, then only the left inequality in (4) is imposed. If C is differentiable at $Q_1(m)$ and $Q_2(m)$, then (4) is equivalent to

$$C'(Q_1(m)) = C'(Q_2(m)) = \frac{C(Q_2(m)) - C(Q_1(m))}{Q_2(m) - Q_1(m)}. \quad (5)$$

That is, the marginal cost is the same at the endpoints of the ironing interval and equal to the average marginal cost over that interval.

The importance of $\underline{C}(Q)$ is that it is the lowest cost at which the firm can procure Q units under an incentive compatible and individually rational mechanism. That the firm cannot do better follows immediately from Myerson (1981) since the convexification procedure considered here is equivalent to Myersonian ironing. If $\underline{C}(Q) = C(Q)$, attainability is also immediate since hiring Q workers at the market-clearing wage yields procurement cost $\underline{C}(Q)$. So suppose instead that $\underline{C}(Q) < C(Q)$. In this case, $\underline{C}(Q)$ can be obtained using a two-wage mechanism. As we established in Section 2, the procurement cost under the two-wage mechanism parameterized by (Q, q_1, q_2) is given by $\beta(Q, q_1, q_2)C(q_2) + (1 - \beta(Q, q_1, q_2))C(q_1)$, where $\beta(Q, q_1, q_2) = (Q - q_1)/(q_2 - q_1)$. Consequently, if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then choosing $q_i = Q_i(m)$ for $i \in \{1, 2\}$ yields the procurement cost $\underline{C}(Q) = \alpha(Q)C(Q_2(m)) + (1 - \alpha(Q))C(Q_1(m))$, as required. Note that the convexification does not involve randomizing over different employment levels $Q_1(m)$ and $Q_2(m)$. Instead, the $Q_1(m)$ workers with the lowest cost are hired with certainty and an additional $Q - Q_1(m)$ mass of workers are selected out of the $Q_2(m) - Q_1(m)$ workers competing to work at the high wage.

Intuitively, the firm would like to procure the Q units of labor from the workers with the lowest marginal costs. When C is not convex, the slope of C is not monotone, which means that the workers with the lowest marginal costs do not in general coincide with those with the lowest costs. Consequently, procuring labor from the workers with the lowest marginal costs can violate the monotonicity constraint implied by incentive compatibility. In particular, when $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, the best the firm can do is hire workers within the ironing interval $(Q_1(m), Q_2(m))$ with equal probability, and the firm achieves this using a two-wage mechanism and involuntary unemployment.

We let $w_1^\ell(Q)$ denote the low wage paid under the optimal mechanism for procuring Q . Thus, $w_1^\ell(Q) = w_1(Q, Q_1(m), Q_2(m))$ if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, where w_1 is

defined in (2), and $w_1^\ell(Q) = W(Q)$ if C is convex at Q . The following proposition summarizes this discussion and introduces some useful properties of w_1^ℓ .

Proposition 1. *The lowest cost of procuring $Q \in [0, 1]$ units of labor is given by $\underline{C}(Q)$. If $\underline{C}(Q) < C(Q)$, the firm achieves this using a two-wage mechanism involving involuntary unemployment, parameterized by $(Q, Q_1(m), Q_2(m))$, where $m \in \mathcal{M}$ satisfies $Q_1(m) < Q < Q_2(m)$. The low wage $w_1^\ell(Q)$ is continuous and strictly increasing in Q . It also satisfies $w_1^\ell(Q) \leq W(Q)$, where this inequality is strict if and only if $\underline{C}(Q) < C(Q)$.*

Since w_1^ℓ is continuous and strictly increasing, its inverse $(w_1^\ell)^{-1}$ is well defined on $[W(0), W(1)]$ and satisfies $(w_1^\ell)^{-1}(w) \geq S(w)$, with equality if and only if $S(w)$ lies outside every ironing interval.

Proposition 1 implies that the firm's laissez-faire profit $\int_0^Q V(y) dy - \underline{C}(Q)$ is concave in Q . Since $V(0) > W(0)$ and $V(1) < W(1)$, every profit-maximizing quantity is interior, and the optimal levels of employment under laissez-faire are therefore the quantities $Q \in (0, 1)$ satisfying $\underline{C}'_-(Q) \leq V(Q) \leq \underline{C}'_+(Q)$. The possibility of multiple profit-maximizing quantities arises because V and the marginal cost of optimal procurement are only weakly monotone in Q . We break ties by selecting the largest profit-maximizing quantity,

$$Q^\ell := \max\{Q \in (0, 1) : \underline{C}'_-(Q) \leq V(Q) \leq \underline{C}'_+(Q)\},$$

as the optimal level of employment under laissez-faire.

Figure 3 illustrates Proposition 1 when the supply side is characterized by the piecewise linear specification

$$W(Q) = \begin{cases} 4Q, & Q \in [0, 1/4], \\ Q/2 + 7/8, & Q \in (1/4, 1]. \end{cases} \quad (6)$$

This implies $C(Q) = 4Q^2$ for $Q \in [0, 1/4]$ and $C(Q) = Q^2/2 + 7Q/8$ for $Q \in (1/4, 1]$, and yields a single ironing interval with $Q_1 = (4 + \sqrt{2})/32 \approx 0.17$ and $Q_2 = (1 + 2\sqrt{2})/8 \approx 0.48$. This supply-side specification will serve as our leading example throughout the paper.

Implementation Attaining $\underline{C}(Q)$ with a two-wage mechanism rests on two assumptions. First, the firm can ration the workers who select the high wage independently of their costs; otherwise, the self-selection underlying the two-wage mechanism would unravel and the mechanism would not achieve the cost $\underline{C}(Q)$. Second, the firm can verify workers' identities, so that workers cannot trade jobs: because rationing is random, some rationed workers have lower costs than some high-wage employees, and a secondary market would reallocate jobs by cost. This is the resale problem familiar from price discrimination. In

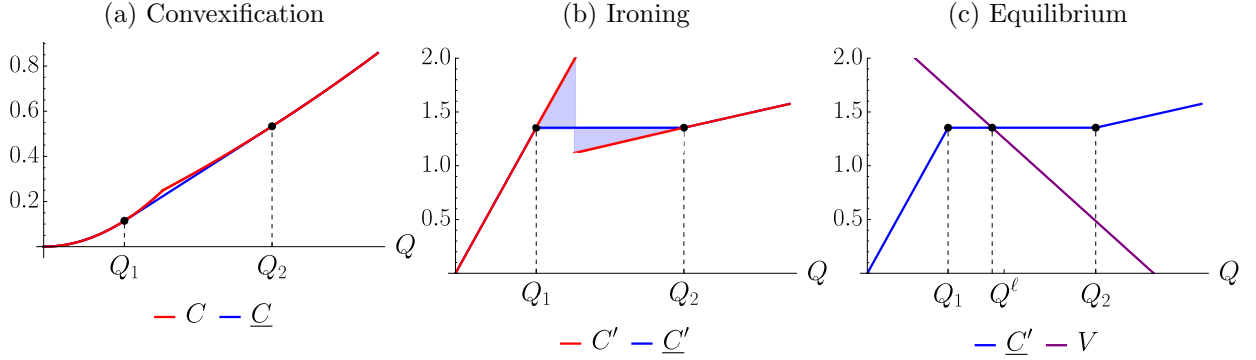


Figure 3: Panel (a) illustrates the convexification $\underline{C}$ of C for our leading example (6). Panel (b) illustrates the corresponding marginal cost C'_- and ironed marginal cost $\underline{C}'_-$. Panel (c) illustrates an example with $V(Q) = 2.4 - 4Q$ and $Q^\ell \in (Q_1, Q_2)$.

short, employment at the high wage must not vary with costs, whether through selection or through trade. Subject to these requirements, the two-wage mechanism can be implemented in a variety of ways as we discuss next.

Up to this point, we have described two-wage mechanisms in their *ex post* form. For the two-wage mechanism parameterized by (Q, q_1, q_2) , every worker is offered the same menu—guaranteed employment at the low wage $w_1(Q, q_1, q_2)$, or employment at the high wage $w_2(q_2)$ with probability $\beta(Q, q_1, q_2)$ —and workers self-select. The lowest-cost workers with $c \leq W(q_1)$ choose the low wage, intermediate-cost workers with $c \in (W(q_1), W(q_2)]$ choose the high-wage lottery, and high-cost workers with $c > W(q_2)$ do not participate. Rationing happens *after* workers have self-selected.

The same allocation can be supported in several alternative ways consistent with ex post implementation. The firm can fill positions dynamically, hiring q_1 workers at the low wage and then rationing the remaining $Q - q_1$ high-wage positions among those who opt for the high wage. Workers may be selected at the high wage on a first-come-first-served basis when the order of arrival is independent of their costs, on the basis of observable characteristics uncorrelated with opportunity cost, or via an explicit lottery—as in the so-called “shape-up” historically used to hire dock workers.²⁴ In each case, rationing is independent of workers’ costs and applied only after self-selection.

A particularly natural real-world interpretation of ex post implementation is in terms of part-time and full-time work. Low-wage workers are full-time employees, hired with certainty at the wage w_1 . High-wage workers are part-time or casual staff who are paid the higher hourly wage w_2 but employed only a fraction $\beta(Q, q_1, q_2)$ of the time. The lottery and the

²⁴This mechanism came to public awareness through the film “On the Waterfront” and the newspaper articles it was based on (see Johnson and Schulberg, 2005).

part-time arrangement are payoff-equivalent from the worker's perspective, and so under this interpretation the model predicts that part-time workers earn a higher hourly wage than their full-time counterparts.

Absent a minimum wage constraint, every decreasing allocation rule admits a dominant-strategy implementation. In particular, given any incentive compatible direct mechanism $\langle x, w \rangle$, the firm can implement the allocation rule x in dominant strategies by making each worker a *randomized* take-it-or-leave-it offer: a wage drawn from a distribution chosen so that, in equilibrium, a worker of type c accepts the offer with probability $x(c)$ and is paid $w(c)$ in expectation conditional on accepting. We refer to this as *ex ante implementation* because the firm randomizes *before* the worker is asked to make a deterministic choice.²⁵ For the two-wage mechanism parameterized by (Q, q_1, q_2) , ex ante implementation takes a particularly simple form: with probability $\beta(Q, q_1, q_2)$ the offer is the wage $W(q_2)$, and otherwise the offer is the wage $W(q_1)$. Workers with $c \leq W(q_1)$ accept either offer, so are paid the average wage $(1 - \beta(Q, q_1, q_2))W(q_1) + \beta(Q, q_1, q_2)W(q_2) = w_1(Q, q_1, q_2)$; workers with $c \in (W(q_1), W(q_2)]$ accept only the high wage; the rest reject both. Absent a minimum wage constraint, ex ante and ex post implementation therefore deliver the same allocation, the same firm profit, and the same worker surplus. As we will see, this equivalence breaks down once a minimum wage constraint is introduced.

3.2 Optimal procurement with a minimum wage

We now characterize optimal procurement in the presence of a minimum wage. Throughout, we restrict attention to two-wage mechanisms under ex post implementation. Restricting attention to ex post implementation is without loss of generality, as we explain next, and restricting attention to two-wage mechanisms is without loss of generality under the separation condition introduced below. Deferring the formal arguments to the appendix, we provide intuition and an overview of the analysis here.

Under a minimum wage constraint the *form* of implementation matters in a way that it does not under laissez-faire and in many other mechanism design problems. The difference arises because a minimum wage directly constrains the transfer rule, and different implementations of the same monotone allocation rule generate different transfer rules. Consequently, the minimum wage constraint is tighter under some implementations than others, and the firm cares which one it uses. To illustrate, we contrast ex post and ex ante implementation. Recall that for the two-wage mechanism parameterized by (Q, q_1, q_2) , the lowest wage paid

²⁵This implementation involving random take-it-or-leave-it offers also resonates with the assumptions made in empirical research in labor economics on wage discrimination; see, for example, Cullen and Pakzad-Hurson (2023).

in equilibrium under ex post implementation is $w_1(Q, q_1, q_2) = (1 - \beta(Q, q_1, q_2))W(q_1) + \beta(Q, q_1, q_2)W(q_2)$, while under ex ante implementation it is $W(q_1)$. Because W is strictly increasing, we have $w_1(Q, q_1, q_2) > W(q_1)$, and the minimum wage constraint is *strictly tighter* under ex ante implementation. For two-wage mechanisms, then, the firm prefers ex post implementation. This superiority generalizes. Under any implementation, a worker's realized wage conditional on employment must be at least $\underline{w}$, so the worker's expected wage conditional on employment is at least $\underline{w}$ as well. Paying that expected wage deterministically, which is what ex post implementation does, therefore satisfies the minimum wage constraint whenever the original implementation does, and leaves the allocation and the firm's cost unchanged. Restricting attention to ex post implementation is thus without loss of generality.

With two-wage mechanisms and ex post implementation in hand, we organize the analysis by the quantity Q being procured. Three cases arise, which Figure 4 illustrates for our leading example.

Case 1: $Q \leq S(\underline{w})$. Here, the firm hires fewer workers than are willing to work at the minimum wage. The best it can do is hire Q workers at $\underline{w}$ and randomly ration the excess supply $S(\underline{w}) - Q$. The minimum cost of procurement is $\underline{w}Q$. This is the situation captured by the neoclassical model from Section 2.

Case 2: $Q \in (S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}))$. This case is non-empty if and only if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ (i.e., the quantity $S(\underline{w})$ is contained within an ironing interval of C). Here, the minimum wage is below the market-clearing wage $W(Q)$ —and so would not bind under uniform wage setting—but above the laissez-faire low wage $w_1^\ell(Q)$ that the firm would like to set. This is the case that requires further analysis.

Case 3: $Q \geq (w_1^\ell)^{-1}(\underline{w})$. Here, the laissez-faire low wage is at least $\underline{w}$, so the minimum wage does not bind. The firm proceeds as under laissez-faire and procures at minimum cost $\underline{C}(Q)$.

We now analyze Case 2. Here the restriction to two-wage mechanisms reduces the firm's choice to the quantities (q_1, q_2) , and the restriction to ex post implementation means that we can express the minimum wage constraint as a constraint on the corresponding equilibrium low wage $w_1(Q, q_1, q_2) = (1 - \beta(Q, q_1, q_2))W(q_1) + \beta(Q, q_1, q_2)W(q_2)$. Formally, the firm chooses $q_1 \in [0, Q)$ and $q_2 \geq Q$ to minimize the procurement cost

$$(1 - \beta(Q, q_1, q_2))C(q_1) + \beta(Q, q_1, q_2)C(q_2)$$

subject to

$$(1 - \beta(Q, q_1, q_2))W(q_1) + \beta(Q, q_1, q_2)W(q_2) \geq \underline{w}.$$

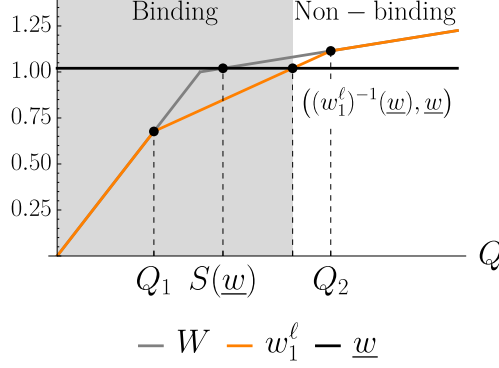


Figure 4: This figure illustrates, for our leading example (6) with $\underline{w} = 1.02$, the quantities $Q \in [0, (w_1^\ell)^{-1}(\underline{w})]$ at which the minimum wage constraint binds (shaded region) and the quantities $Q \geq (w_1^\ell)^{-1}(\underline{w})$ at which it does not. For $Q \in [0, S(\underline{w})]$ the firm optimally rations workers at the minimum wage $\underline{w}$, and for $Q \in (S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}))$ the firm optimally uses a two-wage mechanism.

The cost being minimized and the constraint share the *same functional form*—both are convex combinations, with the same weight $\beta(Q, q_1, q_2)$, of a function evaluated at q_1 and q_2 . The objective uses C ; the constraint uses W .

Letting $\lambda \geq 0$ denote the Lagrange multiplier on the minimum wage constraint and defining

$$C_\lambda(Q) := C(Q) - \lambda W(Q) = (Q - \lambda)W(Q),$$

the firm's dual problem is

$$\mathcal{D}^*(Q, \underline{w}) := \max_{\lambda \geq 0} \min_{q_1 \in [0, Q], q_2 \geq Q} \{(1 - \beta(Q, q_1, q_2))C_\lambda(q_1) + \beta(Q, q_1, q_2)C_\lambda(q_2) + \lambda \underline{w}\}. \quad (7)$$

Because the constraint and the objective share the same functional form, the inner minimization is structurally identical to the laissez-faire convexification problem of Section 3.1, only with C replaced by C_λ . The same convexification argument that delivered Proposition 1 therefore applies: for any fixed λ , the inner minimum equals $\underline{C}_\lambda(Q) + \lambda \underline{w}$, where $\underline{C}_\lambda$ denotes the convexification of C_λ .²⁶

The function C_λ has a simple interpretation. Recall that the procurement cost is a convex

²⁶This is not the same as convexifying the uniform-wage cost $C_U(\cdot, \underline{w})$ from (3), as the laissez-faire analysis might suggest. Consider the leading example with the minimum wage $\underline{w} = 1.02$ of Figure 4, which lies above the kink of W . There $C_U(\cdot, \underline{w})$ is convex, so convexifying it would leave no scope for improvement. Yet the minimum wage does not bind for $Q \in [(w_1^\ell)^{-1}(\underline{w}), Q_2] \approx [0.41, 0.48]$, where the firm procures at the laissez-faire cost $\underline{C}(Q) < C(Q) = C_U(Q, \underline{w})$. The convexification of $C_U(\cdot, \underline{w})$ is instead the minimal cost of procurement under ex ante implementation, where the firm mixes over posted wages, each of which must be at least $\underline{w}$.

combination of $C(q_1)$ and $C(q_2)$ and that the low wage is the same convex combination of $W(q_1)$ and $W(q_2)$, so the mechanism parameters q_1 and q_2 enter both with the same weights. A higher low wage relaxes the minimum wage constraint, and λ is the shadow value of that constraint. Thus, $C_\lambda(q) = C(q) - \lambda W(q)$ subsidizes the cost $C(q)$ with the value $\lambda W(q)$ of the corresponding contribution to the low wage.

The dual problem is set up over two-wage mechanisms and therefore characterizes the optimal two-wage mechanism. To see when this mechanism is optimal among all mechanisms, fix λ at its solution value and consider the supporting line of C_λ at Q . Any mechanism that employs Q workers and whose employment probabilities change only at points where this line touches C_λ attains the same value of the objective in (7). If the line touches C_λ only at the endpoints of the ironing interval containing Q , the two-wage mechanism is the only such mechanism and is therefore optimal. If the line touches C_λ at a third point, a mechanism paying three wages attains the same value. Such a mechanism may satisfy the minimum wage constraint when the two-wage mechanism does not, in which case it is optimal and the two-wage mechanism is not. Under laissez-faire there is no constraint to satisfy, so these ties are inconsequential and Proposition 1 requires no condition on C . Under a minimum wage, we rule them out with the following assumption.

Assumption 1 (Separation condition). *For every $\lambda \in [0, 1]$, no affine function that lies weakly below C_λ on $[0, 1]$ coincides with C_λ at more than two points.*

The restriction to $\lambda \leq 1$ in Assumption 1 is without loss of generality, since the shadow value of the minimum wage constraint is the marginal cost to the firm of a higher minimum wage, which cannot exceed the number of workers it employs. Assuming that W is twice continuously differentiable, a simple sufficient condition for Assumption 1 in terms of primitives is that W is concave, the marginal cost C' is decreasing on at most one interval and increasing elsewhere, and $-W''/W'$ is log-concave on that interval.²⁷ These conditions parallel those of Dworczak et al. (2021): concavity of W fixes the curvature of the term that the multiplier scales, as condition (ii) of their Theorem 4 does, and log-concavity of $-W''/W'$ is the counterpart of the quasi-concavity in their Assumption 1, strengthened so that it holds for every value of the multiplier. Our leading example (6), in which W is piecewise linear with a single concave kink at $Q = 1/4$, satisfies the separation condition by direct inspection, since C_λ is then a strictly convex parabola on each linear piece of W .

Under Assumption 1, the two-wage mechanism characterized by the dual problem is the unique optimal mechanism, and restricting attention to two-wage mechanisms is without

²⁷Under these conditions, for every $\lambda \in [0, 1]$ there exist $0 \leq a \leq b \leq 1$ such that C_λ is strictly convex on $[0, a]$, strictly concave on $[a, b]$ and strictly convex on $[b, 1]$, which implies the separation condition. See Appendix B.1 for the proof.

loss of generality. Unless otherwise stated, we maintain Assumption 1 for the remainder of the paper. Without it, an optimal mechanism may pay three wages, but never more.²⁸ We return to three-wage mechanisms in Section 5.1.

This completes the analysis of Case 2. Let $\underline{C}_R(Q, \underline{w})$ denote the firm's minimal cost of procuring $Q \in [0, 1]$ workers under the minimum wage $\underline{w} \in [W(0), W(1)]$. Putting the three cases together, we have

$$\underline{C}_R(Q, \underline{w}) = \begin{cases} \underline{w}Q, & Q \in [0, S(\underline{w})], \\ \mathcal{D}^*(Q, \underline{w}), & Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right), \\ \underline{C}(Q), & Q \in \left[(w_1^\ell)^{-1}(\underline{w}), 1\right]. \end{cases} \quad (8)$$

For $Q \leq S(\underline{w})$ (Case 1), the firm hires Q workers at $\underline{w}$ and randomly rations the excess supply. For $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right)$ (Case 2), it uses the non-degenerate two-wage mechanism just characterized, whose low wage equals $\underline{w}$. For $Q \geq (w_1^\ell)^{-1}(\underline{w})$ (Case 3), the minimum wage does not bind and the firm uses the laissez-faire mechanism, which at $Q = (w_1^\ell)^{-1}(\underline{w})$ pays exactly the minimum wage.²⁹

The following theorem formally summarizes the analysis and establishes useful properties of $\underline{C}_R(Q, \underline{w})$ and the corresponding marginal cost envelope $\underline{C}'_R(Q, \underline{w})$, defined as the left derivative of $\underline{C}_R$ with respect to Q .³⁰

Theorem 1. *The minimal cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity $Q \in [0, 1]$ under the minimum wage $\underline{w} \in [W(0), W(1)]$ over general direct mechanisms is given by (8). It is attained by a mechanism involving ex post implementation that pays no more than two wages, and for $Q > 0$ the optimal allocation and expected transfers are unique up to sets of workers of measure zero. The function $\underline{C}_R(Q, \underline{w})$ is continuous, separately convex, and increasing in both Q and $\underline{w}$. The marginal cost envelope $\underline{C}'_R(Q, \underline{w})$ is bounded and satisfies $\underline{C}'_R(Q, \underline{w}) \geq W(Q)$, with strict inequality whenever $Q > S(\underline{w})$. Moreover, for all $(Q, \underline{w})$ with $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$, $\underline{C}'_R(Q, \underline{w})$ is strictly increasing in Q and strictly decreasing in $\underline{w}$.*

²⁸Any incentive compatible allocation rule is a weighted sum of market-clearing allocations at different quantities. The firm's problem imposes two linear constraints on this sum: the quantity procured and the minimum wage. By Carathéodory's theorem, an optimal weighted sum therefore involves at most three quantities, each of which corresponds to one wage.

²⁹If the laissez-faire mechanism on $Q \geq (w_1^\ell)^{-1}(\underline{w})$ involves two wages, the firm cannot necessarily implement it ex ante without violating the minimum wage, even though ex post implementation is feasible.

³⁰Throughout the paper, we represent marginal cost envelopes by left derivatives. The one-sided derivatives differ only at kinks of the cost envelope, and there the left derivative is the economically relevant one: it is the marginal cost of the last unit procured under the mechanism that is optimal at that quantity (e.g., $\underline{w}$ at the kink $Q = S(\underline{w})$), and, given our convention of selecting the largest profit-maximizing quantity, equilibrium employment is simply the largest Q satisfying $V(Q) \geq \underline{C}'_R(Q, \underline{w})$ —a characterization that would fail under the right-derivative convention whenever the optimum is at a kink, as in the Robinson region.

Panel (a) of Figure 5 illustrates the marginal cost envelope $\underline{C}'_R(\cdot, \underline{w})$ associated with $\underline{C}_R(\cdot, \underline{w})$ for our leading example (6) and a representative minimum wage satisfying $S(\underline{w}) \in (Q_1, Q_2)$. For $Q < S(\underline{w})$, the marginal cost envelope is constant and equal to the minimum wage $\underline{w}$. At the quantity $Q = S(\underline{w})$, there is generally a discontinuity in the marginal cost envelope. For $Q \in (S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}))$, the marginal cost envelope $\underline{C}'_R(\cdot, \underline{w})$ is strictly increasing. Finally, for $Q \geq (w_1^\ell)^{-1}(\underline{w})$, the minimum wage is not binding and we have $\underline{C}'_R(Q, \underline{w}) = \underline{C}'(Q)$.

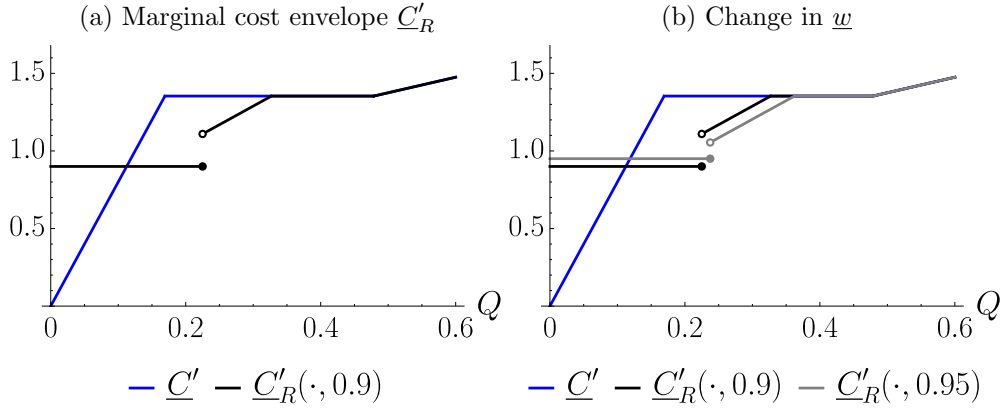


Figure 5: Panel (a) illustrates the laissez-faire marginal cost function $\underline{C}'$, and the marginal cost envelope $\underline{C}'_R$ under the minimum wage $\underline{w} = 0.9$, which satisfies $S(\underline{w}) \in (Q_1, Q_2)$ for our leading example (6). Panel (b) illustrates how the marginal cost envelope $\underline{C}'_R$ changes as $\underline{w}$ increases from 0.9 to 0.95.

Equilibrium employment To determine the optimal level of employment under the minimum wage $\underline{w}$, denoted $Q^*(\underline{w})$, we need to account for the firm's marginal revenue product of labor V . If there are multiple profit-maximizing quantities, we set $Q^*(\underline{w})$ to be the largest profit-maximizing quantity.³¹ Letting $\partial_-/\partial Q$ and $\partial_+/\partial Q$ respectively denote left and right derivatives with respect to Q , we then have the following corollary to Theorem 1.

Corollary 1. *Any quantity $Q \in (0, 1)$ satisfying*

$$\frac{\partial_- \underline{C}_R(Q, \underline{w})}{\partial Q} \leq V(Q) \leq \frac{\partial_+ \underline{C}_R(Q, \underline{w})}{\partial Q} \quad (9)$$

is an optimal level of employment under the minimum wage $\underline{w} \in [W(0), W(1)]$. Every optimal level of employment is either such a quantity or $Q = 0$, and $Q = 0$ is optimal if

³¹This is in line with our convention of letting $D(w)$ denote the largest profit-maximizing quantity demanded by a firm that takes the wage w as given, and letting Q^ℓ denote the largest profit-maximizing quantity under laissez-faire.

and only if $\underline{w} \geq V(0)$. If $\underline{w} < W(Q^p)$, then every optimal level of employment satisfies $Q < Q^p$. The largest profit-maximizing quantity $Q^*(\underline{w})$ maximizes social surplus among all profit-maximizing quantities.

Interestingly, the fact that, among profit-maximizing quantities, social surplus is maximized at the largest quantity also means that social surplus can be larger with involuntary unemployment than without it.³²

Geometric intuition In Section 4, we derive comparative statics concerning the equilibrium level of employment $Q^*(\underline{w})$, involuntary unemployment $S(\underline{w}) - Q^*(\underline{w})$ and the equilibrium wage schedule with respect to the minimum wage $\underline{w}$. For these purposes, the comparative statics from Theorem 1 concerning the marginal cost envelope $\underline{C}'_R(Q, \underline{w})$ on the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage will play an important role. We now leverage the geometric properties of $\underline{C}_R$ to develop intuition for these comparative statics.

Recall that $\underline{C}_\lambda$ denotes the convexification of the function $C_\lambda(Q) = C(Q) - \lambda W(Q)$ and let $\lambda^*(Q, \underline{w})$ denote the solution value of the Lagrange multiplier.³³ Since $\underline{C}_R(Q, \underline{w})$ is the maximum over λ of $\underline{C}_\lambda(Q) + \lambda \underline{w}$, an envelope-theorem argument (see the proof of Theorem 1) gives

$$\underline{C}'_R(Q, \underline{w}) = \left. \frac{\partial \underline{C}_\lambda(Q)}{\partial Q} \right|_{\lambda=\lambda^*(Q, \underline{w})} := s_{\lambda^*(Q, \underline{w})}(Q). \quad (10)$$

Since we are considering the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage, we are restricting attention to minimum wage and quantity combinations such that Q lies within an ironing interval of $\underline{C}_{\lambda^*(Q, \underline{w})}$. Thus, fixing $\lambda^*(Q, \underline{w})$, the right-hand side of (10) does not vary with Q within this region. Consequently, the comparative statics of the marginal cost envelope $\underline{C}'_R(Q, \underline{w})$ in this region

³²For example, suppose that the minimum wage is not binding and that, for some $m \in \mathcal{M}$ and interval $[a, b]$ with $a \leq Q_1(m) < b < Q_2(m)$, we have $V(Q) = \underline{C}'_-(b)$ if and only if $Q \in [a, b]$. The profit-maximizing quantities are then exactly the interval $[Q_1(m), b]$. The firm can hire $Q_1(m)$ workers at the market-clearing wage $W(Q_1(m))$ without involuntary unemployment, whereas hiring the largest profit-maximizing quantity b requires a two-wage mechanism and induces involuntary unemployment. By Corollary 1, social surplus is nevertheless larger at b .

³³The outer maximization over λ in (7) may admit multiple solutions. As shown in the proof of Theorem 1, the set of solutions is a closed interval. Throughout, $\lambda^*(Q, \underline{w})$ denotes the largest solution, paralleling our selection of the largest profit-maximizing quantity. The monotonicity of λ^* in Lemma 1 holds for every solution, not only the largest: at a given quantity, every solution lies strictly below every solution at any smaller quantity, and under a given minimum wage, every solution lies strictly above every solution under a lower minimum wage. This selection is also what makes (10) hold for the left derivative—the right derivative of $\underline{C}_R$ with respect to Q is the slope $s_\lambda(Q)$ at the smallest solution.

depend entirely on how $\lambda^*(Q, \underline{w})$ varies with Q and $\underline{w}$, and the geometry of how the relevant linear segment of $\underline{C}_\lambda$ varies with λ .

The comparative statics of $\lambda^*(Q, \underline{w})$ are very intuitive because $\lambda^*(Q, \underline{w})$ reflects the shadow cost of the minimum wage constraint, and this constraint becomes *tighter* as $\underline{w}$ increases or Q decreases. Formally, we have the following lemma, the proof of which is contained within the proof of Theorem 1. Here and throughout the rest of the paper, $(Q_1(m), Q_2(m))$ denotes an ironing interval of the laissez-faire cost function C , as defined in Section 3.1.

Lemma 1. *Fix $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$ so that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. Then the solution value $\lambda^*(Q, \underline{w})$ of the Lagrange multiplier from the firm's dual problem defined in (7) is strictly decreasing in Q and strictly increasing in $\underline{w}$.*

To derive the comparative statics of the marginal cost envelope $\underline{C}'_R(Q, \underline{w})$, it only remains to investigate how the relevant linear segment of $\underline{C}_\lambda$ varies with λ . Since we are restricting attention to the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint, Q is in the interior of an ironing interval of C_λ for λ sufficiently close to $\lambda^*(Q, \underline{w})$. Let $(q_1^\lambda, q_2^\lambda)$ denote this ironing interval and s_λ the slope of the corresponding linear segment of $\underline{C}_\lambda$. As we formally show in the proof of Theorem 1, we have

$$\frac{ds_\lambda}{d\lambda} = -\frac{W(q_2^\lambda) - W(q_1^\lambda)}{q_2^\lambda - q_1^\lambda} < 0$$

at every such λ , and s_λ is strictly decreasing in λ . This result is geometrically intuitive. Since W is strictly increasing, as λ increases, the function $C_\lambda(Q) = C(Q) - \lambda W(Q)$ shifts down by more at higher values of Q . As a result, the linear segments of $\underline{C}_\lambda$ over ironing intervals become flatter. Putting all of this together and returning to (10), as Q increases, $\lambda^*(Q, \underline{w})$ decreases, which steepens the relevant linear segment of $\underline{C}_{\lambda^*(Q, \underline{w})}$. Consequently, the marginal cost envelope $\underline{C}'_R$ increases in Q . This comparative static is illustrated in Panel (a) of Figure 5. Similarly, as $\underline{w}$ increases, $\lambda^*(Q, \underline{w})$ increases, which flattens the relevant linear segment of $\underline{C}_{\lambda^*(Q, \underline{w})}$. Consequently, the marginal cost envelope $\underline{C}'_R$ decreases in $\underline{w}$. This is illustrated in Panel (b) of Figure 5.

4 Comparative statics

This section derives the comparative statics of optimal procurement under a minimum wage. Section 4.1 distinguishes the four regions that can arise in equilibrium as the minimum wage

varies—the non-binding, constrained two-wage, Robinson and neoclassical regions—and establishes how the parameters of the optimal two-wage mechanism respond to the quantity procured and to the minimum wage (Lemma 2). Section 4.2 derives the effects of marginal minimum wage increases on employment, involuntary unemployment and the wage schedule within each region. Its main result, Theorem 2, is the three-case characterization highlighted in the introduction, and shows how the effects of a marginal increase in the minimum wage are determined by whether there is involuntary unemployment, and whether there is wage dispersion, under the prevailing minimum wage. In particular, wage dispersion can be used to distinguish involuntary unemployment caused by market power from that caused by the minimum wage. Section 4.3 characterizes the sequence of regions that the equilibrium passes through as the minimum wage increases (Proposition 2), and shows that a binding minimum wage can induce wage dispersion and involuntary unemployment even when neither arises under laissez-faire. Section 4.4 turns to welfare and workers' pay. Proposition 3 shows that employment, workers' pay, social surplus and worker surplus increase continuously in the minimum wage up to the competitive wage, including at transitions at which wage dispersion and involuntary unemployment increase discontinuously. Proposition 4 shows that whenever involuntary unemployment and wage dispersion are observed, raising the minimum wage to the highest wage currently paid increases employment and workers' pay and, if the prevailing minimum wage does not bind, decreases involuntary unemployment.

4.1 Preliminaries

Whereas the previous section characterized optimal procurement for an arbitrary quantity $Q \in [0, 1]$ and minimum wage $\underline{w} \in [W(0), W(1)]$, equilibrium employment $Q^*(\underline{w})$ now ties the quantity to the minimum wage, so that the equilibrium configuration is a function of the minimum wage alone. This induces four distinct regions of minimum wages, summarized in the following definition.

Definition 1. *We say that the equilibrium is in*

- (i) *the non-binding region for $\underline{w}$ such that $Q^*(\underline{w}) > (w_1^\ell)^{-1}(\underline{w})$ or, equivalently, $\underline{w} < w_1^\ell(Q^*(\underline{w}))$,*
- (ii) *the constrained two-wage region for $\underline{w}$ such that $Q^*(\underline{w}) \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$ or, equivalently, $\underline{w} \in \left[w_1^\ell(Q^*(\underline{w})), W(Q^*(\underline{w}))\right)$,*
- (iii) *the Robinson region for $\underline{w}$ such that $Q^*(\underline{w}) = S(\underline{w})$ or, equivalently, $\underline{w} = W(Q^*(\underline{w}))$,*
- (iv) *the neoclassical region for $\underline{w}$ such that $Q^*(\underline{w}) < S(\underline{w})$ or, equivalently, $\underline{w} > W(Q^*(\underline{w}))$.*

The form of the optimal mechanism in each region was derived in the previous section. In the non-binding region, the minimum wage constraint does not bind, and the firm may optimally set a market-clearing wage or use a two-wage mechanism, depending on the curvature of C . In the constrained two-wage region—which is non-empty only if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ —the firm optimally uses a two-wage mechanism in which the low wage equals the minimum wage $\underline{w}$. In the Robinson region, the firm optimally sets a market-clearing wage equal to the minimum wage $\underline{w}$; this region corresponds to the “gap case,” in which the discontinuity in the marginal cost envelope $\underline{C}'_R$ comes into play and $V(S(\underline{w})) \in [\underline{w}, \underline{C}'_{R+}(S(\underline{w}), \underline{w})]$, where $\underline{C}'_{R+}(Q, \underline{w})$ denotes the right derivative of $\underline{C}_R$ with respect to Q . In the neoclassical region, the firm optimally rations workers at a uniform wage equal to the minimum wage $\underline{w}$. Figure 6 provides examples illustrating the three regions in which the minimum wage binds: Panels (a), (b) and (c) respectively illustrate the constrained two-wage region, the Robinson region and the neoclassical region.

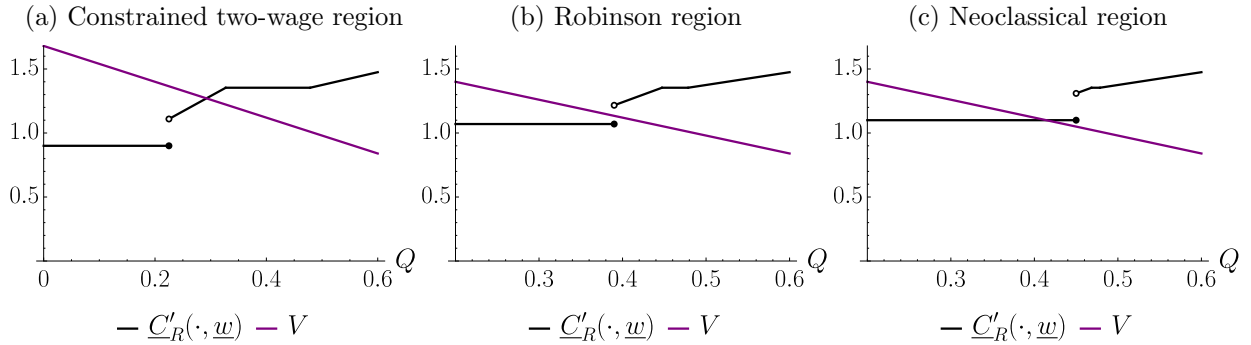


Figure 6: This figure illustrates an example of a binding minimum wage from each of the three possible regions from Definition 1 for our leading example (6) and $V(Q) = 1.68 - 1.4Q$. Panel (a) sets $\underline{w} = 0.9$, which corresponds to the constrained two-wage region. Panel (b) sets $\underline{w} = 1.07$, which corresponds to the Robinson region. Panel (c) sets $\underline{w} = 1.1$, which corresponds to the neoclassical region.

Sections 4.2 and 4.3 focus on deriving comparative statics for the equilibrium level of employment $Q^*(\underline{w})$, involuntary unemployment and the equilibrium wage schedule with respect to the minimum wage $\underline{w}$. While this is a relatively straightforward exercise within the non-binding, Robinson and neoclassical regions, the analysis of the constrained two-wage region is more involved as the behavior of involuntary unemployment and the wage schedule depends on how the optimal mechanism responds to changes in the minimum wage.

In equilibrium, the optimal mechanism depends on the minimum wage in two ways: directly, through the cost-minimization problem of Section 3.2, and indirectly, through equilibrium employment $Q^*(\underline{w})$. Both effects can be traced back to how the optimal mechanism varies with its arguments $(Q, \underline{w})$ in that problem: the direct effect is its response to $\underline{w}$ at a

fixed quantity, and the indirect effect is its response to Q at a fixed minimum wage, combined with the response of $Q^*(\underline{w})$ to the minimum wage. The natural first step, which we take here, is therefore to hold the quantity procured fixed and characterize both responses. Section 4.2 then combines them with the response of $Q^*(\underline{w})$. For any pair $(Q, \underline{w})$ such that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint, let $q_1^*(Q, \underline{w}) < Q < q_2^*(Q, \underline{w})$ denote the parameters of the optimal mechanism of Theorem 1. We then have the following lemma, which is illustrated in Figure 7.

Lemma 2. *For all $(Q, \underline{w})$ with $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in (S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}))$, so that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint, $q_1^*(Q, \underline{w})$ is increasing in $\underline{w}$ and decreasing in Q , and $q_2^*(Q, \underline{w})$ is decreasing in $\underline{w}$ and increasing in Q .*

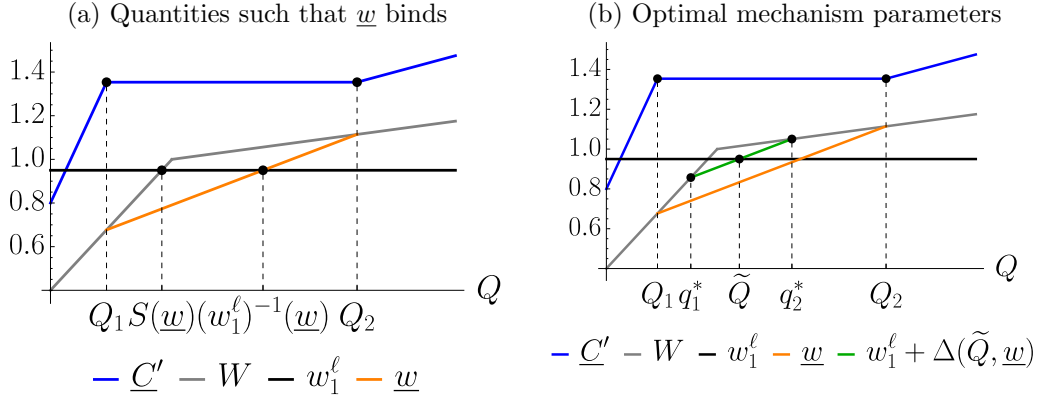


Figure 7: This figure illustrates Lemma 2 for our leading example (6) with $\underline{w} = 0.95$. Panel (a) shows the laissez-faire low wage function w_1^ℓ and the quantities $Q \in (S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}))$ such that $\underline{w}$ binds. Panel (b) shows the optimal mechanism parameters q_1^* and q_2^* for the quantity $\tilde{Q}$, obtained from the parallel shift $w_1^\ell(Q) + \Delta(\tilde{Q}, \underline{w})$ of the low wage function.

That $q_1^*(Q, \underline{w})$ increases in $\underline{w}$ formalizes the intuition that, as $\underline{w}$ increases, the monopsony procures more units at the minimum wage since it is a price-taker on these units. The behavior of q_2^* is best understood jointly with that of q_1^* , through the dual problem and the geometric construction from Section 3.2, which is how the proof of Lemma 2 proceeds. Recall that the parameters q_1^* and q_2^* are the endpoints of the ironing interval of C_{λ^*} containing Q , evaluated at the solution value $\lambda^* = \lambda^*(Q, \underline{w})$ of the multiplier. They therefore depend on $(Q, \underline{w})$ only through $\lambda^*(Q, \underline{w})$, which by Lemma 1 increases as Q decreases or as $\underline{w}$ increases (that is, as the minimum wage constraint becomes tighter). As λ^* increases, the function C_{λ^*} moves closer to its convexification (i.e., C_{λ^*} becomes “more convex”), and each of its

ironing intervals $(q_1^\lambda, q_2^\lambda)$ shrink in the set-inclusion sense: q_1^λ increases and q_2^λ decreases in λ . Together with the monotonicity of λ^* , this delivers the comparative statics of the lemma.

Figure 7 illustrates Lemma 2 for our leading example (6). Panel (a) depicts the range of binding minimum wages: for $Q \in (Q_1, Q_2)$, the lowest wage paid under laissez-faire is $w_1^\ell(Q)$, so a minimum wage $\underline{w} \in (W(Q_1), W(Q_2))$ binds precisely when $\underline{w} > w_1^\ell(Q)$. Panel (b) fixes a quantity $\tilde{Q} \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right)$ for which the minimum wage binds, and shows the parameters $q_1^*(\tilde{Q}, \underline{w})$ and $q_2^*(\tilde{Q}, \underline{w})$ of the optimal two-wage mechanism for procuring it. When W is piecewise linear, these parameters take a particularly tractable form: the wages $W(q_1^*)$ and $W(q_2^*)$ lie on the parallel upward shift $w_1^\ell(Q) + \Delta(\tilde{Q}, \underline{w})$ of the laissez-faire low wage function, where the constant $\Delta(\tilde{Q}, \underline{w}) := \underline{w} - w_1^\ell(\tilde{Q}) > 0$ ensures that the low wage under the resulting mechanism equals $\underline{w}$ (see Appendix B.2 for details). An increase in the minimum wage at fixed $\tilde{Q}$ —or a decrease in $\tilde{Q}$ at fixed $\underline{w}$ —increases the required shift $\Delta(\tilde{Q}, \underline{w})$, shrinking the interval (q_1^*, q_2^*) , as described in Lemma 2.

4.2 Within-region equilibrium effects

Next, we characterize the effects of marginal minimum wage increases on employment, involuntary unemployment and the wage schedule within each region of Definition 1. We first review these effects in the neoclassical and the Robinson region and then turn to the constrained two-wage region, where the effects of marginal minimum wage increases on unemployment and the wage schedule depend on the mechanism parameters q_1^* and q_2^* evaluated along the equilibrium locus $Q^*(\underline{w})$.

For $\underline{w} < w_1^\ell(Q^\ell)$, the equilibrium is within the non-binding region, in which case the minimum wage $\underline{w}$ has no effect on equilibrium outcomes. For $\underline{w} > W(Q^p)$, we are within the neoclassical region as here the firm is essentially a price-taker at the prevailing minimum wage: all employed workers are paid the minimum wage, employment falls below the competitive level, and there is involuntary unemployment. As discussed in relation to the neoclassical model in Section 2, the firm then hires $Q^*(\underline{w}) = D(\underline{w})$ workers, where $D(\underline{w})$ is labor demand at the minimum wage, leading to involuntary unemployment of size $S(\underline{w}) - D(\underline{w})$. Consequently, within this region, employment is decreasing and involuntary unemployment is increasing in $\underline{w}$.

We now turn to the more challenging cases in which $\underline{w} \in [w_1^\ell(Q^\ell), W(Q^p)]$, so that the equilibrium is in either the Robinson region or the constrained two-wage region. Consider first the Robinson region. Recall from Section 2 that in Robinson’s analysis, which assumes uniform wage setting, the employment-increasing effects of minimum wages arise in the gap case—that is, when the marginal revenue product $V(S(\underline{w}))$ lies within the discontinuity in

the marginal cost $C'_U(\cdot, \underline{w})$ at $Q = S(\underline{w})$, as in Panel (b) of Figure 1. The Robinson gap persists when the firm is instead free to choose any mechanism satisfying the minimum wage constraint: it is then the discontinuity in the marginal cost envelope $\underline{C}'_R(\cdot, \underline{w})$ that comes into play, and the minimum wages for which $V(S(\underline{w}))$ falls within this discontinuity constitute precisely the Robinson region, as illustrated in Panel (b) of Figure 6. Within this region, there is no involuntary unemployment and no wage dispersion, and, just as in Robinson's analysis, a marginal increase in the minimum wage increases employment because the firm optimally hires $S(\underline{w})$ workers.

However, minimum wage increases can also have employment-increasing effects outside the Robinson gap case, while at the same time affecting involuntary unemployment. Consider the constrained two-wage region, illustrated in Panel (a) of Figure 6. By Theorem 1, within this region the marginal cost envelope $\underline{C}'_R(Q, \underline{w})$ is strictly decreasing in $\underline{w}$ and strictly increasing in Q . Consequently, a small increase in the minimum wage shifts the marginal cost envelope down, moving its intersection with the decreasing marginal revenue product V to the right, and thereby increasing equilibrium employment $Q^*(\underline{w})$. This argument—which is formally established in the proof of Lemma 3 below—shows that the employment-increasing effects of minimum wages are not limited to the gap case studied by Robinson.

While this settles the direction of the employment effect in the constrained two-wage region, understanding the behavior of involuntary unemployment and the equilibrium wage schedule requires more work. To see why, note that, as we have just established, equilibrium employment $Q^*(\underline{w})$ is increasing in $\underline{w}$ in the constrained two-wage region. This, however, means that the mechanism parameters evaluated along the equilibrium locus, $q_i^*(Q^*(\underline{w}), \underline{w})$, are subject to two countervailing effects. From Lemma 2, we know that $q_1^*(Q, \underline{w})$ is increasing in $\underline{w}$ and decreasing in Q : the direct effect of an increase in the minimum wage raises q_1^* , while the indirect effect operating through the induced increase in $Q^*(\underline{w})$ lowers it. Similarly, $q_2^*(Q, \underline{w})$ is decreasing in $\underline{w}$ and increasing in Q , so the direct effect lowers q_2^* while the indirect effect raises it. The following lemma resolves this tension.

Lemma 3. *Let I be an interval such that $\underline{w} \in [w_1^\ell(Q^*(\underline{w})), W(Q^*(\underline{w}))]$ for all $\underline{w} \in I$, so that the equilibrium is in the constrained two-wage region on I . Then, on I :*

- (i) $Q^*(\underline{w})$ is continuous and strictly increasing in $\underline{w}$;
- (ii) $\lambda^*(Q^*(\underline{w}), \underline{w})$ is increasing in $\underline{w}$;
- (iii) $q_1^*(Q^*(\underline{w}), \underline{w})$ is increasing and $q_2^*(Q^*(\underline{w}), \underline{w})$ is decreasing in $\underline{w}$; and
- (iv) if V is constant on $\{Q^*(\underline{w}) : \underline{w} \in I\}$, then $q_1^*(Q^*(\underline{w}), \underline{w})$ and $q_2^*(Q^*(\underline{w}), \underline{w})$ are constant on I .

The intuition for Lemma 3—in particular, for why the direct effect of the minimum wage on the mechanism parameters dominates the indirect effect operating through equilibrium employment—again runs through the geometric construction from Section 3.2. As discussed, the optimal mechanism parameters q_1^* and q_2^* depend on $(Q, \underline{w})$ only through the solution value of the Lagrange multiplier $\lambda^*(Q, \underline{w})$, so to understand how they vary along the equilibrium locus, it suffices to understand how $\lambda^*(Q^*(\underline{w}), \underline{w})$ varies with $\underline{w}$. By Lemma 1, the two channels again countervail: the direct effect of a higher minimum wage raises the multiplier, while the induced increase in equilibrium employment lowers it. Nevertheless, part (ii) of the lemma establishes that, on net, the multiplier (weakly) rises: the minimum wage constraint becomes tighter along the equilibrium locus. This is natural: re-optimizing employment dampens, but cannot undo, the burden of the higher minimum wage.

The case where V is constant is a limiting case that serves as an insightful benchmark. Suppose $V(Q) = v$ holds for all $Q \in [0, 1]$, and consider equilibria at which the marginal cost envelope is differentiable in Q (which, since $\underline{C}_R(\cdot, \underline{w})$ is convex by Theorem 1, excludes at most countably many quantities). The optimality condition (9) then requires the marginal cost of procurement to equal v at every such equilibrium in the constrained two-wage region. In particular, for two minimum wages $\underline{w} < \underline{w}'$ in this region, $\underline{C}'_R(Q^*(\underline{w}), \underline{w}) = \underline{C}'_R(Q^*(\underline{w}'), \underline{w}') = v$. Recall from Section 3.2 that the marginal cost equals the slope $s_{\lambda^*(Q, \underline{w})}(Q)$ of the relevant linear segment of $\underline{C}_{\lambda^*}$, which is strictly decreasing in the multiplier and, at a fixed multiplier, does not vary with the quantity within that segment. Equality of the marginal costs therefore requires $\lambda^*(Q^*(\underline{w}), \underline{w}) = \lambda^*(Q^*(\underline{w}'), \underline{w}')$. That is, the direct and indirect effects offset exactly and the increase in the multiplier induced by the higher minimum wage is undone by a commensurate increase in the equilibrium quantity. Since the optimal mechanism parameters depend on $(Q, \underline{w})$ only through the multiplier, they are then unchanged, which is part (iv) of Lemma 3. If instead V is decreasing, the same logic runs with inequalities: marginal cost is weakly lower at the higher minimum wage, so the multiplier is weakly higher at the higher minimum wage and the larger equilibrium quantity—the direct effect weakly dominates, which is precisely part (ii).

With the comparative statics of optimal employment and the optimal mechanism parameters in the constrained two-wage region in hand, we can now say what the minimum wage effects are in each of the regions identified in Definition 1. As the discussion in this section shows, these effects are rich and non-monotone across regions. This raises the question of whether a policy maker such as a regulator could predict the effects of a small change to a prevailing minimum wage purely on the basis of observable outcomes, without detailed knowledge of the functions V and W that determine which region the equilibrium is in. Theorem 2 below answers this question affirmatively: it requires only that the policy maker

can separately observe whether there is involuntary unemployment and whether there is wage dispersion under the prevailing minimum wage.

Theorem 2 (Marginal minimum wage effects based on observables). *Fix a prevailing minimum wage $\underline{w} \in [W(0), W(1))$.*

- (i) *If there is involuntary unemployment and wage dispersion under a given minimum wage, then $\underline{w} < W(Q^p)$. A marginal increase in the minimum wage then increases employment and decreases involuntary unemployment and wage dispersion. Moreover, the low wage paid by the firm increases and the high wage decreases as the minimum wage increases marginally.*
- (ii) *If there is no involuntary unemployment under a given minimum wage, then $\underline{w} \leq W(Q^p)$. Provided $\underline{w} \neq W(Q^p)$, a marginal increase in the minimum wage then increases employment.*
- (iii) *If there is involuntary unemployment and no wage dispersion under a given minimum wage, then $\underline{w} > W(Q^p)$. A marginal increase in the minimum wage then decreases employment and increases involuntary unemployment without affecting wage dispersion.*

To see the logic underlying Theorem 2, consider which regions are consistent with each pair of observations. Involuntary unemployment together with wage dispersion (part (i)) arises in the non-binding and constrained two-wage regions, both of which have $\underline{w} < W(Q^p)$. In the former, a marginal increase in the minimum wage leaves the equilibrium unchanged, consistent with the (weak) monotonicity claims. In the latter, the effects are exactly those established in this section: employment rises, and involuntary unemployment $q_2^*(Q^*(\underline{w}), \underline{w}) - Q^*(\underline{w})$ falls. Moreover, the low wage $\underline{w}$ rises mechanically and the high wage $W(q_2^*(Q^*(\underline{w}), \underline{w}))$ falls, so wage dispersion decreases. The absence of involuntary unemployment (part (ii)) identifies either the non-binding region with a market-clearing laissez-faire outcome, or the Robinson region; in both cases, $\underline{w} \leq W(Q^p)$. In the Robinson region employment is $S(\underline{w})$, which is strictly increasing in $\underline{w}$ —the pro-competitive effect driven by the gap case, just as in Robinson’s analysis. The caveat $\underline{w} \neq W(Q^p)$ excludes the single boundary configuration at which the prevailing minimum wage equals the competitive wage and a marginal increase would instead push the equilibrium into the neoclassical region. Finally, involuntary unemployment without wage dispersion (part (iii)) identifies the neoclassical region, where $\underline{w} > W(Q^p)$ and the firm rations workers at the minimum wage. Just as we saw in the neoclassical analysis of Section 2, here a higher minimum wage reduces employment and increases involuntary unemployment, with all employed workers paid the minimum wage throughout.

4.3 Transitions between regions

Section 4.2 established the effects of marginal minimum wage increases within each of the four regions of Definition 1. To complete the analysis, we now characterize the transitions between these regions (i.e., the sequence of regions that the equilibrium passes through as the minimum wage increases). Two boundaries give this sequence a simple overall structure. The equilibrium leaves the non-binding region once $\underline{w}$ reaches $w_1^\ell(Q^\ell)$, the lowest wage paid under laissez-faire, and it enters the neoclassical region once $\underline{w}$ exceeds the competitive wage $W(Q^p)$. In particular, once the minimum wage has become binding, it remains binding, and once the equilibrium has entered the neoclassical region, it never leaves it. For all minimum wages between these two boundaries, the equilibrium is in either the Robinson region or the constrained two-wage region, and the goal of this section is to uncover the structure of this intermediate range, $\underline{w} \in [w_1^\ell(Q^\ell), W(Q^p)]$. As we will show, the region that the equilibrium enters as $\underline{w}$ increases and becomes binding depends on whether there is involuntary unemployment under laissez-faire. If that is the case, then the equilibrium transitions from the non-binding region into the constrained two-wage region. Otherwise, it generically transitions into the Robinson region. Thereafter, the transition patterns can be rich. The equilibrium may alternate between the Robinson and the constrained two-wage regions multiple times, depending on the number of ironing intervals between the laissez-faire and the competitive levels of employment and on the shape of the marginal revenue product over each of them, before ultimately entering the neoclassical region. Generically, this final transition occurs from the Robinson region. Proposition 2 below makes these statements precise and specifies the non-generic exceptions.

Throughout the intermediate range $\underline{w} \in [w_1^\ell(Q^\ell), W(Q^p)]$, the equilibrium is in either the Robinson region or the constrained two-wage region, so the structure of this range is determined by the set of minimum wages at which the Robinson region occurs. At the equilibrium quantity $Q^*(\underline{w}) = S(\underline{w})$, the marginal cost envelope is discontinuous, jumping from $\underline{w}$ (the marginal cost of procuring any quantity below $S(\underline{w})$) to the right derivative $\underline{C}'_{R+}(S(\underline{w}), \underline{w})$. The Robinson region consists of those minimum wages for which $V(S(\underline{w}))$ falls within this gap, that is, $\underline{w} \leq V(S(\underline{w})) \leq \underline{C}'_{R+}(S(\underline{w}), \underline{w})$. For $\underline{w} \leq W(Q^p)$, the first inequality is automatically satisfied,³⁴ so within the intermediate range the Robinson region is delineated by the second inequality alone. Determining where the Robinson region occurs therefore amounts to characterizing the upper edge of this gap—the right derivative of the cost envelope at $Q = S(\underline{w})$ —and relating it to the marginal revenue product V .

³⁴Since S is strictly increasing, $\underline{w} \leq W(Q^p)$ implies $S(\underline{w}) \leq Q^p$, and since V is decreasing, it follows that $V(S(\underline{w})) \geq V(Q^p)$. Recalling that $V(Q^p) = W(Q^p)$ holds by the definition of the competitive quantity Q^p , we conclude that $V(S(\underline{w})) \geq W(Q^p) \geq \underline{w}$.

For the purpose of characterizing the Robinson region, it is useful to first revisit the case in which the firm is restricted to uniform wage setting. From $C_U(Q, \underline{w})$ as defined in (3), the left derivative of $C_U(\cdot, \underline{w})$ at $Q = S(\underline{w})$ is $\underline{w}$, and the upper edge of the Robinson gap is the right derivative of $C_U(\cdot, \underline{w})$ at $Q = S(\underline{w})$, which under uniform wage setting is simply $C'_+(S(\underline{w}))$ and does not directly depend on the minimum wage. Under uniform wage setting, the Robinson region thus consists of all $\underline{w}$ satisfying $\underline{w} \leq V(S(\underline{w})) \leq C'_+(S(\underline{w}))$.

To understand the boundaries of the Robinson region when the firm is not constrained in its choice of mechanism above and beyond the restrictions imposed by $\underline{w}$, we now introduce the function γ , which serves as the analog of C'_+ under optimal procurement. The upper edge of the Robinson gap at $Q = S(\underline{w})$ is again the right derivative of the cost envelope at $Q = S(\underline{w})$. However, in contrast to C'_+ , which only depends on the quantity, this object now directly depends on the minimum wage because the minimum wage constrains the two-wage mechanisms through which the firm procures quantities beyond $S(\underline{w})$. Accordingly, we define

$$\gamma(Q) := \underline{C}'_{R+}(Q, W(Q)).$$

That is, γ is the right derivative of the optimal cost envelope with respect to quantity, evaluated at the minimum wage $\underline{w} = W(Q)$ at which labor supply at the minimum wage is exactly Q . Since $\underline{C}_R(\cdot, \underline{w})$ is convex by Theorem 1, γ is well-defined. At this minimum wage $\underline{w} = W(Q)$, the left derivative of $\underline{C}_R(\cdot, \underline{w})$ at $Q = S(\underline{w})$ is $W(Q)$ itself, so the Robinson gap is $[W(Q), \gamma(Q)]$, which is degenerate if $\gamma(Q) = W(Q)$.³⁵ Within the intermediate range between the non-binding and neoclassical regions, the equilibrium is therefore in the Robinson region if $V(S(\underline{w})) \leq \gamma(S(\underline{w}))$ and in the constrained two-wage region if $V(S(\underline{w})) > \gamma(S(\underline{w}))$.

In general, the number of transitions between the Robinson and the constrained two-wage regions cannot be determined without further information. It depends on how many ironing intervals lie between the laissez-faire and the competitive levels of employment, and on the shapes of V and γ over each of these intervals. Where these transitions can occur is nevertheless simple. Recall from Section 3.2 that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage only if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$. Since $S = W^{-1}$, this holds if and only if $S(\underline{w})$ lies in the laissez-faire ironing interval $(Q_1(m), Q_2(m))$. For minimum wages in the intermediate range $\underline{w} \in [w_1^\ell(Q^\ell), W(Q^p)]$ such that $S(\underline{w})$ lies outside every ironing interval, the equilibrium is therefore in the Robinson region. Within a single laissez-faire ironing interval, by contrast, V may cross γ multiple times, and the equilibrium then transitions into and out of the Robinson region each time it does so. We do not characterize these crossings in general. Instead, we next establish

³⁵The gap is well defined because $\gamma(Q) \geq W(Q)$: the right derivative of the convex function $\underline{C}_R(\cdot, W(Q))$ at Q is at least its left derivative, which is $W(Q)$.

the basic properties of γ and then record, in Proposition 2 below, the transitions that occur independently of these details.

We now characterize the behavior of γ , which is governed by the position of Q relative to the laissez-faire ironing intervals. As we will see, γ coincides with the ironed marginal cost outside the ironing intervals, and dips strictly below it within them. Consider first a quantity Q that lies outside every laissez-faire ironing interval, so that $\underline{C}(Q) = C(Q)$. By Proposition 1, $w_1^\ell(Q) = W(Q)$, so the minimum wage $W(Q)$ does not bind at any quantity above Q and, by (8), $\underline{C}_R(\cdot, W(Q))$ coincides with $\underline{C}$ on $[Q, 1]$. Hence $\gamma(Q) = \underline{C}'_+(Q)$. Because this is a right derivative, its value is determined by the shape of $\underline{C}$ immediately above Q , and two sub-cases arise. If Q is not the left endpoint of an ironing interval, then $\underline{C}$ coincides with C immediately above Q , and we simply have $\gamma(Q) = C'_+(Q)$.³⁶ If instead $Q = Q_1(m)$ for some $m \in \mathcal{M}$, then immediately above Q the function $\underline{C}$ follows the ironed segment over $(Q_1(m), Q_2(m))$, and $\gamma(Q)$ equals its slope. By (4), this slope lies between $C'_-(Q_1(m))$ and $C'_+(Q_1(m))$, and it equals $C''(Q_1(m))$ whenever C is differentiable at this point. Next, consider a quantity Q in the interior of an ironing interval, so that $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Fix this quantity and increase the minimum wage from $w_1^\ell(Q)$, at which the constraint just binds, towards $W(Q)$. Over this range, the firm procures Q using a constrained two-wage mechanism, and by Theorem 1 its marginal cost of procurement strictly decreases in the minimum wage. Correspondingly, γ dips strictly below the ironed marginal cost on every laissez-faire ironing interval, that is, $\gamma(Q) < \underline{C}'_-(Q)$ whenever $\underline{C}(Q) < C(Q)$ (see the proof of Proposition 2).

Figure 8 illustrates the construction of γ for our leading example (6). Panel (a) displays the marginal cost envelope $\underline{C}'_R(\cdot, \underline{w})$ for a range of minimum wages. For each $\underline{w}$, the envelope is discontinuous at $Q = S(\underline{w})$ and strictly increasing on $\left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right)$, where it rejoins the laissez-faire marginal cost $\underline{C}'_-$. The function γ , displayed in Panel (b), traces out the upper edges of these discontinuities.

We are now in a position to collect the results of this section. To state them formally, we first make precise what we mean by a transition. We say that the equilibrium *transitions* from one region into another at a given minimum wage if it is in the first region at all slightly lower minimum wages and in the second region at all slightly higher ones. With this terminology in hand, we have the following proposition.

Proposition 2 (Transitions). *Suppose that the interval (Q^ℓ, Q^p) intersects only finitely many laissez-faire ironing intervals and that γ is left-continuous at Q^p .³⁷ Then:*

³⁶In particular, when C is convex, so that there are no ironing intervals, $\gamma(Q) = C'_+(Q)$ holds for all Q , just like under uniform wage setting.

³⁷These restrictions rule out pathological configurations in which ironing intervals accumulate at Q^ℓ from

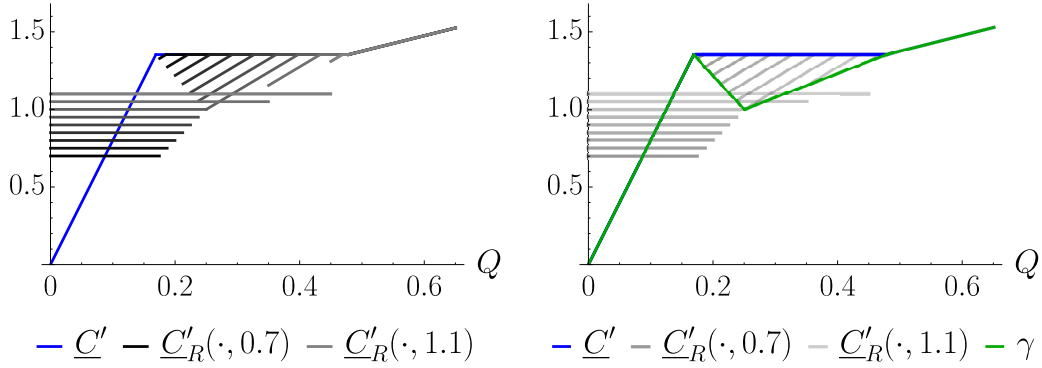


Figure 8: Construction of the function γ for the leading example (6). Panel (a) displays the marginal cost envelope $\underline{C}'_R(\cdot, \underline{w})$ for a range of minimum wages varying from $\underline{w} = 0.7$ to $\underline{w} = 1.1$, where the discontinuities in these envelopes occur at $Q = S(\underline{w})$. Panel (b) displays γ , which traces out the upper edges of these discontinuities in Panel (a).

- (i) *The equilibrium is in the non-binding region if and only if $\underline{w} < w_1^\ell(Q^\ell)$, and it is in the neoclassical region if and only if $\underline{w} > W(Q^p)$. For $\underline{w} \in [w_1^\ell(Q^\ell), W(Q^p)]$, the equilibrium is in the Robinson region if $V(S(\underline{w})) \leq \gamma(S(\underline{w}))$ and in the constrained two-wage region if $V(S(\underline{w})) > \gamma(S(\underline{w}))$.*
- (ii) *If there is involuntary unemployment under laissez-faire, then the equilibrium transitions from the non-binding region into the constrained two-wage region. If there is no involuntary unemployment under laissez-faire, then the equilibrium transitions from the non-binding region into the Robinson region, unless the non-generic condition $Q^\ell = Q_1(m)$ holds for some $m \in \mathcal{M}$.*
- (iii) *The equilibrium transitions into the neoclassical region from the Robinson region, unless the non-generic condition $\gamma(Q^p) = W(Q^p)$ holds. A transition from the constrained two-wage region can occur only in that case.*

The conditions excluded in parts (ii) and (iii) are non-generic. The ironing intervals and the function γ are determined by W alone, whereas Q^ℓ and Q^p also depend on V : the laissez-faire employment level Q^ℓ is determined by where V crosses the ironed marginal cost, and the competitive employment level Q^p by where V crosses W . The condition $Q^\ell = Q_1(m)$ therefore requires an exact coincidence between Q^ℓ and the endpoint of an ironing interval, and the condition $\gamma(Q^p) = W(Q^p)$ requires an exact coincidence between Q^p and a quantity at which the Robinson gap is degenerate. Perturbing V moves Q^ℓ and Q^p without affecting

the right, or the gap $[W(Q), \gamma(Q)]$ becomes arbitrarily small at quantities approaching Q^p from the left. Without them, the conclusions in (ii) and (iii) hold with the non-generic conditions replaced by more cumbersome limiting versions.

the ironing intervals or the function γ , so both conditions fail after any small perturbation of the marginal revenue product. In the non-generic case $\gamma(Q^p) = W(Q^p)$, the transition into the neoclassical region can occur from either the constrained two-wage region or the Robinson region, depending on how V crosses γ at Q^p .

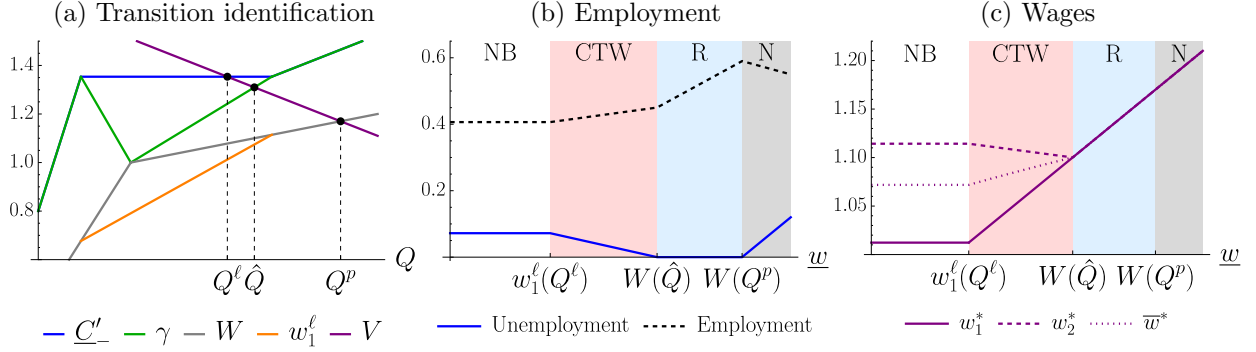


Figure 9: This figure is constructed for our leading example (6) with $V(Q) = 1.76 - Q$, which implies that there is involuntary unemployment under laissez-faire. Panel (a) identifies the three minimum wages at which the equilibrium changes region. The equilibrium transitions from the non-binding (NB) to the constrained two-wage (CTW) region at $w_1^\ell(Q^\ell) \approx 1.01$, where $Q^\ell \approx 0.41$ is given by the intersection of C' and V ; from the constrained two-wage region to the Robinson (R) region at $W(\hat{Q}) \approx 1.10$, where $\hat{Q} \approx 0.45$ is given by the intersection of V and γ ; and from the Robinson region to the neoclassical (N) region at $W(Q^p) = 1.17$, where $Q^p = 0.59$ is given by the intersection of W and V . Panels (b) and (c) display the corresponding paths of equilibrium employment, involuntary unemployment and the wage schedule as functions of w , with the four regions shaded and these three minimum wages marked. In panel (c), w_1^* denotes the lowest wage paid in equilibrium, w_2^* denotes the highest wage paid in equilibrium and $\bar{w}^*$ denotes the average equilibrium wage among employed workers.

Figures 9 and 10 illustrate Proposition 2 for our leading example and two different marginal revenue product functions. In Figure 9, V is strictly decreasing and there is involuntary unemployment under laissez-faire. As Panel (a) shows, V crosses γ once, at the quantity $\hat{Q}$ inside the ironing interval. In line with part (ii) of Proposition 2, the equilibrium transitions from the non-binding region into the constrained two-wage region at $w = w_1^\ell(Q^\ell)$. It then transitions into the Robinson region at $w = W(\hat{Q})$ and, in line with part (iii), into the neoclassical region at $w = W(Q^p)$. Panels (b) and (c) display the resulting paths of employment, involuntary unemployment and wages, with the four regions shaded. Within each region, the comparative statics are those established in Theorem 2.

In Figure 10, V is constant and there is no involuntary unemployment under laissez-faire. As Panel (a) shows, V now crosses γ twice, at the quantities $\hat{Q}_1$ and $\hat{Q}_2$, both of which lie inside the same ironing interval. In line with part (ii), the equilibrium transitions from the

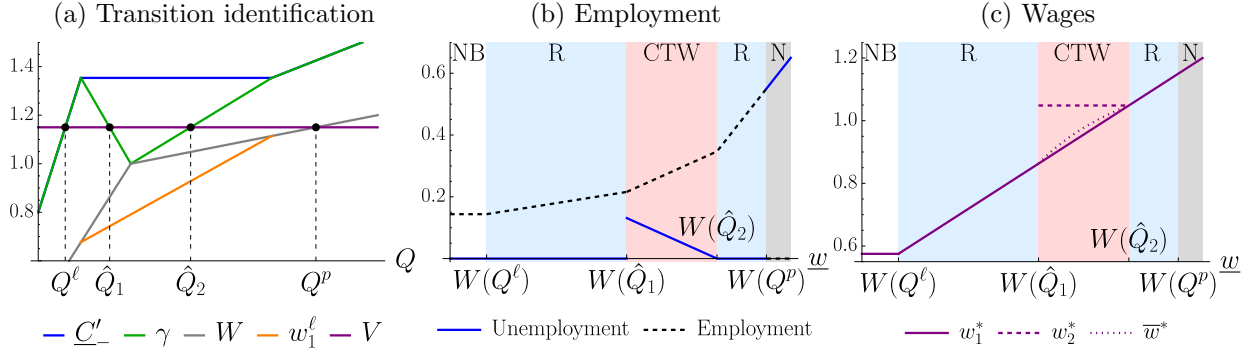


Figure 10: This figure is constructed for our leading example (6) with $V(Q) = 1.15$ for all $Q \in [0, 1]$, which implies that there is no involuntary unemployment under laissez-faire. Panel (a) identifies the minimum wages at which the equilibrium changes region, as in Figure 9. The equilibrium transitions from the non-binding (NB) region into the Robinson (R) region at $w_1^l(Q^\ell) \approx 0.58$, where $Q^\ell \approx 0.14$ is given by the intersection of $\underline{C}'$ and V ; enters and exits the constrained two-wage (CTW) region at $W(\hat{Q}_1) \approx 0.86$ and $W(\hat{Q}_2) \approx 1.05$, where $\hat{Q}_1 \approx 0.22$ and $\hat{Q}_2 \approx 0.35$ are the two intersections of V and γ ; and enters the neoclassical (N) region at $W(Q^p) = 1.15$, where $Q^p = 0.55$ is given by the intersection of W and V . Panels (b) and (c) are constructed as in Figure 9.

non-binding region into the Robinson region. It then transitions into the constrained two-wage region at $\underline{w} = W(\hat{Q}_1)$, back into the Robinson region at $\underline{w} = W(\hat{Q}_2)$, and into the neoclassical region at $\underline{w} = W(Q^p)$. This example shows that the equilibrium can enter and exit the Robinson region within a single ironing interval, so that rich transition patterns do not require multiple ironing intervals.

Figure 10 illustrates a further implication of the analysis. When V is constant, a two-wage mechanism is never strictly optimal under laissez-faire, so the firm sets a market-clearing wage and there is no involuntary unemployment.³⁸ A binding minimum wage can nevertheless make a two-wage mechanism uniquely optimal: in the figure, the firm uses two wages and involuntary unemployment arises for every $\underline{w} \in (W(\hat{Q}_1), W(\hat{Q}_2))$. Minimum wages can therefore induce both wage dispersion and involuntary unemployment in settings where neither arises absent regulation.

³⁸This is the analogue of the observation by Bulow and Roberts (1989) that in a monopoly context, rationing is never strictly optimal when the firm has constant marginal costs. Translated to that setting, our result says that a two-price mechanism can become uniquely optimal in the presence of a price cap even when the firm's marginal cost is constant.

4.4 Welfare effects

We now analyze the effects of binding minimum wages on social surplus and worker surplus. For minimum wages below the competitive wage, an increase in the minimum wage raises employment, workers' pay, social surplus, and worker surplus. Beyond the competitive wage, this alignment breaks down.

Proposition 3. *For all binding minimum wages $\underline{w} \in [w_1^\ell(Q^\ell), W(Q^p)]$, equilibrium employment $Q^*(\underline{w})$, workers' aggregate pay, the average pay per employed worker, social surplus, and worker surplus are continuously increasing in $\underline{w}$. Social surplus is maximized at $\underline{w} = W(Q^p)$, and worker surplus is maximized either at $\underline{w} = W(Q^p)$ or at a higher minimum wage.*

Perhaps the most notable implications of Proposition 3 concern the transitions between regions. In the Robinson region all employed workers are paid the minimum wage, and there is neither involuntary unemployment nor wage dispersion. As the minimum wage crosses from the Robinson region into the constrained two-wage region, there is a discontinuous increase in involuntary unemployment and wage dispersion (see Figure 10). Employment, worker pay and both surpluses nevertheless vary continuously because the probability with which the firm hires at the high wage vanishes as the transition is approached. Beyond the competitive wage, the two surplus objectives can conflict because worker surplus can increase at the expense of social surplus and, of course, the firm's profit.³⁹

The discontinuous increase in involuntary unemployment across a transition at which both worker and social surplus increase continuously is one manifestation of a broader point: within the constrained two-wage region, the amount of involuntary unemployment is not a reliable indicator of the social or worker surplus lost to rationing. This feature already arises under laissez-faire. For some $m \in \mathcal{M}$ and $Q \in (Q_1(m), Q_2(m))$, the social cost associated with the two-wage mechanism,

$$\beta(Q, Q_1(m), Q_2(m)) \int_{Q_1(m)}^{Q_2(m)} W(y) dy - \int_{Q_1(m)}^Q W(y) dy \geq 0, \quad (11)$$

is strictly concave in Q and equal to zero at $Q = Q_1(m)$ and at $Q = Q_2(m)$.⁴⁰ In contrast, involuntary unemployment is $Q_2(m) - Q$, which is strictly decreasing in Q and maximized

³⁹Worker surplus need not be maximized at the competitive wage because, in the neoclassical region, the wage gains of the workers who remain employed can outweigh the losses from reduced employment. This occurs, for example, in our leading example when V is linear and sufficiently steep and the competitive wage lies below the kink of the labor supply schedule. Assuming efficient rationing, Lee and Saez (2012) showed that worker surplus is maximized at a minimum wage above the competitive wage. Under random rationing, whether this occurs depends on the shapes of V and W .

⁴⁰The derivative with respect to Q is $(Q_2(m) - Q_1(m))^{-1} \int_{Q_1(m)}^{Q_2(m)} W(y) dy - W(Q)$, which is strictly decreasing because W is strictly increasing, and which is positive at $Q_1(m)$ and negative at $Q_2(m)$.

as Q approaches $Q_1(m)$. Consequently, involuntary unemployment and the associated social cost are not monotonically related: involuntary unemployment is largest precisely where the social cost vanishes because the mass of workers employed at the high wage vanishes as well. The same behavior arises under a binding minimum wage. By Lemma 3, $Q^*(\underline{w})$ increases and q_2^* decreases in $\underline{w}$, so involuntary unemployment $q_2^* - Q^*(\underline{w})$ is largest at the lower boundary of the constrained two-wage region. When this boundary corresponds to a transition from the Robinson region, $Q^*(\underline{w})$ approaches q_1^* , so involuntary unemployment approaches its maximum $q_2^* - q_1^*$, while the associated social cost vanishes (see the proof of Proposition 3). Equilibrium behavior in the constrained two-wage region is therefore fundamentally different from that in the neoclassical region, where social surplus is lower the larger is involuntary unemployment.

Although every increase in the minimum wage below the competitive wage benefits workers in the aggregate, there is a conflict of interest among them. Workers paid the minimum wage before and after an increase are strictly better off. In contrast, by Lemma 3, the high wage $W(q_2^*(Q^*(\underline{w}), \underline{w}))$ decreases in $\underline{w}$, and strictly so in the example of Figure 9. Whenever the high wage strictly decreases, workers with high opportunity costs of working who were willing to work at the high wage before the increase no longer are after it, and workers paid the high wage before and after the increase are also strictly worse off.

Discrete minimum wage increases Unsurprisingly, setting a minimum wage equal to the competitive wage maximizes social surplus, but doing so requires detailed knowledge of both V and W . The following proposition shows that whenever involuntary unemployment and wage dispersion are observed, a minimum wage equal to the highest wage currently paid always increases employment and workers' total pay. Notably, this guarantee requires no knowledge of the competitive wage: it holds even if the highest wage currently paid—and hence the new minimum wage—exceeds $W(Q^p)$. The improvement from such a discrete increase applies equally when the minimum wage binds before the increase and when it does not. When it does not, the increase also reduces involuntary unemployment and eliminates it whenever the prevailing high wage is below the competitive wage.

Proposition 4. *Suppose there is involuntary unemployment and wage dispersion under a prevailing minimum wage $\underline{w}_0 \in [W(0), W(1))$, and let w_2^* denote the prevailing high wage. Consider increasing the minimum wage to any $\underline{w} \in (\underline{w}_0, w_2^*]$, with all comparisons relative to the outcome under $\underline{w}_0$.*

- (i) *Employment and workers' total pay increase. If $\underline{w} \leq W(Q^p)$, then social surplus and worker surplus also increase.*

- (ii) *If $\underline{w}_0$ does not bind, involuntary unemployment decreases, and it is eliminated at $\underline{w} = w_2^*$ if and only if $w_2^* \leq W(Q^p)$.*

Proposition 4 rests on an observation of independent interest: the firm never pays a high wage above the marginal revenue product of labor, no matter how tightly the minimum wage constrains it. At the highest wage paid, the firm's marginal workers therefore generate at least as much revenue as they cost, so raising the minimum wage to that level induces the firm to expand employment rather than to contract it. A regulator who observes involuntary unemployment and wage dispersion thus needs no knowledge of V or W to improve upon the prevailing outcome. In particular, the regulator is protected against overshooting: employment and workers' total pay increase even when the new minimum wage exceeds the competitive wage.⁴¹ If the prevailing minimum wage does not bind, setting it equal to the highest observed wage also decreases involuntary unemployment, possibly to the point of eliminating it. Elimination obtains precisely when the prevailing high wage lies below the competitive wage, and in that case social surplus and worker surplus are aligned, so both increase; as noted, this alignment is an implication of Proposition 3.

5 Discussion

In this section we discuss the empirical content of our results and the role of our modeling assumptions. We first discuss where the non-convexities that drive our analysis naturally arise, what changes without the separation condition, and whether the wage patterns our mechanisms predict are observed in practice. Then we turn to the robustness of two-wage mechanisms to relaxing our assumptions on preferences and rationing, and to the extent to which our results depend on the firm having monopsony power.

5.1 Non-convex procurement costs and the separation condition

Two-wage mechanisms are only optimal if the firm's procurement cost function C is non-convex, possibly begging the question of how such cost functions emerge in practice. While we are not aware of any fundamental economic forces that would necessarily make C convex, non-convex cost functions naturally arise when workers face a fixed cost of moving, changing

⁴¹Overshooting is, moreover, detectable from the same observables: by Theorem 2, involuntary unemployment accompanied by wage dispersion indicates a minimum wage below the competitive wage, whereas involuntary unemployment without wage dispersion indicates a minimum wage above it. Together with Proposition 4, this suggests a simple procedure for iteratively adjusting the minimum wage toward the competitive wage, at which social surplus is maximized, using only observed market outcomes.

occupation or participating in the labor market.⁴² Cogan (1981) finds such costs to be of first-order importance in accounting for labor supply behavior. Similarly, if two labor markets that differ with respect to the lowest opportunity cost of working are integrated into one, then the integrated labor market always exhibits a non-convex cost function, even when as standalone markets each market exhibits a convex cost function.⁴³ A monopsony that faces a non-convex cost function is analogous to a monopoly that faces a non-concave revenue function. Each of these problems corresponds to a mechanism design problem that fails to be *regular* as defined by Myerson (1981). To our knowledge, the curvature of inverse labor supply schedules W , and of the corresponding cost functions C , has not been investigated empirically. In output markets, where the analogous assumption of concave revenue has been tested, it is frequently rejected.⁴⁴ Regularity is thus an assumption of analytical convenience rather than one with empirical support, and we see little reason to expect it to hold systematically in input markets; our analysis does not impose it.

Our analysis does, however, impose the separation condition, Assumption 1, which limits how an affine function can touch the adjusted cost C_λ and thereby ensures that two wages suffice under a binding minimum wage. Appendix B.3 characterizes optimal procurement without it. Three wages then always suffice, and a third wage can be needed only when the minimum wage binds but the firm employs more workers than are willing to work at it, that is, at equilibria with $w_1^\ell(Q^*(\underline{w})) < \underline{w} < W(Q^*(\underline{w}))$. A third wage can therefore only arise at quantities at which C fails to be convex and, like two-wage mechanisms, three-wage mechanisms are a consequence of non-convex procurement costs. Because Definition 1 classifies equilibria by employment $Q^*(\underline{w})$ relative to labor supply $S(\underline{w})$ and the laissez-faire low wage w_1^ℓ , none of which relies on Assumption 1, the four regions remain well defined, and Proposition 2 continues to describe where they lie and how the equilibrium moves between them.⁴⁵ What changes is the optimal mechanism in the interior of the constrained two-wage region, which is therefore more aptly called the multi-wage region when Assumption 1 is dropped. Where a third wage is needed, the procurement cost is locally affine in the quantity and the minimum wage, so that the marginal cost of labor and the shadow price of the minimum wage are both locally constant (Proposition B.1).

⁴²For example, suppose a mass 1/4 of workers face no fixed cost of moving, with opportunity costs uniformly distributed on $[0, 1]$. The remaining mass 3/4 face a fixed moving cost of 1, with opportunity costs net of that cost uniformly distributed on $[0, 3/8]$. The resulting W and C are those of the leading example.

⁴³This is analogous to the observation made in Loertscher and Muir (2022) that integrating output markets can render revenue non-concave.

⁴⁴See, for example, Celis, Lewis, Mobius, and Nazerzadeh (2014), Appendix D in Larsen and Zhang (2018) and Section 5 in Larsen (2021), as well as the related discussion in Loertscher and Muir (2022).

⁴⁵See Proposition B.2 and the discussion preceding it in Appendix B.3. The boundary case $V(S(\underline{w})) = \gamma(S(\underline{w}))$ in Proposition 2(i) is not covered.

Employment is nonetheless increasing in the minimum wage (Proposition B.2), so that the employment effects in Theorem 2 continue to hold, with the increase in part (i) understood weakly, since employment is locally constant wherever a third wage is paid.⁴⁶ However, the effects of the minimum wage on involuntary unemployment, wage dispersion and welfare are derived from the comparative statics of the two-wage mechanism, and their extension to three wages is considerably more involved—largely because dropping Assumption 1 also means that the optimal mechanism is no longer unique—and beyond the scope of this paper.

5.2 Part-time work

As noted in Section 3.1, the rationing involved in implementing a two-wage mechanism can be interpreted in terms of part-time work, so that high-wage workers are part-time workers who are *underemployed* while low-wage workers are fully employed.

Empirically, part-time workers do sometimes earn strictly more per hour than their full-time colleagues. The clearest evidence comes from Australia, where the Household, Income and Labour Dynamics in Australia (HILDA) survey allows the same individuals to be followed over time as they move between full-time and part-time work. Australian employees classified as *casual* forgo paid leave entitlements and are entitled to a 25 percent loading on the relevant minimum wage, and part-time workers are disproportionately casual, so casual status must be accounted for in any comparison. Doing so, Booth and Wood (2008) and Day and Rodgers (2015) find an hourly premium to part-time work of roughly 10 to 17 percent among non-casual employees, and a premium of comparable magnitude among casual employees.⁴⁷

Importantly, the sustained part-time premium identified in Day and Rodgers (2015) is a within-employer phenomenon. In particular, these authors find that moving from full-time to part-time work while remaining with the same employer produces a large and sustained increase in the hourly wage, whereas moves accompanied by a change of employer produce little evidence of a sustained wage change. The premium is present across a variety of occupations, and is not accounted for by differences in non-wage compensation (i.e., salary sacrifice and measured non-cash benefits). This pattern is consistent with the two-wage mechanisms we characterize, in which a firm with market power optimally pays a strictly

⁴⁶See again the discussion preceding Proposition B.2 in Appendix B.3.

⁴⁷Both studies estimate log hourly wage regressions with individual fixed effects, so that the premium is identified from within-person changes in employment status. Controls include education, quadratics in job tenure and labor market experience, whether the employee holds a single job, works a regular daytime schedule, is on a fixed-term contract or employed through a labor-hire firm, together with demographic and geographic characteristics. Adding workplace size, union membership, sector, industry, and occupation leaves the fixed-effects estimates largely unchanged.

higher per-unit wage to underemployed part-time workers. We show that a monopsonist may optimally use such a mechanism both with and without a binding minimum wage.

The scarcity of comparable evidence elsewhere may be institutional. As Day and Rodgers (2015) discuss, in most OECD countries equivalent full-time and part-time employees are entitled to the same contractual pay and conditions on a pro-rata basis, whereas Australia imposes no such requirement. In the United States, which likewise has no such requirement, a different difficulty arises: employer-provided health insurance is a substantial per-worker cost to which part-time employees typically have less access, and the hourly wage is therefore an incomplete measure of the price paid per unit of labor. Employer-provided health insurance is extremely rare in Australia, while compulsory employer superannuation (i.e., retirement) contributions are proportional to earnings, so hourly wages and total compensation do not diverge in the same way. Even so, a Bureau of Labor Statistics study of 2007 National Compensation Survey data found part-time hourly earnings above full-time earnings in twelve U.S. occupations, concentrated in health services: dental hygienists, physical therapists, speech-language pathologists, and licensed practical nurses, among others.⁴⁸

5.3 Robustness

Like Lee and Saez (2012), our analysis assumes a continuum of risk-neutral workers with quasilinear utility and focuses on the extensive margin in labor supply. The continuum of workers ensures that randomizing over which workers are employed creates no randomness in the aggregate quantity of labor procured.⁴⁹ Together, these assumptions ensure that mechanisms involving at most two wages are optimal absent wage regulation, which in turn, together with Assumption 1, makes the more involved problem of deriving the optimal mechanism under a given minimum wage tractable. If any of these assumptions is relaxed, then alternative techniques will be required to characterize the optimal mechanism. For example, with elastic individual labor supply—that is, when the intensive margin matters—the optimal mechanism need not involve at most two wages. However, the superiority of two-wage mechanisms over uniform-wage mechanisms is robust in three respects.

Consider first risk aversion. The two-wage mechanism that is optimal whenever $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ survives the introduction of risk-averse workers in the following sense. Suppose all workers have the same initial wealth—which without loss of generality can be normalized to zero—and the same strictly concave utility function u , and focus on the case without a minimum wage. A worker with opportunity cost $W(Q)$ working

⁴⁸See *A Comparison of Hourly Wage Rates for Full- and Part-Time Workers by Occupation, 2007*.

⁴⁹With finitely many workers, randomization can induce a lottery over aggregate labor, with implications for the firm’s revenue whenever V is non-linear.

at wage $w \geq W(Q)$ then has a utility of $u(w - W(Q))$, while an unemployed worker has a utility of $u(0)$. The participation constraint for the marginal worker then still requires that $w_2 = W(Q_2(m))$. However, the wage $\hat{w}_1$ that makes workers with opportunity cost $W(Q_1(m))$ indifferent now satisfies $u(\hat{w}_1 - W(Q_1(m))) = \alpha(Q)u(W(Q_2(m)) - W(Q_1(m))) + (1 - \alpha(Q))u(0)$. Under risk neutrality the corresponding condition is $w_1 - W(Q_1(m)) = \alpha(Q)(W(Q_2(m)) - W(Q_1(m)))$, so $\hat{w}_1 < w_1$ holds by Jensen's inequality.⁵⁰ Unsurprisingly, the insurance benefit associated with certain employment works in favor of the firm's scheme, reducing its procurement cost by allowing it to set a lower low wage relative to the case with risk-neutral workers.

Second, the superiority of a two-wage mechanism over a market-clearing wage whenever $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ is robust to small errors in setting the two wages. Specifically, by continuity, if the two-wage mechanism is constructed with sufficiently accurate estimates of $Q_1(m)$ and $Q_2(m)$, then this yields an approximately optimal mechanism with a lower procurement cost than hiring workers at the market-clearing wage $W(Q)$.

Finally, while randomly rationing workers with costs between $W(Q_1(m))$ and $W(Q_2(m))$ is optimal for the monopsony, the superiority of a two-wage mechanism does not hinge on rationing being uniform. Since the workers' incentive compatibility constraints require that workers with lower costs are hired with weakly higher probability, the only alternative rationing schemes are such that the allocation is more efficient than under uniform random rationing. If one parameterizes rationing schemes as convex combinations of the uniform random and the efficient allocation, then keeping the quantity fixed, the scheme with wage dispersion and involuntary unemployment remains optimal if it is optimal with uniform rationing, provided the probability that the efficient allocation obtains is less than one.

5.4 Varying degrees of market power

Our baseline model considers a monopsony firm, which provides a tractable model of the wage-setting power that employers are increasingly recognized to exert in labor markets (Card, 2022a,b). A natural question is to what extent the results extend when the firm has less market power than a monopsonist. A convenient way to conceptualize this is to assume that, for any given $Q \in [0, 1]$, the firm solves the *Ramsey* procurement problem of minimizing

$$C^\alpha(Q) := (1 - \alpha) \int_0^Q W(y) dy + \alpha C(Q),$$

⁵⁰Moreover, the single-crossing condition (see footnote 18) is satisfied.

where $\alpha \in [0, 1]$ parameterizes the firm's market power, with $\alpha = 1$ corresponding to monopsony power and $\alpha = 0$ to price-taking behavior.⁵¹ Because W is strictly increasing, for any $\alpha \in [0, 1)$ and $Q > 0$ we have $C^\alpha(Q) < C(Q)$, and the marginal cost of procurement under a uniform wage satisfies $(C^\alpha)'_-(Q) = (1 - \alpha)W(Q) + \alpha C'_-(Q) < C'_-(Q)$, where $(C^\alpha)'_-$ denotes the left derivative of C^α . By the same arguments as those in Section 3.1, the minimal value of the firm's Ramsey procurement objective is given by the convexification of C^α , denoted $\underline{C}^\alpha$, and for any $\alpha \in [0, 1)$, we have $\underline{C}^\alpha(Q) < \underline{C}(Q)$ and $(\underline{C}^\alpha)'_-(Q) < \underline{C}'_-(Q)$, and both $\underline{C}^\alpha(Q)$ and $(\underline{C}^\alpha)'_-(Q)$ are increasing in α .⁵²

Under laissez-faire, any Q satisfying $(\underline{C}^\alpha)'_-(Q) \leq V(Q) \leq (\underline{C}^\alpha)'_+(Q)$, where $(\underline{C}^\alpha)'_+$ denotes the right derivative of $\underline{C}^\alpha$, is an optimal level of employment, and we let Q^α denote the largest such quantity. Because Q^α is the largest quantity satisfying $V(Q) \geq (\underline{C}^\alpha)'_-(Q)$ and $(\underline{C}^\alpha)'_-$ increases in α , Q^α decreases in α . Thus, the smaller is the firm's market power, the larger is its largest optimal quantity procured. Given a binding minimum wage below the market-clearing wage, the firm's procurement problem is solved as in (7), by convexifying $C^\alpha_\lambda(Q) := C^\alpha(Q) - \lambda W(Q)$ for fixed $\lambda \geq 0$, where the solution value of λ is the Lagrange multiplier associated with the minimum wage constraint. Accordingly, provided that Assumption 1 holds with C^α_λ in place of C_λ , the comparative statics from Section 4 carry over, with C replaced by C^α ; otherwise, the analysis of Appendix B.3 applies in the same way.

6 Conclusions

Minimum wage legislation is at the forefront of public policy debates. This paper studied the effects of minimum wages in the presence of monopsony power, without restricting the contracts the monopsony can offer. The analysis provides a price-theoretic foundation for efficiency wages—involuntary unemployment can arise as an optimal exercise of monopsony power—and shows that an appropriately chosen minimum wage always eliminates such unemployment while increasing employment and workers' pay. It also yields a simple di-

⁵¹Loertscher and Muir (2026) provide an explicit model of imperfect competition between quantity-setting firms, in which the convexification of the procurement cost function or the concavification of the revenue function does not depend on the number of firms. In that framework, just like in a Cournot model, the effects of competition derive from firm-level differences in marginal cost and marginal revenue, respectively, which depend on the number of firms.

⁵²To see this, rewrite $C^\alpha(Q) = C(Q) - (1 - \alpha) \left(C(Q) - \int_0^Q W(y) dy \right) = C(Q) - (1 - \alpha) \int_0^Q y dW(y)$, where the second equality follows from integrating by parts. Since $\int_0^Q y dW(y)$ is increasing in Q , the function C^α , like $C_\lambda(Q) = C(Q) - \lambda W(Q)$, is obtained from C by subtracting an increasing function of Q , scaled by a nonnegative coefficient, with $1 - \alpha$ playing the role of λ . As α decreases, C^α therefore shifts down by more at higher quantities, which lowers both the value and the average slope of C^α over any interval. Consequently, the linear segments of $\underline{C}^\alpha$ flatten as α decreases, paralleling the flattening of the linear segments of $\underline{C}_\lambda$ as λ increases in Section 3.2.

agnostic: involuntary unemployment accompanied by wage dispersion is caused by market power, and a marginal increase in the minimum wage then generically increases employment; involuntary unemployment without wage dispersion is caused by the minimum wage itself, and a further increase then reduces employment.

We conclude with a short discussion of avenues for future research. First, beyond minimum wages, our model lends itself to analyzing the effects of prohibiting wage discrimination.⁵³ An open question is whether total employment, worker surplus and social surplus are larger with or without wage discrimination. Second, one could extend the baseline model to allow for vertically differentiated tasks, which gives rise to a price-theoretic model of multi-tasking. The effects of task-specific minimum wages remain unknown. Third, it would be natural to analyze the effects of introducing unemployment insurance into settings in which involuntary unemployment arises under *laissez-faire*. Finally, our analysis raises the question of what form optimal price regulation more generally takes when a monopoly or monopsony may engage in price discrimination.

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⁵³In concurrent policy debates, there is pressure for greater transparency concerning the wages paid by employers. Policies imposing wage transparency may have effects similar to those of prohibiting wage discrimination. For empirical evidence concerning the effects of requiring wage transparency and a comprehensive list of recent references, see Cullen and Pakzad-Hurson (2023).

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A Omitted proofs

A.1 Proof of Proposition 1

Proof. The first two claims follow from Myerson (1981) and the discussion preceding the proposition. For the properties of w_1^ℓ , observe that this function is obtained from W by replacing W with its chord over each ironing interval $(Q_1(m), Q_2(m))$. Each chord is strictly increasing and agrees with W at the endpoints of its interval, so w_1^ℓ inherits continuity and strict monotonicity from W . It remains to show that $w_1^\ell(Q) < W(Q)$ whenever C is not convex at Q . Let $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. By (2), the low wage $w_1^\ell(Q)$ is a strict convex combination of $W(Q_1(m))$ and $W(Q_2(m))$, so it lies strictly below the high wage $W(Q_2(m))$. The cost of the two-wage mechanism therefore exceeds $Qw_1^\ell(Q)$. That is,

$$Qw_1^\ell(Q) < Q_1(m)w_1^\ell(Q) + (Q - Q_1(m))W(Q_2(m)) = \underline{C}(Q) \leq C(Q) = QW(Q),$$

and hence $w_1^\ell(Q) < W(Q)$. □

A.2 Proof of Theorem 1

Proof. This proof has five parts. Part I shows that ex post implementation and imposing the minimum wage constraint on the lowest type are without loss of generality. It then rewrites the firm's problem as the choice of a probability measure over quantities, subject to a single minimum wage constraint. Part II solves the firm's problem for a fixed value λ of the Lagrange multiplier on the minimum wage constraint, which amounts to convexifying C_λ . Part III derives (8), establishes uniqueness of the optimal mechanism, and shows that the dual problem (7) attains the minimal cost. Part IV collects some useful geometric properties of $\underline{C}_\lambda$ and proves Lemma 1. Part V establishes the stated properties of $\underline{C}_R$.

Part I. By the revelation principle, restricting attention to incentive compatible direct mechanisms is without loss of generality. Such a mechanism specifies, for each reported cost c , an employment probability $x(c) \in [0, 1]$ and an expected total payment $t(c)$. Let $U(c) := t(c) - x(c)c$ denote the worker's equilibrium payoff. By standard arguments, incentive compatibility implies that x is decreasing, and the envelope theorem gives $U(c) = U(\bar{c}) + \int_c^{\bar{c}} x(y) dy$. Thus,

$$t(c) = x(c)c + \int_c^{\bar{c}} x(y) dy + U(\bar{c}), \quad (12)$$

where $U(\bar{c}) \geq 0$ by individual rationality. In Part III we show that $U(\bar{c}) = 0$ under the optimal mechanism.⁵⁴

The minimum wage requires each worker's realized transfer to be at least $\underline{w}$ times the amount of labor supplied, so $t(c) \geq \underline{w}x(c)$ for every type c . Conversely, if $t(c) \geq \underline{w}x(c)$, then paying the deterministic wage $t(c)/x(c)$ conditional on employment when $x(c) > 0$, and paying $t(c)$ conditional on unemployment when $x(c) = 0$, satisfies the minimum wage constraint and generates the same expected payments. Thus, restricting to ex post implementation is without loss of generality. Under ex post implementation, a type c with $x(c) > 0$ is paid the wage $w(c) := t(c)/x(c)$ conditional on employment, and a type with $x(c) = 0$ is paid

⁵⁴That $U(\bar{c}) = 0$ does not follow immediately from cost minimization. By (12), a rent $U(\bar{c}) > 0$ is a lump sum paid to every worker, which relaxes the minimum wage constraint at a marginal cost of one. In Part III we show that the optimal mechanism relaxes the constraint by raising the low wage instead, at a marginal cost of less than one, so the firm never sets $U(\bar{c}) > 0$.

$t(c) = U(\bar{c})$ conditional on unemployment. The firm's problem is

$$\begin{aligned} \underline{C}_R(Q, \underline{w}) &= \min_{x, U(\bar{c})} \int_{\underline{c}}^{\bar{c}} t(c) dG(c), \\ \text{s.t. } x &\text{ decreasing, (12), } U(\bar{c}) \geq 0, \text{ (MW), } \int_{\underline{c}}^{\bar{c}} x(c) dG(c) = Q. \end{aligned} \quad (13)$$

Next, we show that it suffices to impose (MW) on the lowest type $c = \underline{c}$. For $c > \underline{w}$, the minimum wage constraint is implied by individual rationality. So we focus instead on $c \in [\underline{c}, \underline{w}]$, and define $h(c) := \underline{w}x(c) - t(c)$, so that the minimum wage constraint becomes $h(c) \leq 0$. For $c < c'$ in $[\underline{c}, \underline{w}]$, (12) then yields

$$h(c') - h(c) = (\underline{w} - c')(x(c') - x(c)) + \int_c^{c'} x(y) dy - (c' - c)x(c) \leq 0,$$

because x is decreasing and $\underline{w} \geq c'$. Thus, h is decreasing on $[\underline{c}, \underline{w}]$, and $h(\underline{c}) \leq 0$ implies $h(c) \leq 0$ for all $c \in [\underline{c}, \underline{w}]$. Putting all of this together shows that it suffices to impose the minimum wage constraint in (13) on the lowest type $c = \underline{c}$. By (12), this constraint can be rewritten as

$$x(\underline{c})(\underline{w} - \underline{c}) \leq \int_{\underline{c}}^{\bar{c}} x(y) dy + U(\bar{c}). \quad (14)$$

We now rewrite the firm's problem in terms of quantiles of the cost distribution G via the change of variables $z = G(c)$ and $y = x \circ G^{-1}$, so that $W(z) = G^{-1}(z)$ and $C(z) = G^{-1}(z)z$. Since x is decreasing, so too is y . Without loss of generality, we assume that y is right-continuous with $y(1) = 0$.⁵⁵ Any such y induces a measure $\mu \in \Delta([0, 1])$ via $\mu((z, 1]) = y(z)$, where $\Delta([0, 1])$ denotes the set of probability measures on $[0, 1]$. Since $y(1) = 0$, integration by parts gives, for any absolutely continuous function f on $[0, 1]$,

$$\int_0^1 y(z) df(z) = \int_0^1 (f(q) - f(0)) d\mu(q). \quad (15)$$

⁵⁵A monotone function has at most countably many discontinuities, so replacing y by its right-continuous version and setting $y(1) = 0$ only changes y on a set of measure zero, which has no effect on the total procurement cost or total employment. The only constraint that is evaluated at a single point is (14), which involves $x(\underline{c})$. Since $y(0)$ is the right limit at 0 of the decreasing function $x \circ W$, we have $y(0) \leq x(\underline{c})$. Because $\underline{w} \geq \underline{c}$, this gives $y(0)(\underline{w} - \underline{c}) \leq x(\underline{c})(\underline{w} - \underline{c})$, so (14) continues to hold with $y(0)$ in place of $x(\underline{c})$. Conversely, suppose that (14) holds with $y(0)$ in place of $x(\underline{c})$. Consider the allocation rule $\tilde{x} := y \circ G$, which coincides with x except on a set of measure zero and satisfies $\tilde{x}(\underline{c}) = y(0)$. Together with the transfers given by (12), $\tilde{x}$ satisfies every constraint in (13), including (14), at the same cost as x . Thus, replacing $x(\underline{c})$ by $y(0)$ in (14) does not change the value of (13).

We apply (15) to each part of (13). For the objective, substituting (12) and changing the order of integration in the double integral, we have $\int_{\underline{c}}^{\bar{c}} t(c) dG(c) = \int_0^1 y(z)W(z) dz + \int_0^1 zy(z) dW(z) + U(\bar{c}) = \int_0^1 y(z) dC(z) + U(\bar{c}) = \int_0^1 C(q) d\mu(q) + U(\bar{c})$, where the last equality uses (15) with $f = C$ and $C(0) = 0$. For the quantity constraint, (15) with $f(z) = z$ gives $\int_{\underline{c}}^{\bar{c}} x(c) dG(c) = \int_0^1 y(z) dz = \int_0^1 q d\mu(q)$. For (14), (15) with $f = W$ gives $\int_{\underline{c}}^{\bar{c}} x(y) dy = \int_0^1 y(z) dW(z) = \int_0^1 (W(q) - W(0)) d\mu(q)$. Substituting this identity and $y(0) = \mu((0, 1])$ into (14) and using $W(0) = \underline{c}$, the constraint becomes

$$\int_{(0,1]} (W(q) - \underline{w}) d\mu(q) + U(\bar{c}) \geq 0. \quad (16)$$

The firm's problem is therefore

$$\begin{aligned} \underline{C}_R(Q, \underline{w}) = & \min_{\mu \in \Delta([0,1]), U(\bar{c}) \geq 0} \left\{ \int_0^1 C(q) d\mu(q) + U(\bar{c}) \right\} \\ \text{s.t. } & \int_0^1 q d\mu(q) = Q \text{ and } (16). \end{aligned} \quad (17)$$

The correspondence between y and μ is one-to-one. Any $\mu \in \Delta([0, 1])$ defines the allocation rule $y(z) = \mu((z, 1])$, which is decreasing and right-continuous with $y(1) = 0$. This map is linear and sends the Dirac measure δ_q to the indicator $\mathbf{1}_{[0,q]}$, so that $y(z) = \int_0^1 \mathbf{1}_{[0,q]}(z) d\mu(q)$ is the convex mixture of the market-clearing allocations $\{\mathbf{1}_{[0,q]} : q \in [0, 1]\}$ with weights taken from μ . Hence (17) is equivalent to (13). Throughout the rest of the proof, we use that δ_Q maps to $y = \mathbf{1}_{[0,Q]}$, which corresponds to hiring at the market-clearing wage $W(Q)$, and that $(1 - \beta(Q, q_1, q_2))\delta_{q_1} + \beta(Q, q_1, q_2)\delta_{q_2}$ maps to $y = (1 - \beta(Q, q_1, q_2))\mathbf{1}_{[0,q_1]} + \beta(Q, q_1, q_2)\mathbf{1}_{[0,q_2]}$, which corresponds to the two-wage mechanism parameterized by (Q, q_1, q_2) , whose low wage is $\int_0^1 W(q) d\mu(q) = w_1(Q, q_1, q_2)$.

Part II. Fix $\lambda \in [0, 1]$ and $Q \in [0, 1]$ and consider the problem

$$\min_{\mu \in \Delta([0,1])} \int_0^1 C_\lambda(q) d\mu(q) \quad \text{s.t.} \quad \int_0^1 q d\mu(q) = Q. \quad (18)$$

Substituting $C_\lambda = C - \lambda W$, the objective in (18) is, up to the constant $\lambda \underline{w}$, the Lagrangian of (17) with multiplier λ on the minimum wage constraint, evaluated at measures with $\mu(\{0\}) = 0$ and at $U(\bar{c}) = 0$. Part III shows that the optimal measure has these properties, and that the solution to (18) at the right value of λ solves (17). We establish four properties of (18) that Parts III and IV rely on: its value is $\underline{C}_\lambda(Q)$; its minimizer is unique under Assumption 1; the value and the minimizer are jointly continuous in (Q, λ) ; and the value

is differentiable in λ , with a derivative that we compute.

We first show that the value of (18) is $\underline{C}_\lambda(Q)$. For any feasible μ ,

$$\int_0^1 C_\lambda(q) d\mu(q) \geq \int_0^1 \underline{C}_\lambda(q) d\mu(q) \geq \underline{C}_\lambda(Q), \quad (19)$$

where the first inequality holds because $\underline{C}_\lambda$ lies weakly below C_λ , and the second is Jensen's inequality, since $\underline{C}_\lambda$ is convex and μ has mean Q . This bound is attained. If $\underline{C}_\lambda(Q) = C_\lambda(Q)$, it is attained by δ_Q . Otherwise, Q lies in an ironing interval of C_λ , which we denote by $(q_1^\lambda(Q), q_2^\lambda(Q))$, suppressing the argument Q whenever it is clear from the context. On this interval, $\underline{C}_\lambda$ is the chord of C_λ , and the bound is attained by the two-point measure $(1 - \beta(Q, q_1^\lambda, q_2^\lambda))\delta_{q_1^\lambda} + \beta(Q, q_1^\lambda, q_2^\lambda)\delta_{q_2^\lambda}$.

We next show that the minimizer that solves (18) is unique under Assumption 1. Let μ be a minimizer, so that both inequalities in (19) hold with equality. Equality in the second inequality means that $\underline{C}_\lambda$ coincides with one of its supporting lines at Q on the support of μ . Equality in the first means that $C_\lambda = \underline{C}_\lambda$ on the support of μ . Thus, μ is supported on the set of points at which this supporting line, which lies weakly below C_λ , coincides with C_λ . By Assumption 1, this set contains at most two points. If it contains only one, then $\mu = \delta_Q$. If it contains two, then the quantity constraint pins down the weights, and μ is the two-point measure with mean Q on these points.⁵⁶ In either case the minimizer is unique. We denote it by $\mu_{Q,\lambda}$ and let

$$w_1^\lambda(Q) := \int_0^1 W(q) d\mu_{Q,\lambda}(q) \quad (20)$$

denote the corresponding low wage, so that $w_1^\lambda(Q) = W(Q)$ if $\underline{C}_\lambda(Q) = C_\lambda(Q)$ and $w_1^\lambda(Q) = w_1(Q, q_1^\lambda, q_2^\lambda)$ otherwise.

The value $\underline{C}_\lambda(Q)$ is jointly continuous in (Q, λ) . For fixed Q and each feasible μ , the objective in (18) equals $\int_0^1 C(q) d\mu(q) - \lambda \int_0^1 W(q) d\mu(q)$, which is affine in λ with slope in $[-W(1), -W(0)]$. As the pointwise minimum of these functions, $\underline{C}_\lambda(Q)$ is concave in λ and satisfies $|\underline{C}_\lambda(Q) - \underline{C}_{\lambda'}(Q)| \leq W(1)|\lambda - \lambda'|$. Since this bound does not depend on Q , and $\underline{C}_\lambda$ is continuous in Q for each λ , joint continuity follows. The minimizer $\mu_{Q,\lambda}$ of (18) is also jointly continuous in the weak topology. Let $\{(Q_n, \lambda_n)\}_{n \in \mathbb{N}}$ converge to (Q, λ) . By compactness of $\Delta([0, 1])$, every subsequence of $\{\mu_{Q_n, \lambda_n}\}_{n \in \mathbb{N}}$ has a further subsequence converging weakly to some ν with mean Q and $\int_0^1 C_\lambda(q) d\nu(q) = \lim_{n \rightarrow \infty} \underline{C}_{\lambda_n}(Q_n) = \underline{C}_\lambda(Q)$, so $\nu = \mu_{Q,\lambda}$ by

⁵⁶If $\underline{C}_\lambda(Q) = C_\lambda(Q)$, then Q itself is a contact point of the supporting line, so the minimizer is δ_Q whether or not there is a second contact point. If $\underline{C}_\lambda(Q) < C_\lambda(Q)$, then the supporting line is the chord of C_λ over $(q_1^\lambda, q_2^\lambda)$, and its contact points are q_1^λ and q_2^λ .

uniqueness and the whole sequence converges to $\mu_{Q,\lambda}$. By (20), $w_1^\lambda(Q)$ is therefore jointly continuous as well.

We now compute the derivative of $\underline{C}_\lambda(Q)$ with respect to λ . Fix λ and $h > 0$. Since $C_{\lambda+h} = C_\lambda - hW$, evaluating the objective at $\lambda + h$ under $\mu_{Q,\lambda}$ gives $\underline{C}_{\lambda+h}(Q) \leq \underline{C}_\lambda(Q) - h w_1^\lambda(Q)$, and evaluating it at λ under $\mu_{Q,\lambda+h}$ gives $\underline{C}_\lambda(Q) \leq \underline{C}_{\lambda+h}(Q) + h w_1^{\lambda+h}(Q)$. Hence $-w_1^{\lambda+h}(Q) \leq (\underline{C}_{\lambda+h}(Q) - \underline{C}_\lambda(Q)) / h \leq -w_1^\lambda(Q)$. Since $w_1^\lambda(Q)$ is continuous in λ , taking $h \downarrow 0$ shows that the right derivative of $\underline{C}_\lambda(Q)$ with respect to λ equals $-w_1^\lambda(Q)$, and the same argument with $h < 0$ gives the left derivative. Thus,

$$\frac{\partial \underline{C}_\lambda(Q)}{\partial \lambda} = -w_1^\lambda(Q) \quad (21)$$

for every $\lambda \in [0, 1]$, with one-sided derivatives at the endpoints. Concavity in λ then implies that $w_1^\lambda(Q)$ is increasing in λ . Finally, $w_1^\lambda(\cdot)$ is strictly increasing on $[0, 1]$: it equals W outside the ironing intervals of C_λ and at their endpoints, and on each ironing interval it is affine with slope $(W(q_2^\lambda) - W(q_1^\lambda)) / (q_2^\lambda - q_1^\lambda) > 0$.

Part III. We now derive (8) and establish uniqueness of the optimal mechanism. We first introduce an identity relating the chords of C_λ and W over a common interval. We then solve (17) in each of the three cases of Section 3.2, of which only Case 2 requires work. Finally, we show that the dual problem (7) attains the same value.

Fix $\lambda \in [0, 1]$ and $Q \in (0, 1]$ such that Q lies in an ironing interval $(q_1^\lambda, q_2^\lambda)$ of C_λ , and let L_λ^W and L_λ^C denote the chords of W and of C_λ over this interval. That is, for all $q \in [0, 1]$,

$$L_\lambda^W(q) = \frac{q_2^\lambda - q}{q_2^\lambda - q_1^\lambda} W(q_1^\lambda) + \frac{q - q_1^\lambda}{q_2^\lambda - q_1^\lambda} W(q_2^\lambda), \quad L_\lambda^C(q) = \frac{q_2^\lambda - q}{q_2^\lambda - q_1^\lambda} C_\lambda(q_1^\lambda) + \frac{q - q_1^\lambda}{q_2^\lambda - q_1^\lambda} C_\lambda(q_2^\lambda).$$

Note that $w_1^\lambda(Q) = L_\lambda^W(Q)$ and that $\underline{C}_\lambda = L_\lambda^C$ on $[q_1^\lambda, q_2^\lambda]$, so that $C_\lambda \geq L_\lambda^C$ there. These chords satisfy, for all $q \in [0, 1]$,⁵⁷

$$C_\lambda(q) - L_\lambda^C(q) = (q - \lambda) (W(q) - L_\lambda^W(q)) - \frac{(q - q_1^\lambda)(q_2^\lambda - q)}{q_2^\lambda - q_1^\lambda} (W(q_2^\lambda) - W(q_1^\lambda)). \quad (22)$$

We refer to (22) as the chord identity. It has three consequences that we use repeatedly. First,

⁵⁷To see this, substitute $C_\lambda(q) = (q - \lambda)W(q)$ into $C_\lambda(q) - L_\lambda^C(q)$ and add and subtract $(q - \lambda)L_\lambda^W(q)$. The first term on the right-hand side of (22) is then immediate. The second is the remainder $(q - \lambda)L_\lambda^W(q) - L_\lambda^C(q)$, which is a quadratic in q . It vanishes at q_1^λ and at q_2^λ , since L_λ^W agrees with W and L_λ^C agrees with C_λ at these points, and its leading coefficient is the slope $(W(q_2^\lambda) - W(q_1^\lambda)) / (q_2^\lambda - q_1^\lambda)$ of L_λ^W , because L_λ^C is affine. Since a quadratic is determined by its roots and its leading coefficient, the remainder equals $\frac{W(q_2^\lambda) - W(q_1^\lambda)}{q_2^\lambda - q_1^\lambda} (q - q_1^\lambda)(q - q_2^\lambda)$, which is the second term in (22).

evaluating (22) at $q = \lambda$ shows that $\lambda \notin (q_1^\lambda, q_2^\lambda)$, since otherwise the right-hand side would be strictly negative. Thus, λ never lies inside an ironing interval of C_λ , and if $Q > \lambda$, then $\lambda \leq q_1^\lambda$. Second, for any $\tilde{Q} \in (q_1^\lambda, q_2^\lambda)$ with $\tilde{Q} > \lambda$, the terms $\tilde{Q} - \lambda > 0$ and $C_\lambda(\tilde{Q}) \geq L_\lambda^C(\tilde{Q})$ force $W(\tilde{Q}) > L_\lambda^W(\tilde{Q})$ in (22). Third, since $w_1^\lambda(Q) = L_\lambda^W(Q)$ if Q lies in an ironing interval and $w_1^\lambda(Q) = W(Q)$ otherwise, it follows that, for any $Q > \lambda$, $w_1^\lambda(Q) \leq W(Q)$ with strict inequality if and only if Q lies in an ironing interval of C_λ . In particular, $w_1^Q(Q) = W(Q)$ for every $Q \in (0, 1]$, since Q lies outside every ironing interval of C_Q .

We now solve (17) case by case. Since $C_0 = C$, the measure $\mu_{Q,0}$ is the laissez-faire optimum of Proposition 1 and $w_1^0(Q) = w_1^\ell(Q)$. The three cases of Section 3.2 can therefore be stated in terms of the minimum wage as $\underline{w} \geq W(Q)$ (Case 1), $w_1^\ell(Q) < \underline{w} < W(Q)$ (Case 2) and $\underline{w} \leq w_1^\ell(Q)$ (Case 3). Throughout, $(\nu, U(\bar{c}))$ denotes an arbitrary feasible pair in (17). We treat Case 3 before Case 2, since it follows directly from Part II.

Case 1: $Q \leq S(\underline{w})$. Under the minimum wage $\underline{w}$, every worker is paid at least $\underline{w}$ per unit of labor supplied, so hiring Q workers costs at least $\underline{w}Q$. The uniform-wage mechanism that pays $\underline{w}$ and rations the $S(\underline{w})$ workers with cost at most $\underline{w}$ into Q openings costs exactly $\underline{w}Q$, and is therefore optimal. It is the only optimal mechanism. The cost of a mechanism exceeds $\underline{w}Q$ by $\int_{\underline{c}}^{\bar{c}} (t(c) - \underline{w}x(c)) dG(c)$, where the integrand is non-negative by the minimum wage constraint. If the mechanism is optimal, this integral is zero, so $t(c) = \underline{w}x(c)$ for almost every c . Every employed worker is therefore paid exactly $\underline{w}$ and the unemployed are paid nothing. By individual rationality, no worker with cost above $\underline{w}$ is employed, and (12) then forces $U(\bar{c}) = 0$ and x constant and equal to $Q/S(\underline{w})$ below $\underline{w}$.

Case 3: $Q \geq (w_1^\ell)^{-1}(\underline{w})$. Since $Q \geq (w_1^\ell)^{-1}(\underline{w})$ implies that $Q \geq S(\underline{w})$, and $Q = S(\underline{w})$ is covered by Case 1, we suppose that $Q > S(\underline{w})$. Consider the laissez-faire mechanism $\mu_{Q,0}$ of Proposition 1 with $U(\bar{c}) = 0$. Its lowest-cost worker is employed with certainty and paid the low wage $w_1^\ell(Q) \geq \underline{w}$, so it satisfies the minimum wage constraint. Its cost $\underline{C}(Q)$ is, by Part II, the lowest cost of procuring Q workers under any mechanism, so it is optimal. It is the only optimal mechanism, since any pair $(\nu, U(\bar{c}))$ with cost $\underline{C}(Q)$ must have $U(\bar{c}) = 0$ and ν a minimizer of (18) at $\lambda = 0$, which is unique.

Case 2: $Q \in (S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}))$. We first identify the relevant multiplier. Consider the mapping $\lambda \mapsto w_1^\lambda(Q)$ on $[0, Q]$. This mapping is continuous and increasing by Part II, with $w_1^0(Q) = w_1^\ell(Q) < \underline{w}$ and $w_1^Q(Q) = W(Q) > \underline{w}$. Hence there exists $\lambda \in (0, Q)$ with

$$w_1^\lambda(Q) = \underline{w}. \quad (23)$$

Fix any such λ . Since $w_1^\lambda(Q) < W(Q)$ and $\lambda < Q$, the chord identity implies that Q lies in an ironing interval $(q_1^\lambda, q_2^\lambda)$ of C_λ and that $q_1^\lambda \geq \lambda > 0$. Thus, $\mu_{Q,\lambda}$ is the two-point measure

on $\{q_1^\lambda, q_2^\lambda\}$ with mean Q , it has no mass at 0, and its constraint (16) holds with equality by (23). The pair $(\mu_{Q,\lambda}, 0)$ is therefore feasible, and its cost is

$$\int_0^1 C(q) d\mu_{Q,\lambda}(q) = \int_0^1 C_\lambda(q) d\mu_{Q,\lambda}(q) + \lambda \int_0^1 W(q) d\mu_{Q,\lambda}(q) = \underline{C}_\lambda(Q) + \lambda \underline{w}. \quad (24)$$

We now show that no feasible pair does better. Fix a feasible pair $(\nu, U(\bar{c}))$. We bound its cost from below in two steps: first by pricing the minimum wage constraint at λ , and then by replacing C_λ with the chord L_λ^C , which lies weakly below C_λ and satisfies $L_\lambda^C(Q) = \underline{C}_\lambda(Q)$. For the first step, substituting $C = C_\lambda + \lambda W$ into the cost gives $\int_0^1 C d\nu + U(\bar{c}) = \int_0^1 C_\lambda d\nu + \lambda \int_0^1 W d\nu + U(\bar{c})$. By (16), $\lambda \int_{(0,1]} W d\nu \geq \lambda \underline{w} \nu((0,1]) - \lambda U(\bar{c})$. Hence

$$\int_0^1 C d\nu + U(\bar{c}) \geq \int_0^1 C_\lambda d\nu + \lambda \underline{w} + (1 - \lambda)U(\bar{c}) - \lambda(\underline{w} - W(0))\nu(\{0\}). \quad (25)$$

For the second step, since $C_\lambda \geq L_\lambda^C$ on $(0,1]$ and $\int_0^1 L_\lambda^C d\nu = L_\lambda^C(Q) = \underline{C}_\lambda(Q)$,

$$\int_0^1 C_\lambda d\nu \geq \int_{(0,1]} L_\lambda^C d\nu + C_\lambda(0)\nu(\{0\}) = \underline{C}_\lambda(Q) + (C_\lambda(0) - L_\lambda^C(0))\nu(\{0\}).$$

Substituting this into (25) and using $C_\lambda(0) = -\lambda W(0)$ yields

$$\int_0^1 C d\nu + U(\bar{c}) \geq \underline{C}_\lambda(Q) + \lambda \underline{w} + (1 - \lambda)U(\bar{c}) - (L_\lambda^C(0) + \lambda \underline{w})\nu(\{0\}). \quad (26)$$

The last two terms on the right-hand side are non-negative. The first because $\lambda < 1$. The second because the chord identity at $q = 0$, together with $\underline{w} = L_\lambda^W(Q)$, gives

$$L_\lambda^C(0) + \lambda \underline{w} = (\lambda Q - q_1^\lambda q_2^\lambda) \frac{W(q_2^\lambda) - W(q_1^\lambda)}{q_2^\lambda - q_1^\lambda} < 0,$$

where the inequality uses $\lambda \leq q_1^\lambda$ and $Q < q_2^\lambda$. Thus, the cost of every feasible pair is at least $\underline{C}_\lambda(Q) + \lambda \underline{w}$, which is the cost of $(\mu_{Q,\lambda}, 0)$ by (24). Moreover, equality in (26) requires $U(\bar{c}) = 0$, $\nu(\{0\}) = 0$, and $\int_0^1 C_\lambda d\nu = \underline{C}_\lambda(Q)$, so that ν solves (18) and equals $\mu_{Q,\lambda}$ by Part II. Hence $\underline{C}_R(Q, \underline{w}) = \underline{C}_\lambda(Q) + \lambda \underline{w}$, and the optimal pair is unique. It is the two-wage mechanism parameterized by $(Q, q_1^\lambda, q_2^\lambda)$, which is non-degenerate because $q_1^\lambda > 0$ and $q_2^\lambda > Q$. Economically, a positive rent $U(\bar{c})$ is a lump sum paid to every worker, which relaxes the minimum wage constraint at a marginal cost of one, whereas the optimal mechanism relaxes it by raising the low wage at a marginal cost of $\lambda \leq q_1^\lambda < 1$; this is the term $(1 - \lambda)U(\bar{c})$ in (26).

It remains to show that the value in Case 2 agrees with $\mathcal{D}^*(Q, \underline{w})$, that is, that the dual problem (7) attains the minimal cost. The inner minimization in (7) is over two-point measures with mean Q , and the Jensen argument of Part II applies to it for every $\lambda \geq 0$: the bound (19) holds and is attained by a two-point measure, so the inner minimum equals $\underline{C}_\lambda(Q) + \lambda \underline{w}$. Hence $\mathcal{D}^*(Q, \underline{w}) = \max_{\lambda \geq 0} \{\underline{C}_\lambda(Q) + \lambda \underline{w}\}$. The dual objective $\underline{C}_\lambda(Q) + \lambda \underline{w}$ is concave in λ , and on $[0, 1]$ it is differentiable with derivative $\underline{w} - w_1^\lambda(Q)$ by (21). Suppose that $w_1^\ell(Q) \leq \underline{w} < W(Q)$. Then this derivative is decreasing and continuous on $[0, 1]$, non-negative at $\lambda = 0$, and strictly negative on $[Q, 1]$, because $w_1^\lambda(Q) \geq w_1^Q(Q) = W(Q) > \underline{w}$ there. By concavity, the dual objective is then decreasing on $[Q, \infty)$ as well. Thus, its maximizers are exactly the solutions of (23), which form a closed interval in $[0, Q)$, contained in $(0, Q)$ in Case 2. In Case 2, therefore, $\mathcal{D}^*(Q, \underline{w}) = \underline{C}_\lambda(Q) + \lambda \underline{w} = \underline{C}_R(Q, \underline{w})$ at any solution, all of which induce the same measure $\mu_{Q, \lambda}$ by uniqueness of the optimal pair. This establishes (8) and justifies the restriction to $\lambda \in [0, 1]$ in Assumption 1, since every solution value satisfies $\lambda < Q \leq 1$. The same argument shows that

$$\underline{C}_R(Q, \underline{w}) = \max_{\lambda \in [0, Q]} \{\underline{C}_\lambda(Q) + \lambda \underline{w}\} \quad (27)$$

holds for every $(Q, \underline{w}) \in [0, 1] \times [W(0), W(1)]$. In Case 1, the derivative is non-negative on $[0, Q)$ because $w_1^\lambda(Q) \leq W(Q) \leq \underline{w}$ there, so the maximum is attained at $\lambda = Q$ and equals $\underline{w}Q$, since $\underline{C}_Q(Q) = C_Q(Q) = 0$. In Case 3, the derivative at $\lambda = 0$ is non-positive, so the maximum is attained at $\lambda = 0$ and equals $\underline{C}(Q)$. Part V uses (27) repeatedly.

Part IV. We first fix notation. For $Q \in (0, 1]$, let

$$\Lambda(Q) := \{\lambda \in [0, 1] : \underline{C}_\lambda(Q) < C_\lambda(Q)\}$$

denote the set of λ at which Q lies in the interior of an ironing interval of C_λ . This set is open as a subset of $[0, 1]$, since $\lambda \mapsto \underline{C}_\lambda(Q) - C_\lambda(Q)$ is continuous by Part II, and it may be empty. For $\lambda \in \Lambda(Q)$, the relevant ironing interval is $(q_1^\lambda(Q), q_2^\lambda(Q))$ as defined in Part II, and L_λ^W and L_λ^C denote the chords of W and C_λ over this interval, as in Part III, so that $w_1^\lambda(Q) = L_\lambda^W(Q)$. For every $\tilde{Q} \in (0, 1]$, we let $\Delta_\lambda(\tilde{Q}) := C_\lambda(\tilde{Q}) - \underline{C}_\lambda(\tilde{Q}) \geq 0$ denote the convexification gap. Finally, we let

$$s_\lambda(Q) := \frac{\partial_- \underline{C}_\lambda(Q)}{\partial Q}$$

denote the left derivative of $\underline{C}_\lambda$ with respect to Q , which exists and is increasing in Q because $\underline{C}_\lambda$ is convex.

The following lemma collects a number of geometric properties that the remaining proofs draw on. Part (iii) of the lemma restricts attention to pairs at which the firm optimally uses a non-degenerate two-wage mechanism whose low wage equals $\underline{w}$, that is, $(Q, \underline{w})$ such that $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$. The remaining parts hold for arbitrary quantities and multipliers.

Lemma A.1. *Fix $Q \in (0, 1]$. Then:*

- (i) $w_1^\lambda(Q)$ is increasing in $\lambda \in [0, 1]$, and $w_1^\lambda(\cdot)$ is continuous and strictly increasing on $[0, 1]$ for every $\lambda \in [0, 1]$;
- (ii) $\underline{C}_\lambda(Q)$ is concave and Lipschitz in λ , with $\partial \underline{C}_\lambda(Q)/\partial \lambda = -w_1^\lambda(Q)$ for every $\lambda \in [0, 1]$;
- (iii) if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$, then every solution value λ of the Lagrange multiplier in (7) at $(Q, \underline{w})$ satisfies $w_1^\lambda(Q) = \underline{w}$, $\lambda \in \Lambda(Q)$, $\lambda \leq q_1^\lambda$, and, for all $\tilde{Q} \in (q_1^\lambda, q_2^\lambda)$,

$$W(\tilde{Q}) > L_\lambda^W(\tilde{Q}); \quad (28)$$

- (iv) for every $\tilde{Q} \in (0, 1]$, the gap $\Delta_\lambda(\tilde{Q})$ is decreasing in λ on $[0, \tilde{Q})$, so that $[0, \lambda'] \subseteq \Lambda(\tilde{Q})$ whenever $\lambda' \in \Lambda(\tilde{Q})$ and $\lambda' < \tilde{Q}$; consequently, if $\lambda, \lambda' \in \Lambda(Q)$ with $\lambda < \lambda' \leq q_1^{\lambda'}$, then $q_1^\lambda \leq q_1^{\lambda'}$ and $q_2^\lambda \leq q_2^{\lambda'}$; and

- (v) $s_\lambda(Q)$ is decreasing in λ on $[0, 1]$, and if $\lambda < \lambda' < Q$ with $\lambda' \in \Lambda(Q)$, then

$$s_{\lambda'}(Q) - s_\lambda(Q) = - \int_\lambda^{\lambda'} \frac{W(q_2^{\tilde{\lambda}}(Q)) - W(q_1^{\tilde{\lambda}}(Q))}{q_2^{\tilde{\lambda}}(Q) - q_1^{\tilde{\lambda}}(Q)} d\tilde{\lambda} < 0. \quad (29)$$

Consequently, $s_\lambda(Q)$ is continuously differentiable in λ on $\Lambda(Q) \cap [0, Q)$ with

$$\frac{ds_\lambda(Q)}{d\lambda} = - \frac{W(q_2^\lambda(Q)) - W(q_1^\lambda(Q))}{q_2^\lambda(Q) - q_1^\lambda(Q)} < 0, \quad (30)$$

and $s_{\lambda'}(Q) < s_\lambda(Q)$ whenever $\lambda \in \Lambda(Q) \cap [0, Q)$ and $\lambda' \in (\lambda, 1]$.

Proof of Lemma A.1. Fix $Q \in (0, 1]$. Parts (i) and (ii) were established in Part II. For (iii), suppose $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$, so that $w_1^\ell(Q) \leq \underline{w} < W(Q)$, and let λ be a solution value at $(Q, \underline{w})$. By Part III, λ solves (23) and $\lambda < Q$. Since $w_1^\lambda(Q) = \underline{w} < W(Q)$ and $\lambda < Q$, the chord identity implies that Q lies in an ironing interval $(q_1^\lambda, q_2^\lambda)$ of C_λ , that $\lambda \leq q_1^\lambda$, and that $W(\tilde{Q}) > L_\lambda^W(\tilde{Q})$ for every $\tilde{Q} \in (q_1^\lambda, q_2^\lambda)$, since $\tilde{Q} > q_1^\lambda \geq \lambda$. In particular, $\lambda \in \Lambda(Q)$. This establishes (iii).

For (iv), fix $\tilde{Q} \in (0, 1]$. Since $C_\lambda(\tilde{Q}) = C(\tilde{Q}) - \lambda W(\tilde{Q})$ is affine in λ and $\underline{C}_\lambda(\tilde{Q})$ is concave in λ by (ii), the gap $\Delta_\lambda(\tilde{Q})$ is convex in λ , with $\partial \Delta_\lambda(\tilde{Q}) / \partial \lambda = w_1^\lambda(\tilde{Q}) - W(\tilde{Q})$ by (21). By the chord identity, $w_1^\lambda(\tilde{Q}) \leq W(\tilde{Q})$ whenever $\lambda < \tilde{Q}$, so this derivative is non-positive there and $\Delta_\lambda(\tilde{Q})$ is decreasing in λ on $[0, \tilde{Q}]$. In particular, if $\lambda' \in \Lambda(\tilde{Q})$ and $\lambda' < \tilde{Q}$, then $\Delta_\lambda(\tilde{Q}) \geq \Delta_{\lambda'}(\tilde{Q}) > 0$ for all $\lambda \leq \lambda'$, so $[0, \lambda'] \subseteq \Lambda(\tilde{Q})$.

Now let $\lambda, \lambda' \in \Lambda(Q)$ with $\lambda < \lambda' \leq q_1^{\lambda'}$. Since $\lambda' \in \Lambda(Q)$, we have $q_1^{\lambda'} < Q$ and hence $\lambda < Q$; as Q lies in an ironing interval of C_λ , the chord identity gives $\lambda \leq q_1^\lambda$. Suppose, seeking a contradiction, that $q_1^{\lambda'} < q_1^\lambda$. Then $\Delta_\lambda(q_1^\lambda) = 0$, since q_1^λ is an endpoint of an ironing interval of C_λ . On the other hand, $q_1^{\lambda'} < q_1^\lambda < Q < q_2^{\lambda'}$, so q_1^λ lies in the interior of an ironing interval of $C_{\lambda'}$ and $\Delta_{\lambda'}(q_1^\lambda) > 0$. But $\lambda < \lambda' \leq q_1^{\lambda'} < q_1^\lambda$, so both multipliers lie in $[0, q_1^\lambda]$, on which the gap at q_1^λ is decreasing. This is a contradiction. Now suppose that $q_2^{\lambda'} > q_2^\lambda$. Then $\Delta_\lambda(q_2^\lambda) = 0$, since q_2^λ is an endpoint of an ironing interval of C_λ . On the other hand, $q_1^{\lambda'} < Q < q_2^\lambda < q_2^{\lambda'}$, so q_2^λ lies in the interior of an ironing interval of $C_{\lambda'}$ and $\Delta_{\lambda'}(q_2^\lambda) > 0$. But $\lambda \leq q_1^\lambda < Q < q_2^\lambda$ and $\lambda' \leq q_1^{\lambda'} < Q < q_2^\lambda$, so both multipliers lie in $[0, q_2^\lambda]$, on which the gap at q_2^λ is decreasing. This is again a contradiction. This establishes (iv).

For (v), fix $\lambda < \lambda'$ in $[0, 1]$. Integrating (21) gives $\underline{C}_{\lambda'}(Q) - \underline{C}_\lambda(Q) = - \int_\lambda^{\lambda'} w_1^{\tilde{\lambda}}(Q) d\tilde{\lambda}$. Applying this at Q and at $Q - h$ for $h > 0$ and differencing, we have

$$\frac{1}{h} \{ [\underline{C}_{\lambda'}(Q) - \underline{C}_{\lambda'}(Q - h)] - [\underline{C}_\lambda(Q) - \underline{C}_\lambda(Q - h)] \} = - \int_\lambda^{\lambda'} \frac{w_1^{\tilde{\lambda}}(Q) - w_1^{\tilde{\lambda}}(Q - h)}{h} d\tilde{\lambda}. \quad (31)$$

The integrand is non-negative by (i), so letting $h \downarrow 0$ gives $s_{\lambda'}(Q) \leq s_\lambda(Q)$. Now suppose that $\lambda' < Q$ and $\lambda' \in \Lambda(Q)$. By (iv), $[\lambda, \lambda'] \subseteq \Lambda(Q)$, and every $\tilde{\lambda} \in [\lambda, \lambda']$ satisfies $\tilde{\lambda} < Q$ and hence $\tilde{\lambda} \leq q_1^{\tilde{\lambda}}$ by the chord identity, so the ironing intervals are nested by (iv) and $q_1^{\tilde{\lambda}} \leq q_1^{\lambda'} < Q$. For $h < Q - q_1^{\lambda'}$, the low wage $w_1^{\tilde{\lambda}}(\cdot)$ is therefore affine on $[Q - h, Q]$ with slope $(W(q_2^{\tilde{\lambda}}) - W(q_1^{\tilde{\lambda}})) / (q_2^{\tilde{\lambda}} - q_1^{\tilde{\lambda}}) > 0$ for every $\tilde{\lambda} \in [\lambda, \lambda']$, and (31) becomes (29) in the limit. On $\Lambda(Q) \cap [0, Q]$, the endpoints $q_1^\lambda(Q)$ and $q_2^\lambda(Q)$ are the support points of $\mu_{Q,\lambda}$ and therefore vary continuously with λ by Part II, so the integrand in (29) is continuous and (30) follows. Finally, if $\lambda \in \Lambda(Q) \cap [0, Q]$ and $\lambda' > \lambda$, pick $\lambda'' \in (\lambda, \lambda')$ with $\lambda'' \in \Lambda(Q)$ and $\lambda'' < Q$, which exists because $\Lambda(Q)$ is open as a subset of $[0, 1]$; then $s_{\lambda'}(Q) \leq s_{\lambda''}(Q) < s_\lambda(Q)$. $\square$

We conclude this part by proving the comparative statics of the solution value of the Lagrange multiplier in (7) stated in the body of the paper as Lemma 1. Lemma 1 asserts that the solution value $\lambda^*(Q, \underline{w})$ is strictly decreasing in Q and strictly increasing in $\underline{w}$ on the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. As noted in footnote 33, these comparative statics hold for every solution: every solution value at a larger quantity lies strictly below every solution value at

a smaller one, and likewise every solution value at a higher minimum wage lies strictly above every solution value at a lower one.

Proof of Lemma 1. Fix any $(Q, \underline{w})$ with $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}) \right]$. By Part III, the set of solutions to the outer maximization in (7) at $(Q, \underline{w})$ is a closed interval contained in $[0, Q)$, and every solution satisfies $w_1^\lambda(Q) = \underline{w}$. Both comparative statics follow from this together with Lemma A.1(i). First, fix Q and take $\underline{w}' > \underline{w}$ from the region, with solution values $\lambda^*(Q, \underline{w}')$ and $\lambda^*(Q, \underline{w})$. If $\lambda^*(Q, \underline{w}') \leq \lambda^*(Q, \underline{w})$, then since $w_1^\lambda(Q)$ is increasing in λ ,

$$\underline{w}' = w_1^{\lambda^*(Q, \underline{w}')} (Q) \leq w_1^{\lambda^*(Q, \underline{w})} (Q) = \underline{w},$$

a contradiction. Second, fix $\underline{w}$ and take $Q' > Q$ from the region. If $\lambda^*(Q', \underline{w}) \geq \lambda^*(Q, \underline{w})$, then

$$\underline{w} = w_1^{\lambda^*(Q', \underline{w})} (Q') \geq w_1^{\lambda^*(Q, \underline{w})} (Q') > w_1^{\lambda^*(Q, \underline{w})} (Q) = \underline{w},$$

a contradiction, where the two inequalities respectively use that $w_1^\lambda(Q)$ is increasing in λ and that $w_1^\lambda(\cdot)$ is strictly increasing. Since the solution values in both arguments were arbitrary, the conclusions hold in the sense stated in footnote 33. $\square$

Part V. We now establish the properties of $\underline{C}_R$ and of the marginal cost envelope $\underline{C}'_R$ stated in the theorem. The properties of $\underline{C}_R$ follow directly from (17) and (27). For $\underline{C}'_R$, we first identify its one-sided derivatives with respect to Q on the two-wage region with the slopes $s_\lambda(Q)$ at the extreme solution values of the multiplier; this is Lemma A.2, which the remaining proofs also use. We then combine this new lemma with Lemma A.1 and Lemma 1 to obtain the properties of $\underline{C}'_R$.

Fix $\underline{w}$. In (17), the objective and both constraints are linear in $(\mu, U(\bar{c}))$, so a convex combination of pairs that are feasible at quantities Q and Q' is feasible at the corresponding convex combination of quantities, with the corresponding convex combination of costs. Hence $\underline{C}_R(\cdot, \underline{w})$ is convex on $[0, 1]$. It is also increasing, since it is convex with right derivative $\underline{w} \geq 0$ at $Q = 0$. To see that the right derivative at $Q = 0$ is $\underline{w}$, we use (8). If $S(\underline{w}) > 0$, then $\underline{C}_R(Q, \underline{w}) = \underline{w}Q$ for $Q \leq S(\underline{w})$. If $S(\underline{w}) = 0$, then $\underline{w} = W(0)$ and $\underline{C}_R(\cdot, \underline{w}) = \underline{C}$, whose right derivative at 0 is $W(0)$, since $W(0)Q \leq \underline{C}(Q) \leq C(Q) = QW(Q)$ for all Q and W is continuous. Now fix Q . By (27), $\underline{C}_R(Q, \cdot)$ is the pointwise maximum of affine functions of $\underline{w}$ with slopes $\lambda \in [0, Q]$, so it is convex, increasing and Q -Lipschitz on $[W(0), W(1)]$. Finally, substituting $\lambda = Qz$ in (27) gives $\underline{C}_R(Q, \underline{w}) = \max_{z \in [0, 1]} \{ \underline{C}_{Qz}(Q) + Qz\underline{w} \}$, the maximum

over a fixed compact set of a function that is jointly continuous in $(z, Q, \underline{w})$ by Part II. Hence $\underline{C}_R$ is jointly continuous on $[0, 1] \times [W(0), W(1)]$.

We now turn to the marginal cost envelope. Recall that $\underline{C}'_R$ denotes the left derivative of $\underline{C}_R(\cdot, \underline{w})$ with respect to Q , which exists on $(0, 1]$ because $\underline{C}_R(\cdot, \underline{w})$ is convex, and let $\underline{C}'_{R+}$ denote the right derivative, which exists on $[0, 1)$. The following lemma expresses both in terms of the slopes $s_\lambda(Q)$ of Part IV.

Lemma A.2. *Fix $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$. Let $\lambda^{\min}$ and $\lambda^{\max}$ denote the smallest and largest solution values of the Lagrange multiplier in (7) at $(Q, \underline{w})$. Then $\underline{C}'_R(Q, \underline{w}) = s_{\lambda^{\max}}(Q)$ and $\underline{C}'_{R+}(Q, \underline{w}) = s_{\lambda^{\min}}(Q)$.*

Proof of Lemma A.2. For every $q > S(\underline{w})$, the maximizers of $\underline{C}_\lambda(q) + \lambda \underline{w}$ over $\lambda \geq 0$ lie in $[0, q)$ by Part III, so (27) gives $\underline{C}_R(q, \underline{w}) = \max_{\lambda \in [0, 1]} \{\underline{C}_\lambda(q) + \lambda \underline{w}\}$. Since $Q > S(\underline{w})$, this holds for every q in a neighborhood of Q . By Part III and Lemma A.1(iii), the set of solution values at $(Q, \underline{w})$ is a closed interval $[\lambda^{\min}, \lambda^{\max}] \subseteq \Lambda(Q) \cap [0, Q]$. Since the endpoints $q_i^\lambda(Q)$ are continuous in λ on $\Lambda(Q) \cap [0, Q]$ by Part II, there exists $\eta > 0$ such that $[Q - \eta, Q + \eta] \subseteq (q_1^\lambda, q_2^\lambda)$ for every $\lambda \in [\lambda^{\min}, \lambda^{\max}]$. The gap $\Delta_\lambda(q)$ is jointly continuous and strictly positive on the compact set $[\lambda^{\min}, \lambda^{\max}] \times [Q - \eta, Q + \eta]$, so it is strictly positive on $N \times [Q - \eta, Q + \eta]$ for some neighborhood N of $[\lambda^{\min}, \lambda^{\max}]$, which we take to lie in $[0, Q]$ since $\lambda^{\max} < Q$. For $\lambda \in N$, the segment $[Q - \eta, Q + \eta]$ therefore lies in a single ironing interval of C_λ , on which $\underline{C}_\lambda$ is affine with slope $s_\lambda(Q)$, and $s_\lambda(Q)$ is continuous in λ on N by Lemma A.1(v).

Now let $0 < h < \eta$ and let λ_h denote any solution value at $(Q + h, \underline{w})$. For every $\lambda \in [\lambda^{\min}, \lambda^{\max}]$, since $\underline{C}_\lambda$ is affine on $[Q, Q + h]$ and $\underline{C}_\lambda(Q) + \lambda \underline{w} = \underline{C}_R(Q, \underline{w})$,

$$h s_\lambda(Q) = \underline{C}_\lambda(Q + h) - \underline{C}_\lambda(Q) \leq \underline{C}_R(Q + h, \underline{w}) - \underline{C}_R(Q, \underline{w}) \leq \underline{C}_{\lambda_h}(Q + h) - \underline{C}_{\lambda_h}(Q), \quad (32)$$

where the second inequality holds because $\underline{C}_{\lambda_h}(Q) + \lambda_h \underline{w} \leq \underline{C}_R(Q, \underline{w})$. Since $\underline{C}_\lambda(q)$ is jointly continuous and λ_h maximizes $\underline{C}_\lambda(Q + h) + \lambda \underline{w}$ over the compact set $[0, 1]$, every limit point of λ_h as $h \downarrow 0$ maximizes $\underline{C}_\lambda(Q) + \lambda \underline{w}$ and hence lies in $[\lambda^{\min}, \lambda^{\max}]$. For h small enough that $\lambda_h \in N$, the right-hand side of (32) equals $h s_{\lambda_h}(Q)$. Dividing (32) by h , taking $h \downarrow 0$ along a subsequence on which λ_h converges, and using the continuity of $s_\lambda(Q)$ on N , we obtain

$$\max_{\lambda \in [\lambda^{\min}, \lambda^{\max}]} s_\lambda(Q) \leq \underline{C}'_{R+}(Q, \underline{w}) \leq \max_{\lambda \in [\lambda^{\min}, \lambda^{\max}]} s_\lambda(Q).$$

Since $s_\lambda(Q)$ is decreasing in λ by Lemma A.1(v), the maximum is attained at $\lambda^{\min}$, which gives $\underline{C}'_{R+}(Q, \underline{w}) = s_{\lambda^{\min}}(Q)$. The same argument with $-\eta < h < 0$ reverses both inequalities in (32) and gives $\underline{C}'_R(Q, \underline{w}) = \min_{\lambda \in [\lambda^{\min}, \lambda^{\max}]} s_\lambda(Q) = s_{\lambda^{\max}}(Q)$. $\square$

We now establish the stated properties of $\underline{C}'_R$. First, we show that $\underline{C}'_R$ is strictly monotone on the two-wage region. Fix $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$. By Lemma A.2 and the convention that $\lambda^*(Q, \underline{w})$ denotes the largest solution value (footnote 33), $\underline{C}'_R(Q, \underline{w}) = s_{\lambda^*(Q, \underline{w})}(Q)$ on this region. Take $\underline{w}' > \underline{w}$ with $(Q, \underline{w}')$ in the region. By Lemma A.1(iii), $\lambda^*(Q, \underline{w}) \in \Lambda(Q) \cap [0, Q)$, and by Lemma 1, $\lambda^*(Q, \underline{w}') > \lambda^*(Q, \underline{w})$, so Lemma A.1(v) gives

$$\underline{C}'_R(Q, \underline{w}') = s_{\lambda^*(Q, \underline{w}')} (Q) < s_{\lambda^*(Q, \underline{w})} (Q) = \underline{C}'_R(Q, \underline{w}).$$

Thus, $\underline{C}'_R$ is strictly decreasing in $\underline{w}$. Now take $Q' > Q$ with $(Q', \underline{w})$ in the region. By Lemma A.1(iii), $\lambda^*(Q', \underline{w}) \in \Lambda(Q') \cap [0, Q')$, and by Lemma 1, $\lambda^*(Q, \underline{w}) > \lambda^*(Q', \underline{w})$. Applying Lemma A.1(v) at Q' , and then using that $s_\lambda(\cdot)$ is increasing,

$$\underline{C}'_R(Q', \underline{w}) = s_{\lambda^*(Q', \underline{w})} (Q') > s_{\lambda^*(Q, \underline{w})} (Q') \geq s_{\lambda^*(Q, \underline{w})} (Q) = \underline{C}'_R(Q, \underline{w}).$$

Thus, $\underline{C}'_R$ is strictly increasing in Q .

We now show that $\underline{C}'_R$ is bounded. For the lower bound, the right derivative of $\underline{C}_R(\cdot, \underline{w})$ at $Q = 0$ equals $\underline{w}$, so convexity gives $\underline{C}'_R(Q, \underline{w}) \geq \underline{w} \geq W(0)$ for all $Q \in (0, 1]$. For the upper bound, (8) gives $\underline{C}_R(Q, \underline{w}) = \underline{C}(Q)$ for $Q \geq (w_1^\ell)^{-1}(\underline{w})$ when $\underline{w} < W(1)$, and $\underline{C}_R(Q, \underline{w}) = \underline{w}Q$ for all Q when $\underline{w} = W(1)$. In either case, convexity gives $\underline{C}'_R(Q, \underline{w}) \leq \max\{\underline{C}'_-(1), W(1)\}$, and $\underline{C}'_-(1)$ is finite. To see this, note that the slopes $(C(1) - C(q))/(1 - q)$ of the chords of C ending at 1 are bounded above by some M , since they converge to $C'_-(1)$ as $q \uparrow 1$ and C is continuous. The affine function $q \mapsto C(1) + M(q - 1)$ therefore lies weakly below C , hence weakly below $\underline{C}$, and coincides with $\underline{C}$ at $q = 1$. Thus, $\underline{C}'_-(1) \leq M$.

Next, we show that $\underline{C}'_R(Q, \underline{w}) \geq W(Q)$, with strict inequality whenever $Q > S(\underline{w})$. For $Q \leq S(\underline{w})$, the marginal cost is $\underline{w}$ by (8), and $\underline{w} \geq W(Q)$ since W is strictly increasing. For $Q \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$, let λ^* denote the largest solution value at $(Q, \underline{w})$, so that $\underline{C}'_R(Q, \underline{w}) = s_{\lambda^*}(Q)$ by Lemma A.2, $Q \in (q_1^{\lambda^*}, q_2^{\lambda^*})$ and $\lambda^* \leq q_1^{\lambda^*}$ by Lemma A.1(iii). Since $s_{\lambda^*}(Q)$ is the slope of the chord of $C_{\lambda^*}(q) = (q - \lambda^*)W(q)$ over $(q_1^{\lambda^*}, q_2^{\lambda^*})$,

$$s_{\lambda^*}(Q) = W(q_2^{\lambda^*}) + \frac{(q_1^{\lambda^*} - \lambda^*)(W(q_2^{\lambda^*}) - W(q_1^{\lambda^*}))}{q_2^{\lambda^*} - q_1^{\lambda^*}} \geq W(q_2^{\lambda^*}) > W(Q). \quad (33)$$

For $Q > (w_1^\ell)^{-1}(\underline{w})$, $\underline{C}_R(\cdot, \underline{w})$ coincides with $\underline{C}$ on a neighborhood of Q by (8), so $\underline{C}'_R(Q, \underline{w}) = \underline{C}'_-(Q)$. If C is not convex at Q , then Q lies in a laissez-faire ironing interval $(Q_1(m), Q_2(m))$ and (33) with $\lambda^* = 0$ gives $\underline{C}'_-(Q) \geq W(Q_2(m)) > W(Q)$. If C is convex at Q , then $\underline{C} \leq C$ with equality at Q gives $\underline{C}'_-(Q) \geq C'_-(Q) = W(Q) + QW'_-(Q) > W(Q)$, since $W'_- > 0$.

Finally, for later reference, we record that for each fixed $Q > 0$ the marginal cost $\underline{C}'_R(Q, \cdot)$ is decreasing on $[W(0), W(Q))$. For $\underline{w} \leq w_1^\ell(Q)$ it equals $s_0(Q)$ by (8), and for $\underline{w} \in (w_1^\ell(Q), W(Q))$ it equals $s_{\lambda^*(Q, \underline{w})}(Q)$ with $\lambda^*(Q, \underline{w}) > 0$ increasing in $\underline{w}$ by Lemma 1. Since $s_\lambda(Q)$ is decreasing in λ by Lemma A.1(v), the claim follows, including across $\underline{w} = w_1^\ell(Q)$. This completes the proof of Theorem 1. $\square$

A.3 Proof of Corollary 1

Proof. Throughout the remaining proofs, $\Pi(Q, \underline{w}) := \int_0^Q V(y) dy - \underline{C}_R(Q, \underline{w})$ denotes the firm's profit from employing Q workers under the minimum wage $\underline{w}$. Since V is decreasing and $\underline{C}_R(\cdot, \underline{w})$ is convex by Theorem 1, $\Pi(\cdot, \underline{w})$ is concave, with left derivative $V(Q) - \underline{C}'_R(Q, \underline{w})$, and it is jointly continuous by Theorem 1. Condition (9) is the first-order condition for maximizing $\Pi(\cdot, \underline{w})$ at an interior quantity, so by concavity an interior quantity is optimal if and only if it satisfies (9). At $Q = 0$, the right derivative of $\Pi(\cdot, \underline{w})$ is $V(0) - \underline{w}$, since the right derivative of $\underline{C}_R(\cdot, \underline{w})$ at $Q = 0$ is $\underline{w}$ by Part V of the proof of Theorem 1, so $Q = 0$ is optimal if and only if $\underline{w} \geq V(0)$. At $Q = 1$, the left derivative of $\Pi(\cdot, \underline{w})$ is $V(1) - \underline{C}'_R(1, \underline{w}) \leq V(1) - W(1) < 0$ by Theorem 1, so $Q = 1$ is never optimal.

Next, suppose $\underline{w} < W(Q^p)$, so that $S(\underline{w}) < Q^p$. By Theorem 1, $\underline{C}'_R(Q, \underline{w}) > W(Q)$ for every $Q > S(\underline{w})$, and hence for every $Q \geq Q^p$. As $V(Q) \leq W(Q)$ for such Q , by the definition of Q^p and the monotonicity of V and W , no quantity $Q \geq Q^p$ satisfies (9). Since $Q = 1$ is never optimal, every optimal level of employment satisfies $Q < Q^p$.

Finally, we show that the largest profit-maximizing quantity maximizes social surplus among all profit-maximizing quantities. Let $Q < Q'$ be two profit-maximizing quantities. By concavity, every quantity in $[Q, Q']$ is profit-maximizing, so V and $\underline{C}'_R(\cdot, \underline{w})$ coincide on (Q, Q') , and since V is decreasing and $\underline{C}'_R(\cdot, \underline{w})$ is increasing, both are constant there, equal to some v . By Theorem 1, $\underline{C}'_R(\cdot, \underline{w})$ is strictly increasing on $\left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right]$, so $[Q, Q']$ contains no segment of this interval. Thus, $[Q, Q']$ lies in $[0, S(\underline{w})]$, or in $\left[(w_1^\ell)^{-1}(\underline{w}), 1\right]$, or it straddles the point $S(\underline{w}) = (w_1^\ell)^{-1}(\underline{w})$ when these coincide. Since social surplus is continuous in the quantity, it suffices to show that it is increasing on each part of $[Q, Q']$ lying in one of these two regions.

On $[0, S(\underline{w})]$, the firm rations the $S(\underline{w})$ workers with cost at most $\underline{w}$ into the available openings, so at a quantity q in this region workers' aggregate opportunity cost is $\frac{q}{S(\underline{w})} \int_0^{S(\underline{w})} W(y) dy$ and social surplus is $\int_0^q V(y) dy$ minus this amount. Its derivative is $v - \frac{1}{S(\underline{w})} \int_0^{S(\underline{w})} W(y) dy$, which is positive because $v = \underline{w} = W(S(\underline{w}))$ on this region and W is strictly increasing.

On $\left[(w_1^\ell)^{-1}(\underline{w}), 1\right]$, the minimum wage does not bind, so $\underline{C}_R(\cdot, \underline{w}) = \underline{C}$, which is affine

on $[Q, Q']$. By Assumption 1 at $\lambda = 0$, the affine function that coincides with $\underline{C}$ on $[Q, Q']$ touches C at most twice, so $[Q, Q']$ lies in the closure of a single ironing interval $(Q_1(m), Q_2(m))$ of C , and the firm uses the two-wage mechanism parameterized by $(q, Q_1(m), Q_2(m))$ at each q in this region. Workers' aggregate opportunity cost is then

$$\int_0^{Q_1(m)} W(y) dy + \frac{q - Q_1(m)}{Q_2(m) - Q_1(m)} \int_{Q_1(m)}^{Q_2(m)} W(y) dy,$$

whose derivative with respect to q is the average of W over $(Q_1(m), Q_2(m))$, which is strictly below $W(Q_2(m))$. Since $v = \underline{C}'_-(q) \geq W(Q_2(m))$ by (33) at $\lambda = 0$, the derivative of social surplus is positive.

Hence social surplus is increasing on $[Q, Q']$, and in particular is larger at Q' than at Q . Selecting the largest profit-maximizing quantity therefore maximizes social surplus among all profit-maximizing quantities. $\square$

A.4 Proof of Lemma 2

Proof. Fix $(Q, \underline{w})$ satisfying the conditions of the lemma. By Part III of the proof of Theorem 1, every solution value λ of the Lagrange multiplier in (7) at $(Q, \underline{w})$ induces the same optimal measure $\mu_{Q, \lambda}$, and by Lemma A.1(iii) this measure is supported on $\{q_1^\lambda(Q), q_2^\lambda(Q)\}$. Hence $q_i^*(Q, \underline{w}) = q_i^\lambda(Q)$ for $i \in \{1, 2\}$ and every solution value λ at $(Q, \underline{w})$ and, in particular, $q_i^*(Q, \underline{w}) = q_i^{\lambda^*(Q, \underline{w})}(Q)$. Moreover, $q_1^*(Q, \underline{w}) < S(\underline{w}) < Q < q_2^*(Q, \underline{w})$: the low wage of the optimal mechanism equals $\underline{w}$ by Lemma A.1(iii) and is, by (2), a strict convex combination of $W(q_1^*(Q, \underline{w}))$ and $W(q_2^*(Q, \underline{w}))$, so $W(q_1^*(Q, \underline{w})) < \underline{w}$.

We first show that q_1^* is decreasing and q_2^* is increasing in Q . Fix $\underline{w}$ and $Q < Q'$ such that both $(Q, \underline{w})$ and $(Q', \underline{w})$ satisfy the conditions of the lemma, and write $\lambda := \lambda^*(Q, \underline{w})$ and $\lambda' := \lambda^*(Q', \underline{w})$, so that $\lambda' < \lambda$ by Lemma 1. Since $q_1^{\lambda'}(Q') = q_1^*(Q', \underline{w}) < S(\underline{w}) < Q < Q' < q_2^{\lambda'}(Q')$, the quantity Q lies in the ironing interval $(q_1^{\lambda'}(Q'), q_2^{\lambda'}(Q'))$ of $C_{\lambda'}$ containing Q' . Hence $\lambda' \in \Lambda(Q)$ and $q_i^{\lambda'}(Q) = q_i^{\lambda'}(Q')$ for $i \in \{1, 2\}$. By Lemma A.1(iii) at $(Q, \underline{w})$, $\lambda \in \Lambda(Q)$ and $\lambda \leq q_1^\lambda(Q)$. Lemma A.1(iv), applied at Q to the pair $\lambda' < \lambda \leq q_1^\lambda(Q)$, gives $q_1^{\lambda'}(Q) \leq q_1^\lambda(Q)$ and $q_2^\lambda(Q) \leq q_2^{\lambda'}(Q)$, that is, $q_1^*(Q', \underline{w}) \leq q_1^*(Q, \underline{w})$ and $q_2^*(Q, \underline{w}) \leq q_2^*(Q', \underline{w})$.

We next show that q_1^* is increasing and q_2^* is decreasing in $\underline{w}$. Fix Q and $\underline{w} < \underline{w}'$ such that both $(Q, \underline{w})$ and $(Q, \underline{w}')$ satisfy the conditions of the lemma, and write $\lambda := \lambda^*(Q, \underline{w})$ and $\lambda' := \lambda^*(Q, \underline{w}')$, so that $\lambda < \lambda'$ by Lemma 1. By Lemma A.1(iii), $\lambda, \lambda' \in \Lambda(Q)$ and $\lambda' \leq q_1^{\lambda'}(Q)$. Lemma A.1(iv), applied at Q to the pair $\lambda < \lambda'$, gives $q_1^\lambda(Q) \leq q_1^{\lambda'}(Q)$ and $q_2^{\lambda'}(Q) \leq q_2^\lambda(Q)$, as required. $\square$

A.5 Proof of Lemma 3

Recall from the proof of Corollary 1 that $\Pi(Q, \underline{w})$ denotes the firm's profit, which is concave in Q with left derivative $V(Q) - \underline{C}'_R(Q, \underline{w})$, and jointly continuous. Before proving Lemma 3, we record some useful properties of equilibrium employment.

Lemma A.3.

- (i) If $\underline{w} \leq W(Q^p)$, then $Q^*(\underline{w}) \geq S(\underline{w})$. If $\underline{w} \geq W(Q^p)$, then $Q^*(\underline{w}) \leq S(\underline{w})$, and if $\underline{w} > W(Q^p)$, then $Q^*(\underline{w}) = D(\underline{w}) < S(\underline{w})$.
- (ii) If $\underline{w} \leq w_1^\ell(Q^\ell)$, then $Q^*(\underline{w}) = Q^\ell$ and the firm uses the laissez-faire mechanism. In particular, $Q^\ell < Q^p$.
- (iii) The equilibrium is in the non-binding region if and only if $\underline{w} < w_1^\ell(Q^\ell)$.
- (iv) On $[w_1^\ell(Q^\ell), W(Q^p)]$, Q^* is continuous and increasing, and for $\underline{w} < W(Q^p)$ it is the unique maximizer of $\Pi(\cdot, \underline{w})$.

Proof of Lemma A.3. For (i), let $\underline{w} \leq W(Q^p)$. On $[0, S(\underline{w})]$ the marginal cost is $\underline{w} \leq W(Q^p) = V(Q^p) \leq V(Q)$ by (8), so $\Pi(\cdot, \underline{w})$ is increasing there and $Q^*(\underline{w}) \geq S(\underline{w})$. Let $\underline{w} \geq W(Q^p)$. For $Q > S(\underline{w}) \geq Q^p$, Theorem 1 gives $\underline{C}'_R(Q, \underline{w}) > W(Q) > W(Q^p) \geq V(Q)$, so $\Pi(\cdot, \underline{w})$ is strictly decreasing on $[S(\underline{w}), 1]$ and $Q^*(\underline{w}) \leq S(\underline{w})$. If moreover $\underline{w} > W(Q^p)$, the marginal cost is $\underline{w}$ on $[0, S(\underline{w})]$, so the largest maximizer is $D(\underline{w})$, and $D(\underline{w}) \leq Q^p < S(\underline{w})$ because $V(Q) \leq W(Q^p) < \underline{w}$ for $Q > Q^p$. For (ii), let $\underline{w} \leq w_1^\ell(Q^\ell)$. The laissez-faire mechanism at Q^ℓ is feasible and $\underline{C}_R(\cdot, \underline{w}) \geq \underline{C}$ by Proposition 1, so Q^ℓ maximizes $\Pi(\cdot, \underline{w})$. Every $Q > Q^\ell$ yields a laissez-faire profit strictly below the laissez-faire maximum, and hence a profit $\Pi(Q, \underline{w})$ strictly below it as well. Thus $Q^*(\underline{w}) = Q^\ell$, and the mechanism is the laissez-faire one by Case 3 of Part III of the proof of Theorem 1. Finally, $W(0) \leq w_1^\ell(Q^\ell)$, so $Q^\ell = Q^*(W(0)) < Q^p$ by Corollary 1, since $W(0) < W(Q^p)$. For (iii), if $\underline{w} < w_1^\ell(Q^\ell)$, then $\underline{w} < w_1^\ell(Q^*(\underline{w}))$ by (ii). Conversely, if $\underline{w} < w_1^\ell(Q^*(\underline{w}))$, the laissez-faire mechanism is feasible near $Q^*(\underline{w})$ by continuity of w_1^ℓ , so $Q^*(\underline{w})$ is a local, hence global, maximizer of the concave laissez-faire profit, whence $Q^*(\underline{w}) \leq Q^\ell$ and $\underline{w} < w_1^\ell(Q^*(\underline{w})) \leq w_1^\ell(Q^\ell)$. For (iv), fix $\underline{w} \in [w_1^\ell(Q^\ell), W(Q^p)]$. On $[0, S(\underline{w})]$ the marginal cost is $\underline{w} < W(Q^p) = V(Q^p) \leq V(Q)$ by (8), so $\Pi(\cdot, \underline{w})$ is strictly increasing there. On $[S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})]$ the marginal cost is strictly increasing by Theorem 1 and V is decreasing, so $\Pi(\cdot, \underline{w})$ is strictly concave there. On $[(w_1^\ell)^{-1}(\underline{w}), 1]$, where $(w_1^\ell)^{-1}(\underline{w}) \geq Q^\ell$ because w_1^ℓ is strictly increasing, $\Pi(\cdot, \underline{w})$ coincides with the laissez-faire profit by (8), which is concave with largest maximizer Q^ℓ and hence strictly decreasing on $[Q^\ell, 1]$. Together with concavity, this shows that $\Pi(\cdot, \underline{w})$ has a unique maximizer.

For continuity, consider a sequence $\{\underline{w}_n\}_{n \in \mathbb{N}}$ in $[w_1^\ell(Q^\ell), W(Q^p)]$ with $\lim_{n \rightarrow \infty} \underline{w}_n = \underline{w}$, and let $\bar{Q}$ be the limit of $Q^*(\underline{w}_n)$ along a convergent subsequence. Then $\Pi(Q^*(\underline{w}_n), \underline{w}_n) \geq \Pi(Q, \underline{w}_n)$ holds for every Q along the subsequence, and taking limits on both sides, which is permitted because Π is jointly continuous, gives $\Pi(\bar{Q}, \underline{w}) \geq \Pi(Q, \underline{w})$. Hence $\bar{Q}$ maximizes $\Pi(\cdot, \underline{w})$. If $\underline{w} < W(Q^p)$, then $\bar{Q} = Q^*(\underline{w})$ by uniqueness. If $\underline{w} = W(Q^p)$, then $Q^*(\underline{w}) = Q^p$ by (i), and for $\underline{w}_n < W(Q^p)$ we have $S(\underline{w}_n) \leq Q^*(\underline{w}_n) < Q^p$ by (i) and Corollary 1. Since $\lim_{n \rightarrow \infty} S(\underline{w}_n) = Q^p$, it follows that $\bar{Q} = Q^p$ as well. Every convergent subsequence of $Q^*(\underline{w}_n)$ thus has limit $Q^*(\underline{w})$, so $\lim_{n \rightarrow \infty} Q^*(\underline{w}_n) = Q^*(\underline{w})$.

Finally, take $\underline{w} < \underline{w}'$ in $[w_1^\ell(Q^\ell), W(Q^p)]$ and assume, seeking a contradiction, that $Q' := Q^*(\underline{w}') < Q := Q^*(\underline{w})$, so that $Q > Q' \geq S(\underline{w}')$ by (i). For every $y \geq S(\underline{w}')$, the marginal cost $\underline{C}'_R(y, \cdot)$ is decreasing on $[\underline{w}, \underline{w}']$: it is decreasing on $[W(0), W(y))$ by Part V of the proof of Theorem 1, and $\underline{C}'_R(y, W(y)) = W(y) \leq \underline{C}'_R(y, \underline{w})$ by (8) and Theorem 1. Integrating over $[Q', Q]$ yields $\underline{C}_R(Q, \underline{w}') - \underline{C}_R(Q', \underline{w}') \leq \underline{C}_R(Q, \underline{w}) - \underline{C}_R(Q', \underline{w})$. Therefore $\Pi(Q, \underline{w}') - \Pi(Q', \underline{w}') \geq \Pi(Q, \underline{w}) - \Pi(Q', \underline{w}) \geq 0$. Hence Q also maximizes $\Pi(\cdot, \underline{w}')$, contradicting the definition of Q' as the largest maximizer. $\square$

We now turn to the proof of Lemma 3.

Proof. For every $\underline{w} \in I$, the pair $(Q^*(\underline{w}), \underline{w})$ satisfies the conditions of Lemmas A.1(iii) and A.2, and $q_i^*(Q^*(\underline{w}), \underline{w}) = q_i^\lambda(Q^*(\underline{w}))$ for $i \in \{1, 2\}$ and every solution value λ of the Lagrange multiplier at $(Q^*(\underline{w}), \underline{w})$ by the first paragraph of the proof of Lemma 2. Every such solution value satisfies $\lambda \in \Lambda(Q^*(\underline{w}))$ and $\lambda < Q^*(\underline{w})$ by Lemma A.1(iii), so that $[0, \lambda] \subseteq \Lambda(Q^*(\underline{w}))$ by Lemma A.1(iv); we use this repeatedly. For $\underline{w} < \underline{w}'$ in I , we write $Q^* := Q^*(\underline{w})$ and $Q^{*'} := Q^*(\underline{w}')$. By Lemma A.3(i) and (iii), $I \subseteq [w_1^\ell(Q^\ell), W(Q^p))$.

For (i), $Q^*(\cdot)$ is continuous on I by Lemma A.3(iv), so it remains to show that it is strictly increasing. Take $\underline{w} < \underline{w}'$ in I and suppose that $Q^{*'} \leq Q^*$. Let λ' denote the smallest solution value at $(Q^{*'}, \underline{w}')$. By Lemma 1, $\lambda^*(Q^*, \underline{w}) < \lambda'$, and hence $\lambda^*(Q^*, \underline{w}) \in \Lambda(Q^{*'}) \cap [0, Q^{*'}]$. By the optimality condition (9) at $(Q^*, \underline{w})$ and at $(Q^{*'}, \underline{w}')$, Lemma A.2, the monotonicity of $s_{\lambda^*(Q^*, \underline{w})}(\cdot)$ and Lemma A.1(v) applied at $Q^{*'} to the pair $\lambda^*(Q^*, \underline{w}) < \lambda'$,$

$$V(Q^{*'}) \geq V(Q^*) \geq s_{\lambda^*(Q^*, \underline{w})}(Q^*) \geq s_{\lambda^*(Q^*, \underline{w})}(Q^{*'}) > s_{\lambda'}(Q^{*'}) = \underline{C}'_{R+}(Q^{*'}, \underline{w}') \geq V(Q^{*'}),$$

a contradiction. This proves (i).

For each $\underline{w} \in I$, we define $\hat{\lambda}(\underline{w})$ to be the solution value of (7) at $(Q^*(\underline{w}), \underline{w})$ satisfying

$$s_{\hat{\lambda}(\underline{w})}(Q^*(\underline{w})) = V(Q^*(\underline{w})), \quad (34)$$

which exists and is unique.⁵⁸

We now show that $\hat{\lambda}$ is increasing on I . Take $\underline{w} < \underline{w}'$ in I , so that $Q^* < Q^{*'}$, and suppose that $\hat{\lambda}(\underline{w}') < \hat{\lambda}(\underline{w})$, so that $\hat{\lambda}(\underline{w}') \in \Lambda(Q^*)$. Then

$$V(Q^{*'}) = s_{\hat{\lambda}(\underline{w}')}(\underline{w}') \geq s_{\hat{\lambda}(\underline{w}')}(\underline{w}) > s_{\hat{\lambda}(\underline{w})}(\underline{w}) = V(Q^*) \geq V(Q^{*'}),$$

where we use (34) at $\underline{w}'$, the monotonicity of $s_{\hat{\lambda}(\underline{w}')}(\cdot)$, Lemma A.1(v) applied at Q^* to the pair $\hat{\lambda}(\underline{w}') < \hat{\lambda}(\underline{w})$, (34) at $\underline{w}$, and the monotonicity of V . This is a contradiction.

Next we show that, for any $\underline{w} \leq \underline{w}'$ in I , the segment $[Q^*, Q^{*'}]$ lies within a single ironing interval of $C_{\hat{\lambda}(\underline{w})}$. For $z \in [Q^*, Q^{*'}]$, continuity of $Q^*(\cdot)$ yields $\tilde{w} \in [\underline{w}, \underline{w}']$ with $Q^*(\tilde{w}) = z$, and $\hat{\lambda}(\underline{w}) \leq \hat{\lambda}(\tilde{w})$ gives $\hat{\lambda}(\underline{w}) \in \Lambda(z)$. The set $\{\tilde{Q} : C_{\hat{\lambda}(\underline{w})}(\tilde{Q}) < C_{\hat{\lambda}(\underline{w})}(\tilde{Q})\}$ is open with the ironing intervals of $C_{\hat{\lambda}(\underline{w})}$ as its connected components, so the connected set $[Q^*, Q^{*'}]$ lies within one of them. In particular, $q_i^{\hat{\lambda}(\underline{w})}(Q^*) = q_i^{\hat{\lambda}(\underline{w})}(Q^{*'})$ for $i \in \{1, 2\}$ and $s_{\hat{\lambda}(\underline{w})}(Q^*) = s_{\hat{\lambda}(\underline{w})}(Q^{*'})$. We refer to this as the segment property.

We prove (iii) before (ii). Take $\underline{w} < \underline{w}'$ in I . If $\hat{\lambda}(\underline{w}) = \hat{\lambda}(\underline{w}')$, the segment property gives $q_i^*(Q^*, \underline{w}) = q_i^{\hat{\lambda}(\underline{w})}(Q^{*'}) = q_i^*(Q^{*'}, \underline{w}')$ for $i \in \{1, 2\}$. If $\hat{\lambda}(\underline{w}) < \hat{\lambda}(\underline{w}')$, then $\hat{\lambda}(\underline{w}) \in \Lambda(Q^{*'})$ by the segment property, and $\hat{\lambda}(\underline{w}') \in \Lambda(Q^{*'})$ with $\hat{\lambda}(\underline{w}') \leq q_1^{\hat{\lambda}(\underline{w}')}(\underline{w}')$ by Lemma A.1(iii), so Lemma A.1(iv) applied at $Q^{*'}$ gives $q_1^{\hat{\lambda}(\underline{w})}(Q^{*'}) \leq q_1^{\hat{\lambda}(\underline{w}')}(\underline{w}')$ and $q_2^{\hat{\lambda}(\underline{w}')}(\underline{w}') \leq q_2^{\hat{\lambda}(\underline{w})}(Q^{*'})$. Combining with the segment property, $q_1^*(Q^*, \underline{w}) \leq q_1^*(Q^{*'}, \underline{w}')$ and $q_2^*(Q^{*'}, \underline{w}') \leq q_2^*(Q^*, \underline{w})$, proving (iii).

For (ii), take $\underline{w} < \underline{w}'$ in I , write $\lambda^* := \lambda^*(Q^*, \underline{w})$ and $\lambda^{*'} := \lambda^*(Q^{*'}, \underline{w}')$, and suppose that $\lambda^{*'} < \lambda^*$, so that $\lambda^{*'} \in \Lambda(Q^*)$. By (iii), the ironing interval of $C_{\lambda^{*'}}$ containing $Q^{*'}$ is contained in the ironing interval of C_{λ^*} containing Q^* . Since $\lambda^{*'} < \lambda^* \leq q_1^{\lambda^*}(Q^*)$, Lemma A.1(iv) applied at Q^* shows that the latter is in turn contained in the ironing interval of $C_{\lambda^{*'}}$ containing Q^* . The first and third of these intervals are ironing intervals of the same function, and the first is contained in the third, so all three coincide. Hence $w_1^{\lambda^*}(Q^{*'}) = L_{\lambda^*}^W(Q^{*'}) = L_{\lambda^{*'}}^W(Q^{*'}) = w_1^{\lambda^{*'}}(Q^{*'}) = \underline{w}'$, where the last equality is Lemma A.1(iii) at $(Q^{*'}, \underline{w}')$. By Part III of the proof of Theorem 1, the solution values at $(Q^{*'}, \underline{w}')$ are exactly the solutions of (23) there, so λ^* is one of them, contradicting the maximality of $\lambda^{*'}$. This proves (ii).

Finally, suppose V is constant on $\{Q^*(\underline{w}) : \underline{w} \in I\}$. Take $\underline{w} < \underline{w}'$ in I and suppose that $\hat{\lambda}(\underline{w}) < \hat{\lambda}(\underline{w}')$, the reverse ordering being excluded since $\hat{\lambda}$ is increasing. By the segment property, $\hat{\lambda}(\underline{w}) \in \Lambda(Q^{*'})$ and $s_{\hat{\lambda}(\underline{w})}(Q^*) = s_{\hat{\lambda}(\underline{w})}(Q^{*'})$, so Lemma A.1(v) applied at $Q^{*'}$ to the

⁵⁸The set of solution values at $(Q^*(\underline{w}), \underline{w})$ is an interval $[\lambda^{\min}, \lambda^{\max}]$ contained in $\Lambda(Q^*(\underline{w})) \cap [0, Q^*(\underline{w})]$, so $s_{\lambda}(Q^*(\underline{w}))$ is continuous and strictly decreasing in λ on it by Lemma A.1(v), and (9) together with Lemma A.2 places $V(Q^*(\underline{w}))$ between $s_{\lambda^{\max}}(Q^*(\underline{w}))$ and $s_{\lambda^{\min}}(Q^*(\underline{w}))$.

pair $\hat{\lambda}(\underline{w}) < \hat{\lambda}(\underline{w}')$ gives

$$V(Q^{*'}) = s_{\hat{\lambda}(\underline{w}')}(\underline{w}) < s_{\hat{\lambda}(\underline{w})}(\underline{w}) = s_{\hat{\lambda}(\underline{w})}(Q^*) = V(Q^*) = V(Q^{*'}),$$

a contradiction. Thus $\hat{\lambda}$ is constant on I , and the segment property gives $q_i^*(Q^*, \underline{w}) = q_i^{\hat{\lambda}(\underline{w})}(Q^*) = q_i^{\hat{\lambda}(\underline{w})}(Q^{*'}) = q_i^*(Q^{*'}, \underline{w}')$ for $i \in \{1, 2\}$ and all $\underline{w} < \underline{w}'$ in I , proving (iv). $\square$

A.6 Proof of Theorem 2

Proof. By Theorem 1 and Lemma A.1(iii), involuntary unemployment and wage dispersion both arise in the constrained two-wage region, and neither arises in the Robinson region. In the non-binding region, both arise if Q^ℓ lies in a laissez-faire ironing interval, and neither arises if it does not. In the neoclassical region, only involuntary unemployment arises.

For (i), the equilibrium is therefore in the non-binding region with Q^ℓ in an ironing interval, or in the constrained two-wage region. In the former case, $\underline{w} < w_1^\ell(Q^\ell) \leq W(Q^\ell) < W(Q^p)$ by Lemma A.3(ii). A small increase preserves $\underline{w} < w_1^\ell(Q^\ell)$, so by Lemma A.3(ii) the equilibrium is unchanged and the stated effects hold weakly. In the latter case, $Q^*(\underline{w}) > S(\underline{w})$, and hence $\underline{w} < W(Q^p)$ by Lemma A.3(i). Moreover, $\underline{w} \geq w_1^\ell(Q^\ell)$ by Lemma A.3(iii), so no $\underline{w}' > \underline{w}$ is in the non-binding region. By Lemma A.3(iv), $Q^* - S$ is continuous on $[w_1^\ell(Q^\ell), W(Q^p)]$. Since it is positive at $\underline{w}$, it is positive on $[\underline{w}, \underline{w} + \varepsilon]$ for some $\varepsilon > 0$ with $\underline{w} + \varepsilon < W(Q^p)$. The equilibrium therefore remains in the constrained two-wage region on this interval, and Lemma 3 applies there. Employment is strictly increasing by part (i) of that lemma, and $q_2^*(Q^*(\underline{w}), \underline{w})$ is decreasing by part (iii). Hence involuntary unemployment $q_2^*(Q^*(\underline{w}), \underline{w}) - Q^*(\underline{w})$ decreases and the high wage $W(q_2^*(Q^*(\underline{w}), \underline{w}))$ decreases, while the low wage $\underline{w}$ increases mechanically, so wage dispersion strictly decreases.

For (ii), the equilibrium is in the non-binding region with Q^ℓ outside every ironing interval, or in the Robinson region. In the former case, $\underline{w} < w_1^\ell(Q^\ell) = W(Q^\ell) < W(Q^p)$, and a marginal increase leaves the equilibrium unchanged. In the latter case, $Q^*(\underline{w}) = S(\underline{w})$, and hence $\underline{w} \leq W(Q^p)$ by Lemma A.3(i). If $\underline{w} < W(Q^p)$, then for $\varepsilon > 0$ with $\underline{w} + \varepsilon < W(Q^p)$, Lemma A.3(i) gives $Q^*(\underline{w} + \varepsilon) \geq S(\underline{w} + \varepsilon) > S(\underline{w}) = Q^*(\underline{w})$. The restriction $\underline{w} \neq W(Q^p)$ excludes the boundary configuration at which a marginal increase pushes the equilibrium into the neoclassical region.

For (iii), the equilibrium is in the neoclassical region, where $Q^*(\underline{w}) < S(\underline{w})$ and hence $\underline{w} > W(Q^p)$ by Lemma A.3(i). By the same lemma, employment is $D(\underline{w}')$ at every $\underline{w}' > W(Q^p)$, which is decreasing, so involuntary unemployment $S(\underline{w}') - D(\underline{w}')$ is increasing. All employed workers are paid the minimum wage, so wage dispersion remains zero. $\square$

A.7 Proof of Proposition 2

Proof. We start by noting that $\gamma(Q) < \underline{C}'_-(Q)$ whenever $\underline{C}(Q) < C(Q)$, as claimed in Section 4.3.⁵⁹

For (i), the first claim is Lemma A.3(iii), and the second claim follows from Lemma A.3(i): $Q^*(\underline{w}) < S(\underline{w})$ if $\underline{w} > W(Q^p)$ and $Q^*(\underline{w}) \geq S(\underline{w})$ if $\underline{w} \leq W(Q^p)$. Now let $\underline{w} \in [w_1^\ell(Q^\ell), W(Q^p)]$. The equilibrium is then neither in the non-binding nor in the neoclassical region. Thus, $Q^*(\underline{w}) \geq S(\underline{w})$, and the equilibrium is in the Robinson region if $Q^*(\underline{w}) = S(\underline{w})$ and in the constrained two-wage region otherwise. By (8), the left derivative of $\underline{C}_R(\cdot, \underline{w})$ at $S(\underline{w})$ is $\underline{w}$, and its right derivative is $\gamma(S(\underline{w}))$, since $\underline{w} = W(S(\underline{w}))$. As $V(S(\underline{w})) \geq \underline{w}$, the quantity $S(\underline{w})$ satisfies (9) if and only if $V(S(\underline{w})) \leq \gamma(S(\underline{w}))$. If $V(S(\underline{w})) \leq \gamma(S(\underline{w}))$, then $S(\underline{w})$ maximizes $\Pi(\cdot, \underline{w})$. So $Q^*(\underline{w}) = S(\underline{w})$ by Lemma A.3(iv) if $\underline{w} < W(Q^p)$ and by Lemma A.3(i) if $\underline{w} = W(Q^p)$, and the equilibrium is in the Robinson region. If $V(S(\underline{w})) > \gamma(S(\underline{w}))$, then $S(\underline{w})$ is not a maximizer, so $Q^*(\underline{w}) > S(\underline{w})$ and the equilibrium is in the constrained two-wage region.

For (ii), suppose first that there is involuntary unemployment under laissez-faire, so that Q^ℓ lies in a laissez-faire ironing interval and $w_1^\ell(Q^\ell) < W(Q^\ell) < W(Q^p)$, where the last inequality is Lemma A.3(ii). By (i), the equilibrium is in the non-binding region at every $\underline{w} < w_1^\ell(Q^\ell)$, and for $\underline{w} \in [w_1^\ell(Q^\ell), W(Q^\ell))$ it is in the Robinson or the constrained two-wage region. Since $Q^*(w_1^\ell(Q^\ell)) = Q^\ell > S(w_1^\ell(Q^\ell))$ by Lemma A.3(ii) and $w_1^\ell(Q^\ell) < W(Q^\ell)$, and $Q^* - S$ is continuous by Lemma A.3(iv), $Q^*(\underline{w}) > S(\underline{w})$ for all $\underline{w}$ slightly above $w_1^\ell(Q^\ell)$, which excludes the Robinson region there. Hence the equilibrium transitions into the constrained two-wage region.

Suppose instead that there is no involuntary unemployment under laissez-faire, so that $w_1^\ell(Q^\ell) = W(Q^\ell)$ and Q^ℓ lies outside every ironing interval, and suppose that $Q^\ell \neq Q_1(m)$ for all $m \in \mathcal{M}$. Since only finitely many laissez-faire ironing intervals intersect (Q^ℓ, Q^p) , there exists $\varepsilon > 0$ such that $[Q^\ell, Q^\ell + \varepsilon)$ contains no point of any ironing interval. On this interval $\underline{C} = C$, so $\gamma = C'_+$ there, as noted in Section 4.3, and C'_+ is increasing because C is convex there. Optimality of Q^ℓ under laissez-faire gives $V(Q^\ell) \leq C'_+(Q^\ell)$, and V is decreasing, so $V(S(\underline{w})) \leq \gamma(S(\underline{w}))$ whenever $S(\underline{w}) \in (Q^\ell, Q^\ell + \varepsilon)$ (that is, for all $\underline{w}$ slightly above $W(Q^\ell) = w_1^\ell(Q^\ell)$). By (i), the equilibrium is in the Robinson region at these minimum wages and in the non-binding region at minimum wages slightly below $w_1^\ell(Q^\ell)$.

⁵⁹Fix such Q , so that $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, and put $\underline{w} := W(Q)$. Since $w_1^\ell(Q) < \underline{w} < W(Q_2(m)) = w_1^\ell(Q_2(m))$, we have $Q < (w_1^\ell)^{-1}(\underline{w}) < Q_2(m)$. By (8), $\underline{C}_R(Q, \underline{w}) = C(Q) > \underline{C}(Q)$, whereas $\underline{C}_R(\cdot, \underline{w})$ coincides with $\underline{C}$ at $(w_1^\ell)^{-1}(\underline{w})$. As $\underline{C}_R(\cdot, \underline{w})$ is convex, $\gamma(Q) = \underline{C}'_{R+}(Q, \underline{w})$ is at most the slope of its chord over $[Q, (w_1^\ell)^{-1}(\underline{w})]$, which is strictly below the slope of the chord of $\underline{C}$ over the same interval, and the latter is $\underline{C}'_-(Q)$ because $\underline{C}$ is affine on $[Q_1(m), Q_2(m)]$.

For (iii), suppose $\gamma(Q^p) \neq W(Q^p)$, so that $\gamma(Q^p) > W(Q^p)$ by footnote 35, and fix $v \in (W(Q^p), \gamma(Q^p))$. Since γ is left-continuous at Q^p and V is continuous with $V(Q^p) = W(Q^p) < v$, there is $\varepsilon > 0$ such that $V(Q) < v < \gamma(Q)$ for all $Q \in (Q^p - \varepsilon, Q^p)$. Shrinking ε if necessary, we may also assume that $Q^p - \varepsilon \geq Q^\ell$. Every $\underline{w}$ with $S(\underline{w}) \in (Q^p - \varepsilon, Q^p)$ then satisfies $V(S(\underline{w})) < \gamma(S(\underline{w}))$ and $\underline{w} > W(Q^\ell) \geq w_1^\ell(Q^\ell)$. By (i), the equilibrium is therefore in the Robinson region at all minimum wages slightly below $W(Q^p)$, and it is in the neoclassical region at all minimum wages above $W(Q^p)$. Hence the transition into the neoclassical region occurs from the Robinson region. The second claim is the contrapositive. $\square$

A.8 Proof of Proposition 3

Proof. Throughout, the domain is $[w_1^\ell(Q^\ell), W(Q^p)]$. Employment Q^* is continuous and increasing on it by Lemma A.3(iv), and $S(\underline{w}) \leq Q^*(\underline{w}) \leq Q^p$ for every $\underline{w}$ in the domain by Lemma A.3(i) and Corollary 1.

Workers' aggregate pay equals the firm's cost $\underline{C}_R(Q^*(\underline{w}), \underline{w})$. For $\underline{w} < \underline{w}'$, monotonicity of $\underline{C}_R$ in each argument gives $\underline{C}_R(Q^*(\underline{w}'), \underline{w}') \geq \underline{C}_R(Q^*(\underline{w}), \underline{w}') \geq \underline{C}_R(Q^*(\underline{w}), \underline{w})$. Aggregate pay is continuous by the continuity of Q^* and the joint continuity of $\underline{C}_R$. Because $\underline{C}_R(0, \underline{w}) = 0$ and $\underline{C}_R(\cdot, \underline{w})$ is convex, the average pay $\underline{C}_R(Q, \underline{w})/Q$ is increasing in Q . It is also increasing in $\underline{w}$ at fixed Q . Both arguments increase in $\underline{w}$, so the average pay $\underline{C}_R(Q^*(\underline{w}), \underline{w})/Q^*(\underline{w})$ is increasing. It is continuous because aggregate pay is continuous and $Q^*(\underline{w}) \geq S(\underline{w}) > 0$.

Let $K(\underline{w}) := \int_0^1 x(y)W(y) dy$ denote the workers' aggregate opportunity cost under the equilibrium allocation rule x at $\underline{w}$, so that $SS(\underline{w}) = \int_0^{Q^*(\underline{w})} V(y) dy - K(\underline{w})$ and $WS(\underline{w}) = \underline{C}_R(Q^*(\underline{w}), \underline{w}) - K(\underline{w})$. We now show that K is continuous, which implies that SS and WS are continuous. Consider a sequence $\{\underline{w}_n\}_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \underline{w}_n = \underline{w}_0$, and pass to a subsequence along which the parameters of the equilibrium mechanisms converge as $n \rightarrow \infty$. The mechanism with the limiting parameters procures $\lim_{n \rightarrow \infty} Q^*(\underline{w}_n) = Q^*(\underline{w}_0)$ by the continuity of Q^* . It satisfies the minimum wage constraint at $\underline{w}_0$, since its low wage is the limit as $n \rightarrow \infty$ of low wages that are at least $\underline{w}_n$. Since the cost of a two-wage mechanism is continuous in its parameters, it costs $\lim_{n \rightarrow \infty} \underline{C}_R(Q^*(\underline{w}_n), \underline{w}_n)$, which equals $\underline{C}_R(Q^*(\underline{w}_0), \underline{w}_0)$ by the continuity of aggregate pay. It is therefore optimal at $(Q^*(\underline{w}_0), \underline{w}_0)$, and by Theorem 1 its allocation is the unique optimal allocation there. Since every subsequence of $\{\underline{w}_n\}_{n \in \mathbb{N}}$ has a further subsequence along which this holds, the equilibrium allocations at $\underline{w}_n$ converge to the equilibrium allocation at $\underline{w}_0$ as $n \rightarrow \infty$, and $\lim_{n \rightarrow \infty} K(\underline{w}_n) = K(\underline{w}_0)$ because W is continuous and bounded. As noted in Section 4.4, this convergence also implies that, at a transition from the Robinson region into the constrained two-wage region, involuntary unemployment and wage dispersion increase discontinuously while the mass of workers employed

at the high wage vanishes.⁶⁰

For monotonicity of SS and WS , we use the following comparison. Hiring a given quantity through a lottery costs society strictly more than hiring the lowest-cost workers. Specifically, for any allocation x of a non-degenerate two-wage mechanism with employment Q ,

$$\int_0^1 x(y)W(y) dy > \int_0^Q W(y) dy, \quad (35)$$

and, holding Q fixed, this gap is increasing in the width of the lottery range.⁶¹

By Lemma A.3(iii), the equilibrium at a minimum wage in the domain is in the constrained two-wage region if and only if $Q^*(\underline{w}) > S(\underline{w})$, so the set of such minimum wages is open in the domain because Q^* and S are continuous. In the Robinson region, employment is $S(\underline{w})$ and the equilibrium allocation employs exactly the workers willing to work at $\underline{w}$. Thus $SS(\underline{w}) = \int_0^{S(\underline{w})} (V(y) - W(y)) dy$, which is increasing in $\underline{w}$ because S is increasing and $V(y) \geq W(y)$ holds for all $y \leq Q^p$. Similarly, $WS(\underline{w}) = \int_0^{S(\underline{w})} (\underline{w} - W(y)) dy$, which is increasing in $\underline{w}$ because both $\underline{w}$ and $S(\underline{w})$ increase and the integrand is positive.

Lemma 3 applies on each component of the constrained two-wage region. Take $\underline{w} < \underline{w}'$ in the same component, close enough that $S(\underline{w}') < Q^*(\underline{w})$, which is possible because $S(\underline{w}) < Q^*(\underline{w})$ and S is continuous. Write Q^* and q_i^* for the equilibrium quantity and mechanism parameters at $\underline{w}$, and $Q^{*'}$ and $q_i^{*'}$ for those at $\underline{w}'$. We compare the two equilibrium allocations through the intermediate allocation $\tilde{x}$ that uses the lottery range $(q_1^{*'}, q_2^{*'})$ of the allocation at $\underline{w}'$ to procure the quantity Q^* .⁶²

Passing from the allocation at $\underline{w}$ to $\tilde{x}$ holds the quantity fixed at Q^* and narrows the lottery range, since $q_1^* \leq q_1^{*'}$ and $q_2^{*' \leq} q_2^*$ hold by Lemma 3(iii). Because the gap in (35) is

⁶⁰Let the transition occur at $\underline{w}_0$. By Lemma 3(iii), $q_2^*(Q^*(\underline{w}), \underline{w})$ increases as $\underline{w}$ decreases towards $\underline{w}_0$, so its limit exceeds $S(\underline{w}) > S(\underline{w}_0)$ for every $\underline{w}$ in the constrained two-wage region above $\underline{w}_0$. Since the equilibrium allocation at $\underline{w}_0$ is market clearing at $\underline{w}_0$, the probability $\beta(Q^*(\underline{w}), q_1^*, q_2^*)$ of being hired at the high wage vanishes as $\underline{w}$ decreases to $\underline{w}_0$, and $\lim_{\underline{w} \downarrow \underline{w}_0} q_1^*(Q^*(\underline{w}), \underline{w}) = S(\underline{w}_0) = Q^*(\underline{w}_0)$.

⁶¹Both claims follow from the same comparison. Let x and x' be allocations hiring the same total mass Q , with $x - x'$ non-negative on $[0, \hat{q})$ and non-positive on $(\hat{q}, 1]$ for some $\hat{q}$. Then $\int_0^1 (x(y) - x'(y)) dy = 0$. Subtracting $W(\hat{q})$ times this identity from the difference $\int_0^1 (x(y) - x'(y)) W(y) dy$ in aggregate opportunity costs gives $\int_0^1 (x(y) - x'(y)) (W(y) - W(\hat{q})) dy$, whose integrand is non-positive throughout because W is strictly increasing. The integral is strictly negative whenever $x \neq x'$ on a set of positive measure. For (35), take x to be the allocation that hires the lowest-cost Q workers, take x' to be the two-wage allocation, and set $\hat{q} = Q$. For the second claim, take x and x' to be two-wage allocations with lottery ranges $(q_1, q_2) \subseteq (q_1', q_2')$ and lottery probabilities β and β' , respectively. Then $x - x'$ equals 0 on $[0, q_1']$, equals $1 - \beta' \geq 0$ on (q_1', q_1) , equals $\beta - \beta'$ on (q_1, q_2) , equals $-\beta' \leq 0$ on (q_2, q_2') , and equals 0 above q_2' . Setting $\hat{q} = q_1$ if $\beta \leq \beta'$ and $\hat{q} = q_2$ otherwise gives the second claim.

⁶²This is well defined, that is, $q_1^{*' <} Q^* < q_2^{*'}$: the second inequality holds because $Q^* < Q^{*' <} q_2^{*'}$, and the first because $q_1^{*' <} S(\underline{w}') < Q^*$ by the first paragraph of the proof of Lemma 2.

increasing in the width of the lottery range, the aggregate opportunity cost weakly falls, and social surplus weakly rises. Passing from $\tilde{x}$ to the allocation at $\underline{w}'$ holds the lottery range fixed at $(q_1^{*'}, q_2^{*'})$ and raises employment from Q^* to $Q^{*'}$, where $Q^{*'} > Q^*$ holds by Lemma 3(i). With the lottery range fixed, each additional unit of employment generates revenue V and raises the aggregate opportunity cost by the average of W over the lottery range, that is, by $\frac{1}{q_2^{*'} - q_1^{*'}} \int_{q_1^{*'}}^{q_2^{*'}} W(y) dy$ per unit.⁶³ Moreover, V exceeds this average at every employment level in $[Q^*, Q^{*'}]$: since V is decreasing, it suffices to note that $V(Q^{*'}) \geq \underline{C}'_R(Q^{*'}, \underline{w}') \geq W(q_2^{*'})$ by (9) and (33), and that $W(q_2^{*'})$ exceeds the average of the strictly increasing function W over $(q_1^{*'}, q_2^{*'})$. Social surplus therefore rises in this step as well. Social surplus is thus locally increasing at every point of the constrained two-wage region. Hence it is increasing on each component of that region and, by continuity, on the closure of each component. Because the minimum wage binds in the constrained two-wage region, the firm's profit is decreasing in $\underline{w}$ there by revealed preference. Worker surplus equals social surplus minus the firm's profit, so it is likewise increasing on the closure of each component.

Now take arbitrary $\underline{w} < \underline{w}'$ in the domain. If both lie in the closure of the same component of the constrained two-wage region, or both lie in the Robinson region, the required comparison has been established. Otherwise, $[\underline{w}, \underline{w}']$ contains minimum wages in the Robinson region; let a and b denote the smallest and the largest. Then $\underline{w}$ and a lie in the closure of the same component or coincide, and so do b and $\underline{w}'$. Combining the comparisons on the closures of the components with the monotonicity in the Robinson region yields $SS(\underline{w}) \leq SS(a) \leq SS(b) \leq SS(\underline{w}')$, and likewise for WS .

Finally, for any minimum wage and any feasible mechanism with employment Q , the aggregate opportunity cost is at least $\int_0^Q W(y) dy$, because the allocation hiring the lowest-cost Q workers minimizes aggregate opportunity cost at that employment level. Hence $SS \leq \int_0^Q (V(y) - W(y)) dy \leq \int_0^{Q^p} (V(y) - W(y)) dy$, where the second inequality holds because $V(y) \geq W(y)$ for $y \leq Q^p$ and $V(y) < W(y)$ for $y > Q^p$. At $\underline{w} = W(Q^p)$, employment is Q^p and every worker with cost below $W(Q^p)$ is employed. Thus $SS(W(Q^p)) = \int_0^{Q^p} (V(y) - W(y)) dy$ attains the bound, and social surplus is maximized at the competitive wage. For worker surplus, at any $\underline{w} < W(Q^p)$ we have $WS(\underline{w}) \leq \underline{C}_R(Q^*(\underline{w}), \underline{w}) - \int_0^{Q^*(\underline{w})} W(y) dy$. Because $\underline{C}_R$ is increasing in both arguments and $Q^*(\underline{w}) \leq Q^p$, the right-hand side is at most $\underline{C}_R(Q^p, W(Q^p)) - \int_0^{Q^p} W(y) dy = WS(W(Q^p))$. It remains to show that worker surplus attains a maximum on $[W(Q^p), W(1)]$. There, by Lemma A.3(i), the firm

⁶³An allocation with lottery range $(q_1^{*'}, q_2^{*'})$ and employment Q hires the mass $q_1^{*'}$ of lowest-cost workers with probability one and the workers with costs in $(W(q_1^{*'}), W(q_2^{*'}))$ with probability $\beta(Q, q_1^{*'}, q_2^{*'})$. Its aggregate opportunity cost is therefore $\int_0^{q_1^{*'}} W(y) dy + \beta(Q, q_1^{*'}, q_2^{*'}) \int_{q_1^{*'}}^{q_2^{*'}} W(y) dy$, and the derivative of this expression with respect to Q is $\frac{1}{q_2^{*'} - q_1^{*'}} \int_{q_1^{*'}}^{q_2^{*'}} W(y) dy$.

employs $D(\underline{w})$ of the $S(\underline{w})$ workers willing to work at $\underline{w}$, each with probability $D(\underline{w})/S(\underline{w})$, so that

$$WS(\underline{w}) = \frac{D(\underline{w})}{S(\underline{w})} \int_0^{S(\underline{w})} (\underline{w} - W(y)) dy.$$

The integral term is continuous and positive in $\underline{w}$, and D is upper semicontinuous, since $\{\underline{w} : D(\underline{w}) \geq Q\} = \{\underline{w} : V(Q) \geq \underline{w}\}$ is closed for every Q . Hence WS is upper semicontinuous on the closed interval $[W(Q^p), W(1)]$ and attains its maximum there. Worker surplus is therefore maximized either at $W(Q^p)$ or at a higher minimum wage. $\square$

A.9 Proof of Proposition 4

Proof. Write $Q_0 := Q^*(\underline{w}_0)$, and write q_1^* and q_2^* for $q_1^*(Q_0, \underline{w}_0)$ and $q_2^*(Q_0, \underline{w}_0)$, so that $w_2^* = W(q_2^*)$.

We first show that the firm never pays a high wage above the marginal revenue product of labor, that is, $V(Q_0) \geq w_2^*$. By Part V of the proof of Theorem 1, $\underline{C}'_R(Q_0, \underline{w}_0) \geq W(q_2^*)$. If the minimum wage binds, this is (33), since $q_2^* = q_2^{\lambda^*}(Q_0)$ by the first paragraph of the proof of Lemma 2. If it does not bind, then $q_2^* = Q_2(m)$ for the laissez-faire ironing interval $(Q_1(m), Q_2(m))$ containing Q_0 , and the inequality is (33) at $\lambda^* = 0$. Together with the optimality condition (9) at Q_0 , this gives $V(Q_0) \geq \underline{C}'_R(Q_0, \underline{w}_0) \geq W(q_2^*) = w_2^*$.

For (i), fix $\underline{w} \in (\underline{w}_0, w_2^*]$. We show that $\underline{C}'_R(Q_0, \underline{w}) \leq V(Q_0)$, so that the firm does not stop short of Q_0 . If $\underline{w} \geq W(Q_0)$, then $Q_0 \leq S(\underline{w})$, so by (8) the marginal cost at Q_0 is $\underline{w}$, and $\underline{w} \leq w_2^* \leq V(Q_0)$. If instead $\underline{w} < W(Q_0)$, then $\underline{C}'_R(Q_0, \cdot)$ is decreasing on $[W(0), W(Q_0))$ by Part V of the proof of Theorem 1, so $\underline{C}'_R(Q_0, \underline{w}) \leq \underline{C}'_R(Q_0, \underline{w}_0) \leq V(Q_0)$. In either case $Q^*(\underline{w}) \geq Q_0$, so employment increases. Workers' total pay is $\underline{C}_R(Q^*(\underline{w}), \underline{w})$, which increases because $\underline{C}_R$ is increasing in both arguments by Theorem 1, $Q^*(\underline{w}) \geq Q_0$ and $\underline{w} > \underline{w}_0$. This proves (i).

For (ii), suppose the prevailing minimum wage does not bind, so that the optimal mechanism at $(Q_0, \underline{w}_0)$ is the laissez-faire one and $Q_0 = Q^\ell$ by Lemma A.3(ii) and (iii). We then have $q_2^* = Q_2(m)$, where $m \in \mathcal{M}$ is such that $Q_0 \in (Q_1(m), Q_2(m))$, and involuntary unemployment is $Q_2(m) - Q_0$.

Fix $\underline{w} \in (\underline{w}_0, w_2^*]$. We show that the mass of workers entering the employment lottery at $\underline{w}$ is at most $Q_2(m)$; since $Q^*(\underline{w}) \geq Q_0$ by (i), involuntary unemployment at $\underline{w}$ is then at most $Q_2(m) - Q_0$. If the firm sets a uniform wage, this mass is $S(\underline{w}) \leq S(w_2^*) = Q_2(m)$. Suppose instead that the firm uses a non-degenerate two-wage mechanism, so that the equilibrium is in the constrained two-wage region, and let $\lambda^{*'} denote the largest solution value of the$

multiplier at $(Q^*(\underline{w}), \underline{w})$, so that the mechanism's parameters $q_1^{*'} and $q_2^{*'}$ are the endpoints of the ironing interval of $C_{\lambda^{*'}}$ containing $Q^*(\underline{w})$ by the first paragraph of the proof of Lemma 2. By (i) and Definition 1(ii),$

$$Q_1(m) < Q_0 \leq Q^*(\underline{w}) \leq (w_1^\ell)^{-1}(\underline{w}) \leq Q_2(m),$$

where the last inequality uses $\underline{w} \leq w_2^* = w_1^\ell(Q_2(m))$, and $Q^*(\underline{w}) < Q_2(m)$ because the optimal mechanism at $(Q_2(m), \underline{w})$ is the market-clearing laissez-faire one by Case 3 of Part III of the proof of Theorem 1. Thus $Q^*(\underline{w})$ lies in the laissez-faire ironing interval, that is, $0 \in \Lambda(Q^*(\underline{w}))$ in the notation of Part IV of that proof. If $\lambda^{*' = 0$, then $q_2^{*' = Q_2(m)}$. Otherwise, $\lambda^{*' \in \Lambda(Q^*(\underline{w}))$ and $\lambda^{*' \leq q_1^{*'}$ by Lemma A.1(iii), so Lemma A.1(iv) applied at $Q^*(\underline{w})$ to the pair $0 < \lambda^{*'}$ gives $q_2^{*' \leq Q_2(m)}$.

Now set $\underline{w} = w_2^* = W(Q_2(m))$. Since the ironing intervals are disjoint, $S(\underline{w}) = Q_2(m)$ lies outside every ironing interval, so $(w_1^\ell)^{-1}(\underline{w}) = S(\underline{w})$ and, by (8), the firm hires at the uniform wage $\underline{w}$ at every quantity up to $S(\underline{w})$. It does not hire more than $S(\underline{w}) = Q_2(m)$ workers: for $Q \geq Q_2(m)$ the minimum wage does not bind, so by (8) $\Pi(\cdot, \underline{w})$ coincides there with the laissez-faire profit, which is strictly decreasing above $Q^\ell = Q_0 < Q_2(m)$. Employment is thus $\min\{Q_2(m), D(w_2^*)\}$, so involuntary unemployment is eliminated if and only if $D(w_2^*) \geq Q_2(m)$. Since $D(w_2^*) \geq Q_2(m)$ holds exactly when $V(Q_2(m)) \geq W(Q_2(m))$, and V and W cross at Q^p , this is equivalent to $Q_2(m) \leq Q^p$, that is, to $w_2^* \leq W(Q^p)$.

Finally, suppose $w_2^* \leq W(Q^p)$, so that, as just shown, employment at $\underline{w} = w_2^*$ equals $S(\underline{w}) = Q_2(m) \leq Q^p$ and every participant is hired at the market-clearing wage $W(Q_2(m))$. We compare this allocation with the laissez-faire allocation in two steps, as in the proof of Proposition 3. First, holding the quantity fixed at Q_0 , replacing the laissez-faire allocation with the allocation that hires the lowest-cost Q_0 workers lowers the workers' aggregate opportunity cost by (35) and therefore raises social surplus. Second, raising the quantity from Q_0 to $Q_2(m)$ while hiring the lowest-cost workers adds $V(y) - W(y)$ for each additional unit y . Since $y \leq Q_2(m) \leq Q^p$, V is decreasing and W is increasing, $V(y) \geq V(Q^p) = W(Q^p) \geq W(y)$. Social surplus therefore increases relative to laissez-faire. Because the firm's profit is weakly lower than under laissez-faire by revealed preference, worker surplus, which is the difference between social surplus and the firm's profit, increases as well. This proves (ii). $\square$

B Supporting calculations

B.1 Sufficient condition for Assumption 1

Suppose that W is twice continuously differentiable and concave, that $\{Q \in [0, 1] : C'''(Q) < 0\}$ is an interval $(\underline{a}, \bar{a})$, possibly empty, and that $-W''/W'$ is log-concave on $(\underline{a}, \bar{a})$. Fix $\lambda \geq 0$. Since $W'' \leq 0$, $C''_\lambda = 2W' + (Q - \lambda)W'' > 0$ for $Q \leq \lambda$. For $Q > \lambda$, substituting $W'' = (C''' - 2W')/Q$ gives $C''_\lambda = (1 - \lambda/Q)C''' + (\lambda/Q)2W'$, which is positive wherever $C''' \geq 0$. For $Q \in (\underline{a}, \bar{a})$ with $Q > \lambda$, $C''_\lambda < 0$ if and only if $(Q - \lambda)(-W'''(Q)/W'(Q)) > 2$. The left-hand side is a product of log-concave functions of Q , hence quasi-concave, so the set on which it exceeds 2 is an interval (a_λ, b_λ) , possibly empty. Thus C_λ is strictly convex on $[0, a_\lambda]$ and $[b_\lambda, 1]$ and strictly concave on $[a_\lambda, b_\lambda]$.

Now suppose that an affine function lies below C_λ on $[0, 1]$ and agrees with it at three points $x < y < z$. The difference between C_λ and this affine function is non-negative, vanishes at x, y and z , and has the same convexity pattern as C_λ . A strictly convex function that vanishes at two points is negative between them, so $y \leq a_\lambda$ and $y \geq b_\lambda$ are both impossible. But a strictly concave function that is non-negative at two points is positive between them, so the difference cannot vanish at $y \in (a_\lambda, b_\lambda)$ either. Hence no affine function below C_λ touches it at more than two points, which is Assumption 1.

B.2 Supporting calculations for the leading example

This appendix establishes the “parallel shift” property illustrated in Panel (b) of Figure 7. Under the piecewise linear specification of W introduced below, the wages of the optimal constrained two-wage mechanism for procuring $\tilde{Q}$ under the minimum wage $\underline{w}$ lie on the parallel upward shift $w_1^\ell(Q) + \Delta(\tilde{Q}, \underline{w})$ of the laissez-faire low wage function, where $\Delta(\tilde{Q}, \underline{w}) := \underline{w} - w_1^\ell(\tilde{Q})$.

The argument uses two objects from Part III of the proof of Theorem 1. For an ironing interval $(q_1^\lambda, q_2^\lambda)$ of C_λ , the chord L_λ^W is the affine function agreeing with W at q_1^λ and q_2^λ . The low wage of the two-wage mechanism parameterized by $(Q, q_1^\lambda, q_2^\lambda)$ satisfies $w_1^\lambda(Q) = L_\lambda^W(Q)$ for $Q \in (q_1^\lambda, q_2^\lambda)$. We begin with a lemma that holds for a general inverse labor supply schedule W and pins down the slope of L_λ^W .

Lemma B.1. *Fix $\lambda \geq 0$ and let $(q_1^\lambda, q_2^\lambda)$ be an ironing interval of C_λ with $0 < q_1^\lambda$ and $\lambda \leq q_1^\lambda$, and suppose that W is differentiable at q_1^λ and q_2^λ . Then $\lambda < q_1^\lambda$ and*

$$\left(\frac{W(q_2^\lambda) - W(q_1^\lambda)}{q_2^\lambda - q_1^\lambda} \right)^2 = W'(q_1^\lambda) W'(q_2^\lambda). \quad (36)$$

Proof. Write $q_i := q_i^\lambda$, $u_i := q_i - \lambda$, $\Delta q := q_2 - q_1$ and $\Delta W := W(q_2) - W(q_1)$, so that $u_1 \geq 0$, $u_2 = u_1 + \Delta q > 0$ and, since W is strictly increasing, $\Delta W/\Delta q > 0$. Because $C_\lambda(q) = (q - \lambda)W(q)$, the first-order conditions characterizing the endpoints of the ironing interval are

$$W(q_1) + u_1 W'(q_1) = \frac{u_2 W(q_2) - u_1 W(q_1)}{\Delta q} = W(q_2) + u_2 W'(q_2). \quad (37)$$

Substituting $W(q_2) = W(q_1) + \Delta W$ and $u_2 = u_1 + \Delta q$ into the numerator of the middle term gives $u_2 W(q_2) - u_1 W(q_1) = W(q_1)\Delta q + u_2 \Delta W$, so the middle term equals $W(q_1) + u_2 \Delta W/\Delta q$ and the first equality in (37) reduces to $u_1 W'(q_1) = u_2 \Delta W/\Delta q$. Substituting instead $W(q_1) = W(q_2) - \Delta W$ and $u_1 = u_2 - \Delta q$ gives $u_2 W(q_2) - u_1 W(q_1) = W(q_2)\Delta q + u_1 \Delta W$, so the middle term also equals $W(q_2) + u_1 \Delta W/\Delta q$ and the second equality reduces to $u_2 W'(q_2) = u_1 \Delta W/\Delta q$. Since $u_2 \Delta W/\Delta q > 0$, the first of these implies $u_1 > 0$, that is, $\lambda < q_1$. Multiplying the two and dividing by $u_1 u_2 > 0$ yields (36). $\square$

We now apply Lemma B.1 to the piecewise linear specification

$$W(Q) = \begin{cases} aQ, & Q \in [0, \underline{q}], \\ bQ + (a - b)\underline{q}, & Q \in [\underline{q}, 1], \end{cases} \quad (38)$$

where $a > b > 0$ and $\underline{q} \in (0, 1)$. Setting $a = 4$, $b = 1/2$ and $\underline{q} = 1/4$ yields the leading example (6). Under (38), C_λ is quadratic with a positive leading coefficient on each of $[0, \underline{q}]$ and $[\underline{q}, 1]$, and hence is convex on each interval. Any ironing interval of C_λ therefore contains the kink $\underline{q}$ in its interior, so W is differentiable at its endpoints, with $W'(q_1^\lambda) = a$ and $W'(q_2^\lambda) = b$. Since ironing intervals are disjoint, C_λ has at most one. In particular, taking $\lambda = 0$, there is a single laissez-faire ironing interval, which we denote (Q_1, Q_2) . Whenever the remaining hypotheses of Lemma B.1 hold, the chord L_λ^W thus has slope $\sqrt{ab}$.

Now fix $\underline{w} \in (W(Q_1), W(Q_2))$ and $\tilde{Q} \in \left(S(\underline{w}), (w_1^\ell)^{-1}(\underline{w})\right)$, so that the firm optimally procures $\tilde{Q}$ using a non-degenerate two-wage mechanism under a binding minimum wage, and let λ^* denote the largest solution value of the Lagrange multiplier at $(\tilde{Q}, \underline{w})$. The hypotheses of Lemma B.1 hold at λ^* , since $q_1^{\lambda^*} > 0$ by non-degeneracy and $\lambda^* \leq q_1^{\lambda^*}$ by Lemma A.1(iii). They also hold at $\lambda = 0$, since $Q_1 > 0$ by Section 3.1. The chords $L_{\lambda^*}^W$ and L_0^W are therefore parallel with slope $\sqrt{ab}$, and L_0^W coincides with w_1^ℓ on (Q_1, Q_2) . By Lemma A.1(iii), $L_{\lambda^*}^W(\tilde{Q}) = w_1^{\lambda^*}(\tilde{Q}) = \underline{w}$, whereas $L_0^W(\tilde{Q}) = w_1^\ell(\tilde{Q})$. The two parallel chords thus differ by the constant $\Delta(\tilde{Q}, \underline{w})$, which is positive because $\tilde{Q} < (w_1^\ell)^{-1}(\underline{w})$ and w_1^ℓ is strictly increasing. The wages of the optimal mechanism lie on $L_{\lambda^*}^W$, so they lie on the parallel upward shift $w_1^\ell + \Delta(\tilde{Q}, \underline{w})$, as claimed. For (6), the common slope is $\sqrt{2}$.

B.3 Optimal procurement without Assumption 1

This appendix considers optimal procurement without the separation condition (Assumption 1). Three ingredients of the analysis do not rely on this assumption, and we use them freely: Part I of the proof of Theorem 1, which reduces the firm's problem to (17); the identification of the value of (18) with $\underline{C}_\lambda(Q)$ in Part II; and Proposition 1. Without Assumption 1, the optimal mechanism is not necessarily unique. We continue to let $Q^*(\underline{w})$ denote the largest profit-maximizing quantity. As in Section 3.2, the non-trivial case is that in which $w_1^\ell(Q) < \underline{w} < W(Q)$, so that the minimum wage constraint binds but the firm does not simply set a uniform wage. Throughout, $s_\lambda(Q)$ denotes the slope of $\underline{C}_\lambda$ at Q , as in Part IV of the proof of Theorem 1. We also extend the notation of Part III of that proof: L_λ^C denotes the supporting line of $\underline{C}_\lambda$ at Q with slope $s_\lambda(Q)$, that is, $L_\lambda^C(q) = \underline{C}_\lambda(Q) + s_\lambda(Q)(q - Q)$. This line coincides with the chord of Part III whenever Q lies in an ironing interval of C_λ . We say that L_λ^C touches C_λ at q , and call q a contact point of L_λ^C , if $C_\lambda(q) = L_\lambda^C(q)$.

Proposition B.1. *Suppose that Assumption 1 does not hold.*

- (i) *For every $(Q, \underline{w})$, an optimal mechanism paying at most three wages exists. Three wages are required only if $w_1^\ell(Q) < \underline{w} < W(Q)$, and hence only if $\underline{C}(Q) < C(Q)$.*
- (ii) *Suppose that $w_1^\ell(Q_0) < \underline{w}_0 < W(Q_0)$ and that an optimal measure in (17) at $(Q_0, \underline{w}_0)$ places positive probability on three distinct cutoffs $q_1 < q_2 < q_3$. Then there exist $\lambda_0 \in (0, Q_0)$ and a neighborhood N of $(Q_0, \underline{w}_0)$ such that, for all $(Q, \underline{w}) \in N$,*

$$\underline{C}_R(Q, \underline{w}) = \underline{C}_R(Q_0, \underline{w}_0) + s_{\lambda_0}(Q_0)(Q - Q_0) + \lambda_0(\underline{w} - \underline{w}_0), \quad (39)$$

and the measure μ supported on $\{q_1, q_2, q_3\}$ with $\int_0^1 q d\mu(q) = Q$ and $\int_0^1 W(q) d\mu(q) = \underline{w}$ is optimal. Its probabilities are strictly positive and affine in $(Q, \underline{w})$. In particular, $\underline{C}'_R(Q, \underline{w}) = s_{\lambda_0}(Q_0)$ on N .

- (iii) *If the hypothesis of (ii) holds at $(Q^*(\underline{w}_0), \underline{w}_0)$, then $Q^*(\underline{w}) = Q^*(\underline{w}_0)$ for all $\underline{w}$ in a neighborhood of $\underline{w}_0$.*

Proof. If $\underline{w} \geq W(Q)$, the argument of Case 1 in Part III of the proof of Theorem 1 applies verbatim, and the uniform-wage mechanism is optimal. If $\underline{w} \leq w_1^\ell(Q)$, the laissez-faire mechanism of Proposition 1 satisfies the minimum wage constraint, and it is optimal because $\underline{C}_R(Q, \underline{w}) \geq \underline{C}(Q)$. This proves the second claim in (i), since $w_1^\ell(Q) < W(Q)$ implies $\underline{C}(Q) < C(Q)$ by Proposition 1. For the remainder of the proof, fix $(Q, \underline{w})$ with $w_1^\ell(Q) < \underline{w} < W(Q)$.

We first describe the minimizers of (18). Fix $\lambda \in [0, Q]$. Since $L_\lambda^C \leq \underline{C}_\lambda \leq C_\lambda$ and $L_\lambda^C(Q) = \underline{C}_\lambda(Q)$, a measure with mean Q minimizes (18) if and only if it is supported on the

contact points of L_λ^C . Since $C_\lambda(\lambda) = 0$, we have $L_\lambda^C(\lambda) \leq 0$. Substituting $C_\lambda(q) = (q-\lambda)W(q)$ into $C_\lambda(q) = L_\lambda^C(q)$ shows that every contact point $q \neq \lambda$ satisfies

$$W(q) = s_\lambda(Q) + \frac{L_\lambda^C(\lambda)}{q - \lambda}. \quad (40)$$

Thus $W \geq s_\lambda(Q)$ at contact points below λ and $W \leq s_\lambda(Q)$ at contact points above λ , so there are no contact points on both sides of λ . Moreover, if some $q' > \lambda$ is a contact point, then λ is not (otherwise $L_\lambda^C(q) = (q - \lambda)W(q') > C_\lambda(q)$ for all $q \in (\lambda, q')$, contradicting $L_\lambda^C \leq C_\lambda$). Now suppose that L_λ^C has two or more contact points and that some measure with mean $Q > \lambda$ is supported on them. Then all contact points lie in $(\lambda, 1]$. Since W is strictly increasing, (40) then gives $L_\lambda^C(\lambda) < 0$. Consequently, $W < s_\lambda(Q)$ at every contact point, and W is strictly concave on the set of contact points.

We next choose the multiplier. Consider the function $\lambda \mapsto \underline{C}_\lambda(Q) + \lambda \underline{w}$ on $[0, Q]$. Since $\underline{C}_\lambda(Q)$ is the minimum of $\int_0^1 C(q) d\mu(q) - \lambda \int_0^1 W(q) d\mu(q)$ over the compact set of measures with mean Q , this function is concave. By the envelope theorem, its right and left derivatives at λ are $\underline{w} - \max \int_0^1 W(q) d\mu(q)$ and $\underline{w} - \min \int_0^1 W(q) d\mu(q)$, where the maximum and minimum are taken over the minimizers at λ . At $\lambda = Q$, the unique minimizer is δ_Q : indeed, $\int_0^1 (q - Q)W(q) d\mu(q) \geq W(Q) \int_0^1 (q - Q) d\mu(q) = 0$, with equality only if $\mu = \delta_Q$. The left derivative at Q is therefore $\underline{w} - W(Q) < 0$. At $\lambda = 0$, Proposition 1 places Q in an ironing interval $(Q_1(m), Q_2(m))$ of C . Hence L_0^C is the chord of C over this interval, and its contact points lie outside the interval. By (40), every minimizer μ at $\lambda = 0$ satisfies $\int_0^1 W(q) d\mu(q) = s_0(Q) + L_0^C(0) \int_0^1 q^{-1} d\mu(q)$, where $L_0^C(0) < 0$ by the preceding paragraph. Since q^{-1} is convex, it lies weakly above its chord over $[Q_1(m), Q_2(m)]$ at every contact point. The right-hand side is therefore maximized by the measure on $\{Q_1(m), Q_2(m)\}$ with mean Q , which corresponds to the laissez-faire mechanism of Proposition 1. The maximum is $w_1^\ell(Q)$, so the right derivative at 0 is $\underline{w} - w_1^\ell(Q) > 0$. Hence the function attains its maximum over $[0, Q]$ at some $\lambda \in (0, Q)$. At this λ , $\underline{w}$ lies between the smallest and largest values of $\int_0^1 W(q) d\mu(q)$ over the minimizers. Since the minimizers form a convex set, some minimizer μ satisfies $\int_0^1 W(q) d\mu(q) = \underline{w}$. As $\underline{w} < W(Q)$, $\mu \neq \delta_Q$, so L_λ^C has two or more contact points and the preceding paragraph applies. In particular, $\underline{w} = \int_0^1 W(q) d\mu(q) < s_\lambda(Q)$, and hence $L_\lambda^C(0) + \lambda \underline{w} = L_\lambda^C(\lambda) - \lambda (s_\lambda(Q) - \underline{w}) < 0$.

We now verify optimality by a direct bound. Fix $q \in [0, 1]$ and $\tilde{w}$ with $L_\lambda^C(0) + \lambda \tilde{w} < 0$, and let $(\nu, U(\bar{c}))$ be feasible in (17) at $(q, \tilde{w})$. Substituting $C = C_\lambda + \lambda W$ on $(0, 1]$ and

$C(0) = 0$ gives

$$\begin{aligned} \int_0^1 C(z) d\nu(z) + U(\bar{c}) - (L_\lambda^C(q) + \lambda\tilde{w}) &= \int_{(0,1]} (C_\lambda(z) - L_\lambda^C(z)) d\nu(z) \\ - (L_\lambda^C(0) + \lambda\tilde{w}) \nu(\{0\}) + \lambda \left[\int_{(0,1]} (W(z) - \tilde{w}) d\nu(z) + U(\bar{c}) \right] &+ (1 - \lambda)U(\bar{c}) \geq 0. \end{aligned} \quad (41)$$

Each term on the right-hand side is non-negative: the first because $L_\lambda^C \leq C_\lambda$, the second because $L_\lambda^C(0) + \lambda\tilde{w} < 0$, the third because $(\nu, U(\bar{c}))$ satisfies (16), and the last because $\lambda < 1$. Hence $\underline{C}_R(q, \tilde{w}) \geq L_\lambda^C(q) + \lambda\tilde{w}$. Equality holds for $U(\bar{c}) = 0$ and any measure ν supported on the contact points of L_λ^C , which lie in $(\lambda, 1]$, with $\int_0^1 z d\nu(z) = q$ and $\int_0^1 W(z) d\nu(z) = \tilde{w}$. Conversely, any optimal pair at a point where equality is attained has these properties. The measure μ constructed above attains equality at $(Q, \underline{w})$. It is therefore optimal, and $\underline{C}_R(Q, \underline{w}) = \underline{C}_\lambda(Q) + \lambda\underline{w}$. Finally, $(Q, \underline{w}) = \int_0^1 (q, W(q)) d\mu(q)$ lies in the convex hull of the compact set of points $(q, W(q))$ with q a contact point. By Carathéodory's theorem, it is a convex combination of at most three such points. The corresponding measure is supported on contact points, has mean Q and the same value of $\int_0^1 W(q) d\mu(q)$, and is therefore optimal. By Part I, it corresponds to a mechanism with at most three employment probabilities, and hence at most three wages. This proves (i).

For (ii), apply the above at $(Q_0, \underline{w}_0)$ and let λ_0 denote the resulting multiplier. Then $\underline{C}_R(Q_0, \underline{w}_0) = L_{\lambda_0}^C(Q_0) + \lambda_0\underline{w}_0$, and the right-hand side of (39) equals $L_{\lambda_0}^C(Q) + \lambda_0\underline{w}$. Since the given measure is optimal, it attains equality in (41). Hence q_1, q_2, q_3 are contact points of $L_{\lambda_0}^C$, and $\sum_i p_i W(q_i) = \underline{w}_0$, where $p_i > 0$ are the probabilities of the measure. By (40), W is strictly concave on the contact points, so the points $(q_i, W(q_i))$ are not collinear, and $(Q_0, \underline{w}_0) = \sum_i p_i (q_i, W(q_i))$ lies in the interior of their triangle. The linear system $\sum_i p_i = 1$, $\sum_i p_i q_i = Q$, $\sum_i p_i W(q_i) = \underline{w}$ therefore has a unique solution $p(Q, \underline{w})$. This solution is affine in $(Q, \underline{w})$ and strictly positive on a neighborhood N of $(Q_0, \underline{w}_0)$, which we take small enough that $L_{\lambda_0}^C(0) + \lambda_0\underline{w} < 0$ on N . For $(Q, \underline{w}) \in N$, the measure with probabilities $p(Q, \underline{w})$ on $\{q_1, q_2, q_3\}$ attains equality in (41) at $(Q, \underline{w})$. This proves (39) and the optimality claim.

For (iii), let $Q_0 := Q^*(\underline{w}_0)$, and let λ_0 and N be as in (ii). Near Q_0 , the firm's profit $\Pi(\cdot, \underline{w}_0)$ equals $\int_0^Q V(y) dy - L_{\lambda_0}^C(Q) - \lambda_0\underline{w}_0$ by (39). Since Q_0 is an interior maximizer, $V(Q_0) = s_{\lambda_0}(Q_0)$. Since V is decreasing, Q_0 maximizes the concave function $\int_0^Q V(y) dy - s_{\lambda_0}(Q_0)Q$ on $[0, 1]$. It is moreover its largest maximizer: otherwise this function, and hence $\Pi(\cdot, \underline{w}_0)$, would be constant on an interval to the right of Q_0 , contradicting the definition of

Q_0 . Now fix $\underline{w}$ with $(Q_0, \underline{w}) \in N$. For every Q , (41) and (39) give

$$\Pi(Q, \underline{w}) \leq \int_0^Q V(y) dy - L_{\lambda_0}^C(Q) - \lambda_0 \underline{w} \leq \int_0^{Q_0} V(y) dy - L_{\lambda_0}^C(Q_0) - \lambda_0 \underline{w} = \Pi(Q_0, \underline{w}),$$

where the second inequality holds because $L_{\lambda_0}^C$ has slope $s_{\lambda_0}(Q_0)$. Hence Q_0 maximizes $\Pi(\cdot, \underline{w})$. Any maximizer Q attains equality in the second inequality, so $Q \leq Q_0$. Thus $Q^*(\underline{w}) = Q_0$. $\square$

The employment effects in Theorem 2 and Proposition 2, which Section 5.1 asserts hold without Assumption 1, rely on this assumption in two places. First, the proof of Lemma A.3(i) uses $\underline{C}'_R(Q, \underline{w}) > W(Q)$ for $Q > S(\underline{w})$, which follows from Theorem 1. The following proposition shows that this inequality holds for the right derivative without the assumption, which suffices for that proof. Second, the proof of Proposition 2(ii) uses the continuity of Q^* to exclude the Robinson region just above $w_1^\ell(Q^\ell)$. The monotonicity of Q^* established below suffices, since then $Q^*(\underline{w}) \geq Q^\ell > S(\underline{w})$ for $\underline{w}$ slightly above $w_1^\ell(Q^\ell)$ by Lemma A.3(ii) and the continuity of S . The remaining steps in those proofs use only Lemma A.3(i)–(iii), the monotonicity of Q^* , and the strict inequality $V(S(\underline{w})) \neq \gamma(S(\underline{w}))$ in Proposition 2(i), whose proof uses uniqueness of the maximizer only when equality holds.

Proposition B.2. *Suppose that Assumption 1 does not hold. Then (27) holds for all $(Q, \underline{w})$, the right derivative of $\underline{C}_R(\cdot, \underline{w})$ at Q exceeds $W(Q)$ for all $Q > S(\underline{w})$, and $Q^*(\underline{w})$ is increasing on $[W(0), W(Q^p)]$.*

Proof. Fix $(Q, \underline{w})$. As in the proof of Proposition B.1, the function $\lambda \mapsto \underline{C}_\lambda(Q) + \lambda \underline{w}$ is concave on $[0, Q]$, and we show that its maximum is $\underline{C}_R(Q, \underline{w})$. If $\underline{w} \geq W(Q)$, then $\underline{C}_R(Q, \underline{w}) = \underline{w}Q$ by Case 1 of Part III. The left derivative at Q is $\underline{w} - W(Q) \geq 0$, so the function is increasing, and its maximum is its value $Q\underline{w}$ at $\lambda = Q$, since $\underline{C}_Q(Q) = C_Q(Q) = 0$. If $\underline{w} \leq w_1^\ell(Q)$, then $\underline{C}_R(Q, \underline{w}) = \underline{C}(Q)$, which is the value at $\lambda = 0$. The right derivative at 0 is $\underline{w} - w_1^\ell(Q) \leq 0$: if Q lies in an ironing interval of C , this is the computation in the proof of Proposition B.1, and otherwise it follows from (40) and Jensen's inequality, since W is concave on the contact points of L_0^C . So the maximum is the value at 0. In the remaining case, the proof of Proposition B.1 shows that $\underline{C}_R(Q, \underline{w}) = \underline{C}_\lambda(Q) + \lambda \underline{w}$ at a maximizer λ . This proves (27). Consequently, $\underline{C}_R(Q, \cdot)$ is a maximum of affine functions of $\underline{w}$ with slopes in $[0, Q]$, hence convex, and its right derivative at $\underline{w}$ is the largest solution value $\lambda^*(Q, \underline{w})$.

Now let $Q > S(\underline{w})$, so that $\underline{w} < W(Q)$, and let λ be a solution value. By (27), $\underline{C}_R(\cdot, \underline{w}) \geq \underline{C}_\lambda(\cdot) + \lambda \underline{w}$ with equality at Q , so the right derivative of $\underline{C}_R(\cdot, \underline{w})$ at Q is at least $s_\lambda(Q)$. If $\underline{C}_\lambda(Q) < C_\lambda(Q)$, then L_λ^C has contact points $q < Q < q'$, which lie above λ by the second paragraph of the proof of Proposition B.1, and the computation in (33) applied to

q and q' gives $s_\lambda(Q) \geq W(q') > W(Q)$. If $\underline{C}_\lambda(Q) = C_\lambda(Q)$, then for any $q \in (\lambda, Q)$, $s_\lambda(Q) \geq \frac{C_\lambda(Q) - C_\lambda(q)}{Q - q} = W(Q) + \frac{(q - \lambda)(W(Q) - W(q))}{Q - q} > W(Q)$.

We next show that $\lambda^*(\cdot, \underline{w})$ is decreasing on $[S(\underline{w}), 1]$. For $\lambda \in [0, Q]$, let $w_1^\lambda(Q)$ denote the smallest value of $\int_0^1 W(q) d\mu(q)$ over the minimizers of (18). This extends the notation of Part III, where the minimizer is unique. By the envelope theorem, the left derivative of $\lambda \mapsto \underline{C}_\lambda(Q) + \lambda \underline{w}$ at $\lambda \in (0, Q]$ is $\underline{w} - w_1^\lambda(Q)$. By concavity, $\lambda^*(Q, \underline{w})$ is therefore the largest $\lambda \in (0, Q]$ with $w_1^\lambda(Q) \leq \underline{w}$, or 0 if there is none. At $Q = S(\underline{w})$, the left derivative at Q is $\underline{w} - W(Q) = 0$, so $\lambda^*(S(\underline{w}), \underline{w}) = S(\underline{w})$. For $Q > S(\underline{w})$, every solution value $\lambda > 0$ satisfies $w_1^\lambda(Q) \leq \underline{w} < W(Q)$, so L_λ^C has two or more contact points. By the second paragraph of the proof of Proposition B.1, they all lie strictly above λ , and if q' is the smallest, then $W(q') \leq w_1^\lambda(Q) \leq \underline{w}$, so $\lambda < S(\underline{w})$. It therefore suffices to show that, for fixed λ , $w_1^\lambda(Q)$ is increasing in Q on $(\lambda, 1]$. Then, for $S(\underline{w}) < Q < Q'$ and $\lambda' := \lambda^*(Q', \underline{w}) > 0$, we have $\lambda' < S(\underline{w}) < Q$ and $w_1^{\lambda'}(Q) \leq w_1^{\lambda'}(Q') \leq \underline{w}$, whence $\lambda^*(Q, \underline{w}) \geq \lambda'$.

Fix λ and $\lambda < Q < Q'$, so that $s_\lambda(Q) \leq s_\lambda(Q')$. As in the proof of Proposition B.1, the minimizers of (18) at Q are the measures with mean Q supported on the contact points of the supporting line of $\underline{C}_\lambda$ at Q , and likewise at Q' . If $s_\lambda(Q) = s_\lambda(Q')$, the two supporting lines coincide. On the convex hull of their contact points, $w_1^\lambda(y)$ is the minimum of $\int_0^1 W(q) d\mu(q)$ over measures μ with mean y supported on these contact points, which is a convex function of y and, since W is increasing, attains its minimum at the smallest contact point. Hence $w_1^\lambda(\cdot)$ is increasing on the hull, and $w_1^\lambda(Q) \leq w_1^\lambda(Q')$. If $s_\lambda(Q) < s_\lambda(Q')$, then every contact point q' at Q' lies weakly above every contact point q at Q : for $q' < q$, evaluating the two supporting lines at q gives $C_\lambda(q') + s_\lambda(Q')(q - q') > C_\lambda(q') + s_\lambda(Q)(q - q') \geq C_\lambda(q)$, contradicting that the supporting line at Q' lies below C_λ . Since W is increasing, $w_1^\lambda(Q') \geq w_1^\lambda(Q)$.

Finally, let $\underline{w} < \underline{w}'$ in $[W(0), W(Q^p)]$ and suppose that $Q' := Q^*(\underline{w}') < Q := Q^*(\underline{w})$. For $y \leq S(\underline{w}')$, the marginal cost at $\underline{w}'$ is $\underline{w}' \leq W(Q^p) \leq V(y)$ by Case 1 of Part III, so $Q' \geq S(\underline{w}') \geq S(\tau)$ for every $\tau \in [\underline{w}, \underline{w}']$. By the first paragraph, $\underline{C}_R(y, \underline{w}') - \underline{C}_R(y, \underline{w}) = \int_{\underline{w}}^{\underline{w}'} \lambda^*(y, \tau) d\tau$ for every y , and $\lambda^*(\cdot, \tau)$ is decreasing on $[Q', Q] \subseteq [S(\tau), 1]$. Hence $\underline{C}_R(Q, \underline{w}') - \underline{C}_R(Q, \underline{w}) \leq \underline{C}_R(Q', \underline{w}') - \underline{C}_R(Q', \underline{w})$. Rearranging, $\Pi(Q, \underline{w}') - \Pi(Q', \underline{w}') \geq \Pi(Q, \underline{w}) - \Pi(Q', \underline{w}) \geq 0$, so Q maximizes $\Pi(\cdot, \underline{w}')$, contradicting the definition of Q' as the largest maximizer. $\square$