Wage dispersion, minimum wages and involuntary unemployment: A mechanism design perspective *

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Abstract

Adopting a mechanism design approach, we show that wage dispersion and involuntary unemployment are optimal for a monopsony whenever the cost of procurement under market-clearing wages is not convex at the optimal level of employment. A minimum wage between the lowest equilibrium wage and the market-clearing wage decreases involuntary unemployment and increases employment. Whenever a minimum wage induces wage dispersion and involuntary unemployment, a small increase in that wage increases employment and decreases involuntary unemployment. Absent involuntary unemployment, a small increase in the minimum wage still generically increases employment. Extensions analyze quantity competition, horizontal differentiation, migration and unemployment insurance.

Keywords: Monopsony power, efficiency wages, quantity competition, wage regulation, employment
JEL-Classification: C72, D47, D82

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1 Introduction

Minimum wage legislation—which is once again featuring prominently in public policy proposals in the United States—has been around for over a century.[1] So too have debates among economists and policy makers concerning the effects of minimum wages on total employment, involuntary unemployment, and workers' pay. In models with price-taking firms and workers, minimum wages have either no effect, or induce involuntary unemployment and inefficiently low employment. However, as pointed out by Stigler (1946), if employers exert monopsony power over the labor market then an appropriately chosen minimum wage can increase workers' pay and employment without creating involuntary unemployment. The effects Stigler identified are consistent with the empirical findings of Card and Krueger (1994).

In this paper, we offer a novel perspective on the effects of minimum wages that nests the aforementioned approaches. We first analyze a model in which a monopsony employer faces a continuum of workers. Without a minimum wage, the monopsony's optimal procurement mechanism involves wage dispersion and induces involuntary unemployment whenever the procurement cost function—that is, the quantity procured multiplied by the market-clearing wage—is not convex at the optimal level of employment. Such a mechanism is optimal because it minimizes the total procurement cost subject to workers' incentive compatibility and individual rationality constraints. We then show that introducing a minimum wage between the lowest wage offered in equilibrium and the market-clearing wage at the equilibrium level of employment increases total employment and decreases involuntary unemployment. In fact, it is always possible to eliminate involuntary unemployment with an appropriately chosen minimum wage. Even a minimum wage equal to the highest wage offered absent regulation increases total employment relative to the case without wage regulation. However, it eliminates involuntary unemployment if and only if the perfectly competitive equilibrium wage is larger than this wage.

Figure 1 illustrates the effects of minimum wages for the textbook model that assumes price-taking behaviour; for Stigler's analysis of minimum wages under monopsony power; and for the case analyzed in this paper, in which the monoposony is allowed to use the optimal mechansim subject to workers' incentive compatibility and individual rationality constraints. If the cost of hiring labor at the market-clearing wage is not convex in the level of employment, then the optimal mechanism involves an efficiency wage—a wage exceeding the market clearing wage—and a low wage at which the workers with the lowest opportunity cost of working are employed. In such cases, randomly rationing the associated excess

[1] While precursors to minimum wage legislation date back to the Hammurabi Code (c. 1755–1750 BC), New Zealand became the first country to implement a minimum wage in 1894, followed by the Australian state of Victoria in 1896, and the United Kingdom in 1909 (Starr, 1981).

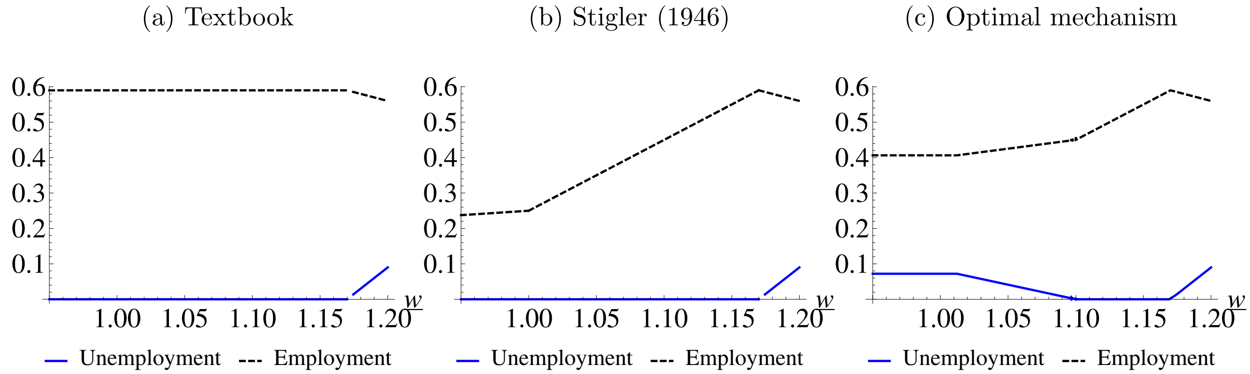

Figure 1

supply of labor allows the employer to procure labor at the lowest marginal cost, subject to the monotonicity constraint implied by the incentive compatibility constraints that workers with lower costs cannot be hired with lower probability than workers with higher costs.[2]

The richness and—at first glance—counterintuitive nature of the minimum wage effects when the employer uses an optimal mechanism raise the question of how a regulator or legislator could tell whether the problem at hand is such that increasing the minimum wage increases employment and decreases involuntary unemployment or achieves the contrary effects. As we show, the answer relates to whether or not there is wage dispersion before the minimum wage is increased marginally: If there is involuntary unemployment and wage dispersion, then a sufficiently small increase of the minimum wage will increase employment and decrease involuntary unemployment. Similarly, if there is no wage dispersion and no involuntary unemployment, then a sufficiently small increase in the minimum wage will generically increase employment.[3]

For a model of quantity competition in which the aggregate quantity is procured at minimal cost, we also show that total employment and involuntary unemployment can move in the same direction and that there is no intrinsic relationship between the intensity of competition and the level of involuntary unemployment. Indeed, perfect competition is consistent with a positive level of involuntary unemployment. The main insights from the monopsony model with regard to minimum wage effects carry over to the model with quantity competition. In particular, an appropriately chosen minimum wage still eliminates involuntary

[2]The fact that the mechanism involving an efficiency wage and involuntary unemployment resonates with the dictum often attributed to Henry Ford that the Five-Dollar Day was "the best cost-cutting measure ever undertaken." Contrary to perceived wisdom, a wage of five dollars per day was not uniformly applied across all workers from the time of its introduction in 1914. See, for example, Sward (1948) who notes that according to the company's financial statement 30% of the overall workforce were paid less than that in 1916.

[3]The non-generic, knife-edge case arises when the minimum wage is equal to the wage that would prevail under price-taking behaviour.

unemployment. With horizontally differentiated workers and jobs, optimal procurement may involve deliberate and inefficient mismatches of workers and jobs, in addition to involuntary unemployment.

Our paper also sheds new light on the effects of costly migration from one region, country or sector to another (or, equivalently, of fixed costs associated with joining the work force). We show that costly migration causes the cost of procurement to be non-convex even if it is convex absent any migration. In this sense, migration caused by an increase in demand in one region can lead to involuntary unemployment because using an efficiency wage and inducing involuntary unemployment may become optimal due to the non-convexity of the procurement cost. In another extension, we also show that the introduction of a small amount of unemployment insurance increases unemployment and decreases total employment if the equilibrium without government intervention involves involuntary unemployment.

Our paper is closely related to three strands of literature: efficiency wage theory, monopoly and monopsony models under price regulation, and mechanism design problems that involve ironing. That involuntary unemployment is beneficial for businesses and detrimental for workers is a popular idea whose origins date back at least to Friedrich Engels' and Karl Marx' notion of a *reserve army of labor*.[4] More recently, it appears in the guise of the efficiency-wage theory of involuntary unemployment. According to this theory firms deliberately offer wages that exceed their market-clearing level so that the resulting excess supply of labor (and corresponding level of involuntary unemployment) can be used to discipline their workforce. For example, firms may offer efficiency wages to increase workers' effort or reduce churn. The collection of essays in Akerlof and Yellen (1986) provides an overview of the early literature that formalized these ideas, while Krueger and Summers (1988) provide empirical evidence on industry wage structure. Notwithstanding their popular appeal, one major drawback of shirking and labor market turnover models of efficiency wages is that they rest on implicit or explicit restrictions on the contracting space. As Yellen (1984, p. 202) put it: "All these models suffer from a similar theoretical difficulty—that employment contracts more ingenious than the simple wage schemes considered, can reduce or eliminate involuntary unemployment." Our paper contributes to this literature by developing a model in which an efficiency wage that is optimal, subject only to individual rationality and incentive compatibility constraints, induces involuntary unemployment. Because the mechanism design approach we use is free of institutional assumptions and does not restrict the contracting space, in our setting efficiency wages and involuntary unemployment arise from the primitives of the problem.

Stigler (1946) observed that equilibrium employment can be increased with a minimum

[4]See Engels (1845) and Marx (1867).

wage in the presence of monopsony power.[5] The basic logic extends to imperfectly competitive markets, as shown, for example, by Bhaskar et al. (2002). Our paper shares the feature that minimum wages can increase employment. However, while these models can explain inefficiently low employment due to market power on the demand side, they cannot say anything about effects on involuntary unemployment because all unemployment is voluntary in models with market-clearing wages. By allowing the monopsony to use an optimal procurement mechanism, we obtain wage dispersion and involuntary unemployment absent wage regulation in equilibrium, thereby combining insights from Stigler's analysis and the mechanism design approach pioneered by Roger Myerson. From a methodological perspective, our paper is thus most closely related to the literature on monopoly pricing and mechanism design that fail to satisfy the regularity condition of Myerson (1981) and involve ironing. Of course, the idea that monopolies may benefit from bunching when faced with non-concave optimization problems is not novel and dates back to Hotelling (1931), with subsequent contributions by Mussa and Rosen (1978), Myerson (1981), Bulow and Roberts (1989), and a recent upsurge of interest driven by the applications considered in Condorelli (2012), Dworczak et al. (2021), Loertscher and Muir (2021a) and Akbarpour et al. (2020). That said, to the best of our knowledge, the connection between irregular mechanism design problems, involuntary unemployment and minimum wage effects that are made in this paper have never been touched upon before.[6]

Our model of quantity competition in Section 5 is related to the literature on Cournot competition (Cournot, 1838), while our discussion of optimal mechanisms in the Hotelling model builds on Balestrieri et al. (2021) and Loertscher and Muir (2021b). It also relates to mechanism design problems involving endogenous worst-off types (see, for example, Loertscher and Wasser, 2019).

The remainder of this paper is structured as follows. Section 2 introduces the baseline procurement setup. In Section 3, we relate the monopsony's optimal procurement mechanism to efficiency wages and involuntary unemployment. In Section 4, we analyze the effects of minimum wages. While we do not pursue it in this paper, with the appropriate adjustments the methodology developed there is also applicable to the analysis of price caps imposed on a monopoly seller who faces a non-concave revenue function. Section 5 extends the model to quantity competition. Section 6 provides extensions to horizontally and vertically

[5] As mentioned, there is empirical evidence consistent with these effects. The classic paper is Card and Krueger (1994). More recently, Wiltshire (2021) provides an analysis of the labor market effects of Walmart supercenters and the effects of minimum wages in the presence of monopsony power, as well as a comprehensive list of references.

[6] The analysis of the model with heterogeneous tasks and endogenous multi-tasking in Section 6.2 is related to the seminal multi-tasking model of Holmström and Milgrom (1991) and the mechanism design analyses of Condorelli (2012) and Loertscher and Muir (2021a), which involve the allocation of heterogeneous goods.

differentiated jobs, analyzes the effects of prohibiting wage discrimination, and discusses the effects of costly migration and of introducing unemployment insurance. Section 7 concludes the paper.

2 Setup

We consider the procurement problem of a monopsony whose willingness to pay for $Q \in [0, 1]$ units of labor is $V(Q)$. For simplicity the function V is assumed to be a strictly decreasing and continuously differentiable function on $Q \in [0, 1]$.[7,8] Let W denote the inverse supply function faced by the the monopsony so that $W(Q)$ is then the market-clearing wage for procuring the quantity $Q \in [0, 1]$. We denote by

$$C(Q) := W(Q)Q$$

the cost of procuring $Q \in [0, 1]$ units at the market-clearing wage. We assume that the supply side consists of a continuum of workers of mass 1 each of whom supplies one unit of labor inelastically, with $W(Q)$ representing the opportunity cost of working for the worker with the Q-th lowest opportunity cost. We further assume that the opportunity cost of working is the private information of each worker and that W is a strictly increasing (so that the monopsony faces an upward sloping labor supply schedule) and continuously differentiable function.[9] This in turn implies that the function C is strictly increasing and continuously differentiable. The input (or labor) supply function is denoted by S so that $S(w) = W^{-1}(w)$ holds for all wages $w \in [W(0), W(1)]$.

We assume that $V(0) > W(0)$ and $V(1) < W(1)$, which implies that under the optimal procurement mechanism there is a strictly positive mass of workers employed by the monopsony and a strictly positive mass of workers that are not employed. We also allow the monopsony to offer multiple wages $(w_1, \ldots, w_n)$ and to specify the amount that it is willing to procure at each wage w_i. We refer to $(w_1, \ldots, w_n)$ as the *wage schedule*. We say that the monopsony uses an *efficiency wage* to procure a total quantity Q if its wage schedule involves a wage w_i that is larger than the market-clearing wage (i.e. $w_i > W(Q)$) and if it

[7] As will become clear, by assuming that V is strictly decreasing we avoid having to deal with the possibility that the optimal quantity procured is not unique. Introducing this assumption ensures that whenever a procurement mechanism involving an efficiency wage is optimal, it is the uniquely optimal mechanism.

[8] If the firm uses the input to generate revenue $R(Q)$, where R is concave and increasing for Q sufficiently small, then the firm's willingness to pay for the Q-th unit of input is given by $V(Q) = R'(Q)$. The firm could be a monopoly on the output market with a technology that transforms one unit of input into one unit of output or a price-taking firm, in which case the concavity is derived from a production function that exhibits decreasing marginal products in the input.

[9] The assumption that W is continuously differentiable is made purely for expositional convenience.

procures a positive quantity at w_i.

In short, the setting we consider makes two important departures from an otherwise completely standard monopsony pricing problem. First, we do not restrict the monopsony to setting the market-clearing wage $w = W(Q)$ when it procures the quantity Q. Second, we do not assume that the cost of procurement function C is convex. As we shall see, these assumptions go hand-in-hand: It is without loss of generality to restrict attention to market-clearing wages when the cost function C is convex. However, when the cost function C fails to be convex, the monopsony may strictly benefit from offering an efficiency wage and inducing involuntary unemployment.

Given the important role that non-convex cost functions play in our analysis, this may beg the question of why such functions might arise in practice. In Section 6.4 we show that non-convexities arise naturally when workers face a fixed cost of moving, changing occupation or participating in the labor market. The assumption that a monopsony faces a convex procurement cost is analogous to the assumption that a monopoly faces a concave revenue function. As discussed in Loertscher and Muir (2021a), while this assumption is widely maintained in both theoretical and empirical work in Industrial Organization it is frequently rejected when tested empirically. However, more fundamentally, there is simply no theoretical reason for why convex cost functions should arise in the first place.

3 Optimal procurement mechanism

We begin by introducing a function that will play a central role in our analysis: the *convexification* $\underline{C}$ of the cost function C, which is the largest convex function that is weakly less than C at every point $Q \in [0, 1]$. If C is a convex function then we have $\underline{C} = C$. If C fails to be convex then its convexification is characterized by a countable set $\mathcal{M}$ and a set of disjoint open intervals $\{(Q_1(m), Q_2(m))\}_{m\in\mathcal{M}}$ such that

$$\underline{C}(Q) = \begin{cases} C(Q_1(m)) + \frac{(Q-Q_1(m))(C(Q_2(m))-C(Q_1(m)))}{Q_2(m)-Q_1(m)}, & \exists\, m \in \mathcal{M} \text{ s.t. } Q \in (Q_1(m), Q_2(m)) \\ C(Q), & Q \notin \bigcup_{m\in\mathcal{M}}(Q_1(m), Q_2(m)). \end{cases}$$

Moreover, since C is continuously differentiable, for each $m \in \mathcal{M}$, $Q_1(m)$ and $Q_2(m)$ satisfy the first-order condition

$$C'(Q_1(m)) = \frac{C(Q_2(m)) - C(Q_1(m))}{Q_2(m) - Q_1(m)} = C'(Q_2(m)). \tag{1}$$

Observe that on each interval $(Q_1(m), Q_2(m))$, $\underline{C}$ is a linear function given by a convex combination of $C(Q_1(m))$ and $C(Q_2(m))$ that exhibits constant marginal cost. In particular, if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then $\underline{C}(Q)$ can equivalently be written

$$\underline{C}(Q) = (1 - \alpha_m(Q))C(Q_1(m)) + \alpha_m(Q)C(Q_2(m)),$$

where $\alpha_m(Q) := \frac{Q - Q_1(m)}{Q_2(m) - Q_1(m)}$. If $|\mathcal{M}| > 1$, then without loss of generality we can index the intervals $(Q_1(m), Q_2(m))$ in increasing order so that, for all $m \geq 2$, we have $Q_2(m-1) < Q_1(m)$. Note that since C is increasing, so too is $\underline{C}$, and since W is strictly increasing, we must have $Q_1(1) > 0$.[10]

For the monopsony problem with a fixed marginal benefit function V, the focus on the case where the function C fails to be convex on a single interval is without loss of generality.[11] Consequently, for the remainder of this section we assume that C exhibits only one interval of non-convexity (i.e. $|\mathcal{M}| = 1$) and simply write Q_1 and Q_2 in lieu of $Q_1(1)$ and $Q_2(1)$. As an illustration, consider the following piecewise linear input supply function and its corresponding cost function

$$W(Q) = \begin{cases} 4Q, & Q \in [0, 1/4) \\ Q/2 + 7/8, & Q \in [1/4, 1] \end{cases} \quad \text{and} \quad C(Q) = \begin{cases} 4Q^2, & Q \in [0, 1/4) \\ Q^2/2 + 7Q/8, & Q \in [1/4, 1] \end{cases}. \qquad (2)$$

This non-convex cost function is illustrated in Figure 2.[12] Straightforward computations show that

$$Q_1 = \frac{4 + \sqrt{2}}{32} \approx 0.169 \quad \text{and} \quad Q_2 = \frac{1 + 2\sqrt{2}}{8} \approx 0.478.$$

The importance of the convexification $\underline{C}$ is made clear by Lemma 1 and Proposition 1.

Lemma 1. *The monopsony can procure the quantity Q at cost $\underline{C}(Q)$. Moreover, if $\underline{C}(Q) < C(Q)$, this is achieved using an efficiency wage.*

[10] To see this, assume to the contrary that $Q_1(1) = 0$. Because $C'(0) = W(0)$ and $C(Q_2(1)) = Q_2(1)W(Q_2(1))$, the first equality in the first-order condition (1) becomes $W(0) = W(Q_2(1))$, which contradicts the assumption that W is strictly increasing.

[11] The reason for this sufficiency is is that the function V is decreasing, which implies a unique point of intersection of the functions $\underline{C}'$ and V.

[12] The example in (2) is a special case of the piecewise linear specification in which, for $a > b > 0$ and $\underline{q} \in (0, 1)$, $W(Q)$ is given by

$$W(Q) = \begin{cases} aQ, & Q \in \left[0, \underline{q}\right) \\ bQ + (a-b)\underline{q}, & Q \in \left[\underline{q}, 1\right] \end{cases}, \qquad (3)$$

which gives rise to $C(Q) = aQ^2$ for $Q \in \left[0, \underline{q}\right)$ and $C(Q) = bQ^2 + Q(a-b)\underline{q}$ for $Q \in \left[\underline{q}, 1\right]$. Example (2) arises by setting $a = 4$, $b = 1/2$ and $\underline{q} = 1/4$.

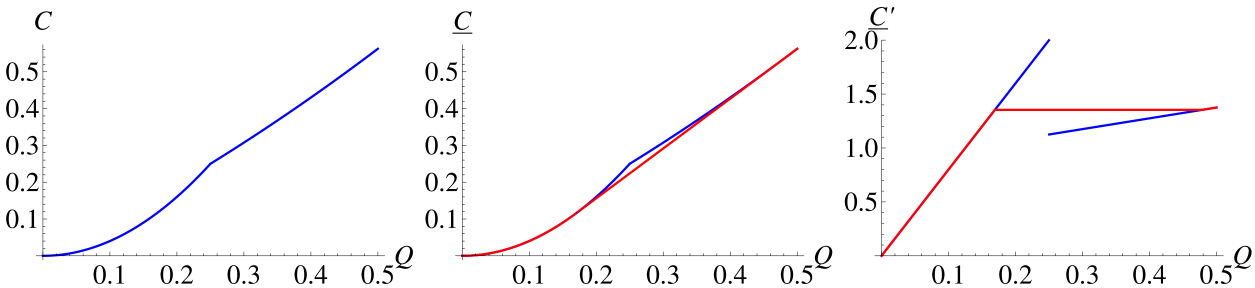

Figure 2: C (blue), $\underline{C}$ (red), C' (blue) and $\underline{C}'$ (red) for our leading example (2).

Since the proof of this lemma is instructive, we provide it in the main body of the paper. The first part of the lemma is evidently true if $Q \notin (Q_1, Q_2)$, since in this case $\underline{C}(Q) = C(Q)$. So assume that $Q \in (Q_1, Q_2)$, which implies that $\underline{C}(Q) < C(Q)$.

Suppose that the monopsony sets the wage schedule (w_1, w_2), where $w_1 < w_2$ and $w_2 = W(Q_2)$. The monopsony procures $Q - Q_1$ units of labor at the wage w_2 and Q_1 units of labor at the wage w_1. In equilibrium, Q_2 is the total mass of workers that participate in the procurement mechanism and $Q_2 - Q_1$ is the mass of workers competing for the $Q - Q_1$ jobs or openings at the high wage w_2. Consequently, the probability α that any given worker who competes for these $Q - Q_1$ jobs at the high wage is given by $\alpha = \frac{Q-Q_1}{Q_2-Q_1}$. This probability is independent of workers' opportunity cost of working (i.e. rationing is random). The Q_1 workers who in equilibrium apply for a job at the low wage w_1 are hired with certainty.

To implement this procurement mechanism, the marginal worker with opportunity cost $W(Q_1)$ must be indifferent between working with certainty at the low wage of w_1 and taking the gamble of working at the higher wage w_2 with probability α.[13] That is, the incentive compatibility constraints require that

$$w_1 - W(Q_1) = \alpha(W(Q_2) - W(Q_1)),$$

which in turn implies that

$$w_1 = (1 - \alpha)W(Q_1) + \alpha W(Q_2).$$

Note that w_1 increases in Q because α increases in Q and $W(Q_1) < W(Q_2)$.[14]

[13] If the worker with opportunity cost $W(Q_1)$ is indifferent, then all workers with lower opportunity costs strictly prefer to work for sure at w_1 over taking the gamble associated with the higher wage w_2. Moreover, all workers with higher opportunity costs strictly prefer taking the gamble to working for sure at w_1.

[14] This mechanism is robust to the introduction of risk-averse workers in the following sense. Suppose all workers have the same initial wealth level, which without loss of generality can be normalized to zero, and the same, strictly concave utility function U. So a worker with opportunity cost $W(Q)$ working at wage $w \geq W(Q)$ has a utility of $U(w - W(Q))$ while an unemployed worker has a utility of $U(0)$. To replicate the equilibrium above, the participation constraint for the marginal worker still requires $w_2 = W(Q_2)$

We are left to show that this results in the cost $\underline{C}(Q)$. To see that this is the case, notice first that the total wage payment to the low-wage workers is $Q_1 w_1 = (1-\alpha)C(Q_1) + \alpha W(Q_2)Q_1$, while the total wage bill for the high-wage workers is $(Q - Q_1)w_2 = W(Q_2)(Q - Q_1)$. Summing up these components and using $\alpha Q_1 + Q - Q_1 = \alpha Q_2$ then yields

$$Q_1 w_1 + (Q - Q_1)w_2 = (1-\alpha)C(Q_1) + \alpha C(Q_2) = \underline{C}(Q)$$

as required. While this implementation involving a Nash equilibrium does not preclude the possibility of other equilibria, there is also a dynamic implementation that has a dominant strategy equilibrium: the monopsony first hires workers at wage w_1 and opens $Q - Q_1$ vacancies at w_2 only after Q_1 workers have been hired at w_1.

Lemma 1 shows that for $Q \in (Q_1, Q_2)$ the monopsony can do better by using a procurement mechanism involving an efficiency wage instead of using a market-clearing wage. Moreover, in such cases we have also constructed an explicit mechanism, parameterized by the quantities (Q_1, Q, Q_2), that achieves a procurement cost of $\underline{C}(Q)$. We will refer to this class of mechanisms as *two-price mechanisms*.[15] The next proposition shows that two-price mechanisms involving an efficiency wage are optimal in the sense that $\underline{C}(Q)$ is the minimum cost for procuring Q in an incentive compatible and individually rational mechanism.

Proposition 1. *Under incentive compatible and individually rational mechanism that minimizes the cost of procuring the quantity Q the cost of procurement is $\underline{C}(Q)$.*

Proposition 1 can be established using the mechanism design approach and ironing procedure of Myerson (1981). Together with Lemma 1, this proposition implies it is without loss of generality to restrict attention to two-price mechanisms involving an efficiency wage when $Q \in (Q_1, Q_2)$. Intuitively, the convexification $\underline{C}$ is constructed from C by replacing C' with the average slope of C over the interval (Q_1, Q_2) since

$$\frac{\int_{Q_1}^{Q_2} C'(Q)\,dQ}{Q_2 - Q_1} = \frac{C(Q_2) - C(Q_1)}{Q_2 - Q_1}.$$

as in the risk-neutral case. In contrast, the wage $\hat{w}_1$ that makes workers with opportunity cost $W(Q_1)$ indifferent now satisfies $U(\hat{w}_1 - W(Q_1)) = \alpha U(W(Q_2) - W(Q_1)) + (1-\alpha)U(0)$. Since U is strictly concave, $\hat{w}_1 < w_1 = (1-\alpha)W(Q_1) + \alpha W(Q_2)$ follows. Moreover, the single-crossing condition is satisfied a fortiori because U is concave. Not surprisingly, the additional benefit of insurance offered by certain employment works in favor of the firm's scheme. However, with risk-averse agents it is not clear whether the optimal mechanism only involves two wages.

[15] Notice that the equilibrium construction does not require that Q_1 and Q_2 are the quantities corresponding to the convexification of C. Indeed, one can construct a similar equilibrium for arbitrary quantities $\tilde{Q}_1$ and $\tilde{Q}_2$ satisfying $\tilde{Q}_1 \le Q \le \tilde{Q}_2$, which yields a cost of $(1-\tilde{\alpha})C(\tilde{Q}_1) + \tilde{\alpha}C(\tilde{Q}_2)$ with $\tilde{\alpha} = (Q - \tilde{Q}_1)/(\tilde{Q}_2 - \tilde{Q}_1)$. Minimizing over $\tilde{Q}_1$ and $\tilde{Q}_2$ then yields the minimizers Q_1 and Q_2. By continuity, if $\underline{C}(Q) < C(Q)$, the two-price mechanism with $\tilde{Q}_1$ and $\tilde{Q}_2$ sufficiently close to Q_1 and Q_2, respectively, yields a lower procurement cost than hiring at a market-clearing wage.

Therefore, $\underline{C}$ is the smallest function that can be constructed from C by taking a weighted average, subject to the monotonicity constraint that, whenever $\hat{Q} < Q$, $C'(Q)$ is not given more weight than $C'(\hat{Q})$. This is achieved by assigning all $Q \in (Q_1, Q_2)$ equal weight. The monotonicity constraint corresponds to the incentive compatibility constraints of the mechanism design problem. These constraints imply that a worker with a lower opportunity cost of working, say $\hat{w} = W(\hat{Q})$, cannot be employed with lower probability than a worker with the higher opportunity cost $w = W(Q)$. Otherwise, the worker of type $\hat{w}$ could profitably imitate the worker of type w if w were employed with higher probability.[16] Since the firm would prefer to hire workers with high types whose marginal cost is small, to hiring workers with low types whose marginal cost is high, the best it can do is to hire them with equal probability.

Let Q^* be such that

$$V(Q^*) = \underline{C}'(Q^*).$$

Observe that Q^* is unique because V is strictly decreasing by assumption and $\underline{C}'$ is weakly increasing since $\underline{C}$ is a convex function.

Proposition 2. *The monopsony optimally employs Q^* workers. It optimally uses wage dispersion and induces involuntary unemployment if and only if $Q^* \in (Q_1, Q_2)$.*

Proposition 2 can be proven by adapting arguments from Loertscher and Muir (2021a), which analyzes optimal monopoly pricing under a non-concave revenue function. With homogeneous goods, randomization takes the form of rationing and it provides scope for resale if the goods are transferable. Resale harms the seller by reducing its equilibrium level of revenue. In a labor market context, resale (or subcontracting) among workers is not an issue if the employer has the ability to verify workers' identities and control which individuals actually perform a job. Using the terminology of Marx and Engels, Proposition 2 provides a formalization of why a firm can benefit from a reserve army of the unemployed. The excess supply induced by the efficiency wage allows it to randomize over workers and thereby to reduce its procurement cost.

If the monopsony optimally uses an efficiency wage, then the level of involuntary unemployment is $Q_2 - Q^*$, and the mass of workers who are employed is Q^*. The rate of involuntary unemployment, measured as fraction of unemployed over the total number of individuals willing to work, is $(Q_2 - Q^*)/(Q_2 - Q_1)$. An equivalent interpretation of involun-

[16] The formal proof is elementary (see e.g. Börgers, 2015). Denote by $t(w)$ the expected transfer an agent receives when, in a direct mechanism, he reports that his type is w, and by $q(w)$ the probability that he has to work upon the same report. Incentive compatibility for types w and $\hat{w}$ then implies $t(w) - q(w)w \geq t(\hat{w}) - q(\hat{w})w$ and $t(w) - q(w)\hat{w} \leq t(\hat{w}) - q(\hat{w})\hat{w}$, respectively. Subtracting the second from the first implies $q(w)(\hat{w} - w) \geq q(\hat{w})(\hat{w} - w)$, which for $\hat{w} < w$ holds if and only if $q(w) \leq q(\hat{w})$.

tary unemployment in this model is that the mass $Q_2 - Q_1$ of workers who want to work at the efficiency wage $W(Q_2)$ are all employed but only work part-time (they have a fraction α of a full-time job). Viewed in this way, these workers are *underemployed*, while the workers who are employed at the lower wage of w_1 work full-time.

4 Minimum wage effects

We are now going to show that when involuntary unemployment occurs in equilibrium without a minimum wage, an appropriately chosen minimum wages increase employment and decrease involuntary unemployment. Moreover, setting a minimum wages within a specific range eliminates involuntary unemployment. Throughout this section, the efficient employment level that would emerge under price-taking behaviour will play an important role in the analysis. Denoting this quantity by Q^p, it satisfies the equation $V(Q^p) = W(Q^p)$. Since V is strictly decreasing and W is strictly increasing and these functions satisfy $V(0) > W(0)$ and $V(1) < W(1)$, Q^p exists and is unique.

We will show that, roughly speaking, the implications of introducing a minimum wage for employment, wage dispersion, and involuntary unemployment vary depending on whether the minimum wage lies within one of three regions. These regions are illustrated in Figure 3 (where we drop the dependence of the various quantities on the index m to simplify notation). In the first region, which is characterized by $\underline{w} \in (w_1(Q^*), W(\hat{Q}))$ and plotted in red in Figure 3, the minimum wage is accompanied by wage dispersion and involuntary unemployment.[17] In this region, increasing the minimum wage will decrease involuntary unemployment and wage dispersion and increase employment. The second region, plotted in blue and characterized by $\underline{w} \in [W(\hat{Q}), W(Q^p))$, has the pure effects identified by Stigler (1946) that increasing the minimum wage increases employment without causing involuntary unemployment. In this region, and beyond, there is no wage dispersion. The last region, plotted in black and characterized by $\underline{w} \geq W(Q^p)$, corresponds to the textbook model with price-taking behaviour in which increasing the minimum wage increases involuntary unemployment and decreases employment. Figure 3 provides a rough (schematic) summary only insofar as there may be additional regions inside the interval $(W(Q^*), W(Q^p)]$ with and without wage dispersion, and $W(\hat{Q})$ need not be strictly less than $W(Q^p)$ if $Q^p < Q_2$.

To build intuition, we first consider a minimum wage equal to the efficiency wage. We then address the general problem. Although we do not pursue this in the present paper, the methodology developed in this section also applies to the analysis of price caps imposed on a

[17] Here, $w_1(Q^*)$ is the lower wage paid in equilibrium, absent any minimum wage regulation. The definition of the quantity $\hat{Q}$ is somewhat involved and is provided in Section 4.2 (see, in particular, Footnote 21).

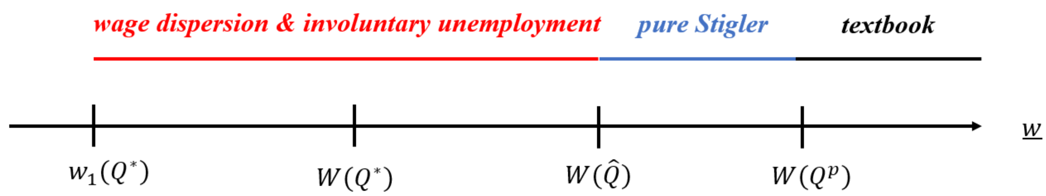

Figure 3: Three regions of minimum wage effects (schematic).

monopoly seller who faces a non-concave revenue function (see Loertscher and Muir, 2021a).

4.1 Setting a minimum wage equal to the efficiency wage

We begin by considering the case in which a regulator imposes a minimum wage equal to the efficiency wage that prevails absent regulation. This case is reasonably straightforward. In particular, since both $C'(Q) > W(Q)$ and $V(Q^*) = C'(Q_2(m))$ hold, we know that $V(Q^*) = C'(Q_2(m)) > W(Q_2(m))$. Consequently, a minimum wage equal to the efficiency wage $W(Q_2(m))$ will increase employment in equilibrium since the firm will demand the quantity $Q^*(\underline{w})$ such that $V(Q^*(\underline{w})) = \underline{w}$ with $\underline{w} = W(Q_2(m))$.

Whether or not this eliminates involuntary unemployment depends on whether Q^p is larger or smaller than $Q_2(m)$. If $Q^p \geq Q_2(m)$, then we have $V(Q_2(m)) \geq W(Q_2(m))$. This means that a firm facing a minimum wage of $\underline{w} = W(Q_2(m))$ will optimally hire $Q_2(m)$ workers at $\underline{w}$, which is the market-clearing wage for the quantity $Q_2(m)$. It will not hire any additional workers because $\underline{C}'(Q) > V(Q_2(m))$ for $Q > Q_2(m)$. In contrast, if $Q^p < Q_2(m)$, then $V(Q_2(m)) < W(Q_2(m))$, and a minimum wage equal to $W(Q_2(m))$ will still produce involuntary unemployment (whereas, for a minimum wage of, say, $\underline{w} = W(Q^p)$, there would be no involuntary unemployment).

Figure 4 illustrates the effects of a minimum wage of $\underline{w} = W(Q_2(m))$ for $Q^p < Q_2(m)$ (right-hand panel) and $Q^p > Q_2(m)$ (left-hand panel). Without a minimum wage, there is involuntary unemployment and if $\underline{w} = W(Q_2)$ is imposed, then the monopsony always hires more workers. When $Q^p < Q_2(m)$, we have $W(Q^p) < W(Q_2(m))$ and $V(Q_2) < W(Q_2(m))$. Consequently, the firm hires less than $Q_2(m)$ workers and involuntary unemployment is not eliminated because a minimum wage equal to $W(Q_2(m))$ already induces the textbook region from Figure 3. In contrast, if $Q^p \geq Q_2(m)$ then $V(Q_2(m)) \geq W(Q_2(m))$ holds and the firm optimally employs $Q_2(m)$ workers. Consequently, involuntary unemployment is eliminated and here we end up in the Stigler region from Figure 3.

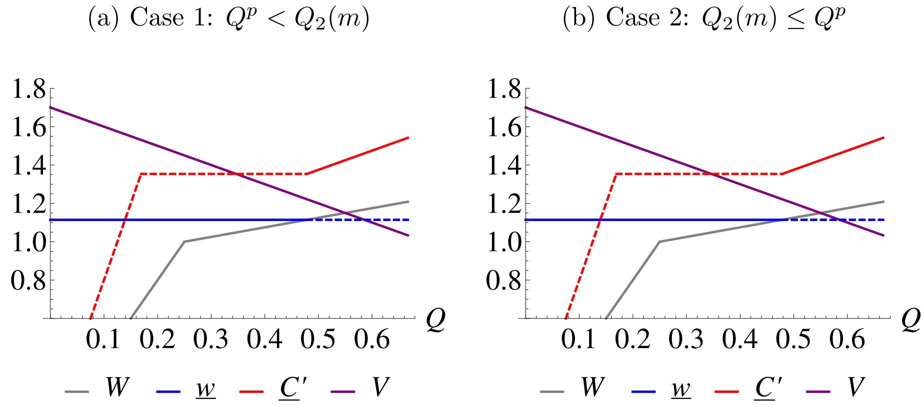

Figure 4: Illustration of the effects associated with imposing a minimum wage of $\underline{w} = W(Q_2)$ in our leading example (2). The solid sections of the $\underline{w}$ (blue) and $\underline{C}'$ (red) curves indicate the marginal cost schedule induced by optimal procurement under the minimum wage. The quantity Q^p is given the intersection of the W (grey) with V (purple) curves. In Panel (a), $Q^p < Q_2$ and $\underline{w}$ exceeds the wage $W(Q^p)$ that would prevail under price-taking behaviour. In Panel (b), $Q^p > Q_2$ and $\underline{w}$ eliminates involuntary unemployment.

4.2 General minimum wage effects

In this section we analyze the general effects of minimum wages on employment, wage dispersion, and involuntary unemployment. In order to state and prove our results, we first need to determine the minimum cost $\underline{C}(Q, \underline{w})$ of procuring the quantity Q—and the associated optimal procurement mechanism—for a given minimum wage $\underline{w}$. Recall that S denotes the labor supply function. For any $Q \leq S(\underline{w})$, the minimum cost of procuring the quantity Q is $\underline{w}Q$ because this cost cannot be reduced by randomizing over wages that are all at least as high as $\underline{w}$. Likewise, when $S(\underline{w}) \notin (Q_1(m), Q_2(m))$ for any $m \in \mathcal{M}$ (or, equivalently, when $\underline{w} \notin (W(Q_1(m)), W(Q_2(m)))$ for any $m \in \mathcal{M}$) the minimum cost of procuring the quantity $Q > S(\underline{w})$ is simply $\underline{C}(Q)$.[18] Thus, if $\underline{w} \notin (W(Q_1(m)), W(Q_2(m)))$ for any $m \in \mathcal{M}$, the minimum cost of procuring the quantity Q is $\underline{C}(Q, \underline{w}) = \underline{w}Q$ if $Q \in [0, S(\underline{w})]$ and $C(Q, \underline{w}) = \underline{C}(Q)$ if $Q > S(\underline{w})$.

Matters become more complicated when $S(\underline{w}) \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ (or, equivalently, when $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$). Given such an $m \in \mathcal{M}$, recall that $\alpha_m(Q) = \frac{Q - Q_1(m)}{Q_2(m) - Q_1(m)}$ and let

$$w_1(Q; m) := (1 - \alpha_m(Q))W(Q_1(m)) + \alpha_m(Q)W(Q_2(m)) \tag{4}$$

[18] If $Q \notin (Q_1(m), Q_2(m))$ for any $m \in \mathcal{M}$, then the minimum cost of procuring Q absent a minimum wage is $\underline{C}(Q) = W(Q)Q$ with $W(Q) > \underline{w}$. Alternatively, if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then the lower wage absent wage regulation is no smaller than $W(Q_1(m)) \geq \underline{w}$. In either case the minimum wage does not affect the minimum cost of procurement.

denote the lower wage that is paid under the optimal mechanism for procuring the quantity Q, absent wage regulation. The minimum wage does not constrain the optimal wages the monopsony uses absent wage regulation and the minimum cost of procuring Q given $\underline{w}$ is still $\underline{C}(Q)$ if

$$\underline{w} \leq w_1(Q;m).$$

Since $Q_1(m) < S(\underline{w}) < Q_2(m)$ and $w_1(Q;m)$ is an increasing and continuous function in Q on $[Q_1(m), Q_2(m)]$ satisfying $w_1(Q_i(m);m) = W(Q_i(m))$, for any $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$, $w_1^{-1}(\underline{w};m)$ is well-defined. Consequently, we have $\underline{C}(Q,\underline{w}) = \underline{C}(Q)$ for any $Q \geq w_1^{-1}(\underline{w};m)$ and $Q \in (S(\underline{w}), w_1^{-1}(\underline{w};m))$ is the only case that requires further analysis.

We first state in general terms what the solution is. Then we discuss its key technical properties and their economic implications. For $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$, consider the cost function

$$\underline{C}(Q,\underline{w}) = \begin{cases} \underline{w}Q, & Q \in [0, S(\underline{w})] \\ \mathcal{L}^*(Q,\underline{w}), & Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w};m)\right) \\ \underline{C}(Q), & Q \geq w_1^{-1}(\underline{w};m), \end{cases} \tag{5}$$

where $\mathcal{L}^*(Q,\underline{w})$ is the value (which is written in terms of a Lagrangian in the proof of Lemma 2) of the cost-minimization problem

$$\mathcal{L}^*(Q,\underline{w}) := \min_{q_1 \in [0,Q],\, q_2 \geq Q} \{(1-\alpha)C(q_1) + \alpha C(q_2)\} \tag{6}$$

$$\text{s.t.} \quad (1-\alpha)\,W(q_1) + \alpha W(q_2) \geq \underline{w}, \quad \alpha = \frac{Q - q_1}{q_2 - q_1}.$$

Whenever $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w};m)\right)$ the function $\underline{C}(Q,\underline{w})$ computes the cost-minimizing two-wage procurement mechanism, subject to the constraint that the lower wage is no less than the minimum wage. The following lemma shows that it is without loss of generality to restrict attention to two-price procurement mechanisms when $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w};m)\right)$. It also establishes a number of useful properties of the minimal cost of procurement function and of the marginal cost function $\underline{C}'(Q,\underline{w}) := \frac{\partial \underline{C}(Q,\underline{w})}{\partial Q}$.

Lemma 2. *The minimal cost $\underline{C}(Q,\underline{w})$ of procuring the quantity Q under the minimum wage $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ is given by (5). This function is convex in Q and increasing in both Q and $\underline{w}$. It also satisfies $\lim_{Q \downarrow S(\underline{w})} \underline{C}(Q,\underline{w}) = \underline{w}S(\underline{w})$, $\lim_{Q \uparrow w_1^{-1}(\underline{w};m)} \underline{C}(Q,\underline{w}) = \underline{C}(Q)$ and, for $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w};m)\right)$,*

$$\frac{\partial \underline{C}'(Q,\underline{w})}{\partial Q} > 0 > \frac{\partial \underline{C}'(Q,\underline{w})}{\partial \underline{w}}.$$

We have now formally shown that for a given minimum wage $\underline{w}$ and quantity $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w}; m)\right)$, the optimal procurement mechanism is a two-price mechanism where workers are hired with certainty at the minimum wage and rationed at a higher wage. From a theoretical perspective, the fact that $\frac{\partial \underline{C}'(Q,\underline{w})}{\partial Q} > 0$ holds for this region of the parameter space (illustrated in Figure 5) is noteworthy. It implies that over the interval $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w}; m)\right)$, the minimum cost of procurement is strictly convex and the "ironed" marginal cost function is strictly increasing. In standard irregular mechanism design problems the ironed marginal cost functions are constant on such ironing intervals. Here, the slope of the function $\underline{C}(\cdot, \underline{w})$ varies with Q over the interval $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w}; m)\right)$ because the Lagrange multiplier (i.e. shadow price) associated with the minimum wage constraint decreases as Q increases.

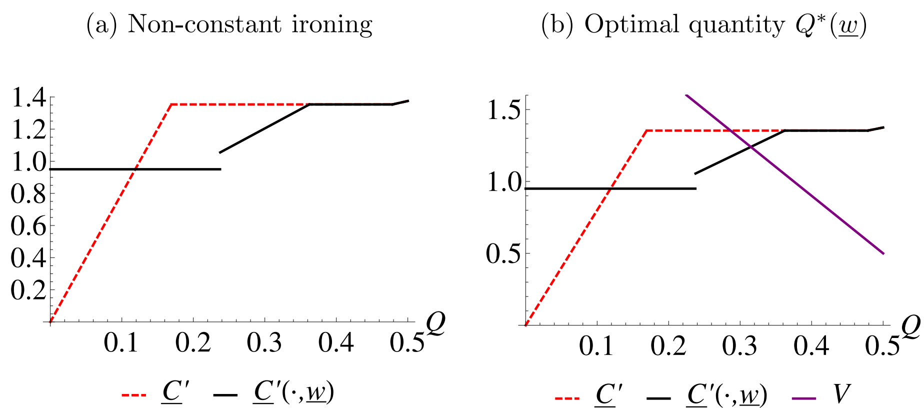

Figure 5: Panel (a) illustrates $\underline{C}'(\cdot, \underline{w})$ and $\underline{C}'$ for (2) with $\underline{w} = 0.95$. Panel (b) shows the optimal quantity, given by the intersection of V and $\underline{C}'(\cdot, \underline{w})$.

From an economic perspective, the fact that the marginal cost of procurement decreases in $\underline{w}$ for $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}; m))$—the second inequality in the display in Lemma 2—is key for the possibility that an increase in the minimum wage increases employment. This is illustrated in the right-hand panel in Figure 5. Here, the optimal quantity given the minimum wage $\underline{w}$, which we denote $Q^*(\underline{w})$, is given by the point of intersection of $V(Q)$ and $\underline{C}'(Q, \underline{w})$. Since $\underline{C}'(Q, \underline{w})$ decreases in $\underline{w}$ and $V(Q)$ decreases in Q, it follows that a local increase in the minimum wage increases employment. The following lemma characterizes the optimal quantity procured given $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ when $Q^* \in (Q_1(m), Q_2(m))$ formally and completely. It also relates the minimum wage and the resulting optimal quantity procured to wage dispersion and involuntary unemployment.

Lemma 3. *Assume $Q^* \in (Q_1(m), Q_2(m))$. Then, for $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$, the* *optimal quantity procured, denoted $Q^*(\underline{w})$, is such that*

$$V(Q^*(\underline{w})) = \underline{C}'(Q^*(\underline{w}), \underline{w}),$$

provided such a quantity exists. If there is no Q such that $V(Q) = \underline{C}'(Q, \underline{w})$, we have $Q^*(\underline{w}) = S(\underline{w})$. For $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$, the optimal procurement mechanism given $\underline{w}$ involves wage dispersion if and only if $Q^*(\underline{w}) > S(\underline{w})$. Moreover, provided $Q^*(\underline{w}) \neq S(\underline{w})$, this mechanism induces involuntary unemployment.

To state more detailed comparative statics concerning these minimum wage effects, some additional notation is required. We let $\gamma(Q; m)$ denote the marginal cost of procuring the quantity Q as the minimum wage $\underline{w}$ approaches $W(Q)$ from below. That is,

$$\gamma(Q; m) := \lim_{\underline{w} \uparrow W(Q)} \underline{C}'(Q, \underline{w}).$$

As is illustrated in Figure 5, the marginal cost function $\underline{C}'(\cdot, \underline{w})$ may be discontinuous at the point $Q = S(\underline{w})$, where the optimal procurement mechanism involves posting a market-clearing wage of $\underline{w}$. Given any sufficiently small $\epsilon > 0$, the optimal mechanism for procuring the quantity $S(\underline{w}) - \epsilon$ is a single-price mechanism according to which all $S(\underline{w}) - \epsilon$ workers are hired at the minimum wage $\underline{w}$, and the optimal mechanism for procuring the quantity $S(\underline{w}) + \epsilon$ is a two-price mechanisms in which some workers are hired with certainty at the minimum wage $\underline{w}$ and others are rationed at an efficiency wage. This difference between the left-hand and right-hand mechanisms explains why the marginal cost function $\underline{C}'(\cdot, \underline{w})$ is not necessarily continuous at $Q = S(\underline{w})$. When $Q = S(\underline{w})$ (or, equivalently, $\underline{w} = W(Q)$), $\gamma(Q; m)$ corresponds to the left-hand value of $\underline{C}'(Q, \underline{w})$.

The significance of this function is illustrated in Figure 4 for the special case where $\underline{w} = W(Q_2(m))$. The intersections between the functions γ and V dictate when we enter regions such as the one illustrated in the right-hand panel of Figure 4, where wage dispersion and involuntary unemployment are eliminated. When stating general comparative statics concerning minimum wage effects, we have to account for these regions. To that end, we introduce two important quantity cutoffs in the following lemma.

Lemma 4. *Assuming that $Q^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, there exists quantity cutoffs $\hat{Q}_L(m)$ and $\hat{Q}_H(m)$ given by*

$$\hat{Q}_L(m) := \min\{Q : \gamma(Q; m) = V(Q)\} \quad \text{and} \quad \hat{Q}_H(m) := \max\{Q : \gamma(Q; m) = V(Q)\}$$

satisfying $\hat{Q}_H(m) \leq Q^p$ and $Q^* < \hat{Q}_L(m) \leq \hat{Q}_H(m) < Q_2(m)$.

We are now in a position to state and prove a series of propositions which state comparative statics that specify how wage dispersion, involuntary unemployment, and employment vary as the minimum wage $\underline{w} \in (w_1(Q^*; m), W(Q_2(m))]$ increases.

Proposition 3. *Suppose that $Q^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Then for all $\underline{w} \in (w_1(Q^*; m), W(\hat{Q}_L(m)))$,*

(i) there is wage dispersion and involuntary unemployment; and

(ii) increasing $\underline{w}$ decreases involuntary unemployment and wage dispersion, and increases employment.

Proposition 3 covers the first region from Figure 3. The fact that an increase in the minimum wage at $\underline{w} = w_1(Q^*; m)$ has a positive effect on employment is noteworthy in itself because $w_1(Q^*; m) < W(Q^*)$. That is, $w_1(Q^*; m)$ is below the market-clearing wage for the quantity Q^*. In models in which market-clearing wages are imposed, minimum wages that are so low are typically ineffective as is a minimum wage equal to $W(Q^*)$. The reason for the positive quantity effect of such "small" minimum wages here is that the minimum wage makes the firm a price-taker on the amount of workers hired at the low wage even though it still exerts market power on the segment of workers employed at the high wage.

To gain intuition as to why wage dispersion decreases in $\underline{w}$ in this region, the following lemma is useful. It describes some formal properties of the optimal procurement mechanism in the region where the minimum wage constraint is binding and the monopsony optimally uses a two-price mechanism.

Lemma 5. *Given any $m \in \mathcal{M}$, suppose that $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ and $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w}; m)\right)$. For $i \in \{1, 2\}$, let $q_i^*(Q, \underline{w})$ denote the solution value of q_i in (6). Then $q_1^*(Q, \underline{w})$ increases in $\underline{w}$ and decreases in Q and $q_2^*(Q, \underline{w})$ decreases in $\underline{w}$ and increases in Q.*

In the proof of Proposition 3 we show that an increase in $\underline{w}$ reduces wage dispersion by both decreasing the high wage paid in equilibrium and increasing the low wage paid in equilibrium, provided there is wage dispersion in equilibrium. That the low wage increases in $\underline{w}$ is trivial since this wage is simply the minimum wage itself. That the high wage $W(q_2^*(Q^*(\underline{w}), \underline{w}))$ decreases in $\underline{w}$ is less obvious due to the countervailing effects that changes in $\underline{w}$ have on $q_2^*(Q^*(\underline{w}), \underline{w})$.[19] However, since $q_2^*(Q^*(\underline{w}), \underline{w})$ decreases in $\underline{w}$ and $Q^*(\underline{w})$ increases in $\underline{w}$, this implies that involuntary unemployment decreases in $\underline{w}$.[20]

[19]From Lemma 5, we know that $q_2^*(Q, \underline{w})$ increases in Q and decreases in $\underline{w}$. Since $Q^*(\underline{w})$ increases in $\underline{w}$ showing that that $q_2^*(Q^*(\underline{w}), \underline{w})$ is decreasing in $\underline{w}$ requires showing that the latter effect dominates the former effect.

[20]It further implies that the unemployment rate, defined as $\frac{q_2^*(Q^*(\underline{w}), \underline{w}) - Q^*(\underline{w})}{q_2^*(Q^*(\underline{w}), \underline{w})} = 1 - \frac{Q^*(\underline{w})}{q_2^*(Q^*(\underline{w}), \underline{w}))}$, decreases in $\underline{w}$ because $Q^*(\underline{w})$ increases and $q_2^*(Q^*(\underline{w}), \underline{w})$ decreases.

Proposition 4. *Suppose that $Q^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Then, for all $\underline{w} \in (W(\hat{Q}_H(m))), W(Q_2(m))]$, there is no wage dispersion. Moreover,*

(i) for all $\underline{w} \in [W(\hat{Q}_H(m)), \min\{W(Q^p), W(Q_2(m))\})$, there is no involuntary unemployment and employment increases in $\underline{w}$; and

(ii) for all $\underline{w} \in (\min\{W(Q^p), W(Q_2(m))\}, W(Q_2(m))]$, there is involuntary unemployment, which increases in $\underline{w}$, while employment decreases in $\underline{w}$, provided $\underline{w} < V(0)$.

Proposition 4 applies to the second and third regions from Figure 3.[21] It distinguishes between whether the efficient quantity, which as mentioned is denoted by Q^p, is smaller or larger than $Q_2(m)$. If $Q^p \leq Q_2(m)$ then statement (a) describes the comparative statics for the second region (the "pure Stigler" region) where there is no wage dispersion, no involuntary unemployment and employment is increasing in the minimum wage $\underline{w}$.[22] Moreover, statement (b) describes the comparative statics for the third region (the "textbook" region) where there is no wage dispersion, involuntary unemployment increases in $\underline{w}$ and employment decreases in $\underline{w}$.

If in addition to W being piecewise linear V is weakly concave, then $\hat{Q}_L = \hat{Q}_H$. Consequently, Propositions 3 and 4 provide a complete characterization of the minimum wage effects for $\underline{w} \in (w_1(Q^*; m), W(Q_2(m))]$ and there is no region in which increases in the minimum wage induce wage dispersion and involuntary unemployment (while still increasing employment).[23] Figure 6 illustrates these effects and the comparative statics from Propositions 3 and 4 for the piecewise linear specification given in (2) with $Q^*, Q^p \in (Q_1, Q_2)$. Here we can see that if there is wage dispersion in equilibrium, increasing the minimum wage will increase employment and decrease both involuntary unemployment and wage dispersion. Note that wage dispersion and involuntary unemployment vanish before the minimum wage reaches $W(Q^p)$, which is typically the case since $\hat{Q}_H(m) = Q^p$ is non-generic as described in Footnote 22. For $\underline{w} \in [W(\hat{Q}_H(m)), W(Q^p))$, increasing the minimum wage has the effect of increasing employment as observed by Stigler (1946). For $\underline{w} > W(Q^p)$, increasing the minimum wage has the textbook effect of decreasing employment and increasing involuntary unemployment.

[21] If there is a single point of intersection between the functions γ and V then we have $\hat{Q}_L(m) = \hat{Q}_H(m) =: \hat{Q}(m)$. This case is illustrated in Figure 3.

[22] If $W(Q)$ is piecewise linear of the form in (3) then, aside from knife-edge cases in which $V(\underline{q}) = W(\underline{q})$, we have $\hat{Q}_H < Q^p$ and such a region exists. (It can be shown that for W piecewise linear, $\underline{q}$ is the only point between Q_1 and Q_2 at which $\underline{C}'(\cdot, \underline{w})$ is continuous at $Q = S(\underline{w})$.)

[23] The function γ is piecewise linear and convex when W is piecewise linear. Moreover, if V is concave then these functions can only intersect once on $(Q_1(m), Q_2(m))$. (It might seem that V, if linear, could coincide with the downward sloping part of γ, which would mean that there is a continuum of points of overlap; but that is not possible because $Q^* \in (Q_1(m), Q_2(m))$ implies $V(Q^*) > \gamma(Q^*; m)$.)

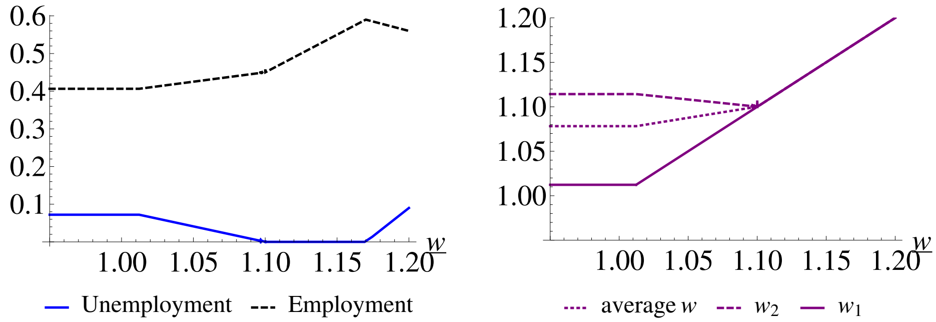

Figure 6: Equilibrium employment, involuntary unemployment and wages for a case with $Q^* \in (Q_1(m), Q_2(m))$ and $Q^p < Q_2(m)$. All functions are constant for $\underline{w} \leq w_1(Q^*; m)$.

In Appendix B.1 we discuss the effects of minimum wages above $W(Q_2(m))$. These results are needed for the proof of the general theorem stated at the end of this subsection. When $Q_2(m) \geq Q^p$, statement (ii) from Proposition 4 still applies in this case. However, when $Q^p > Q_2(m)$, what happens past $W(Q_2(m))$ depends on whether or not there exists another ironing range before one reaches $W(Q^p)$.

Proposition 5. *Suppose that $Q^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ and that $\hat{Q}_L(m) < \hat{Q}_H(m)$. Then the set $\{Q : \gamma(Q; m) = V(Q)\}$ contains an even number k of quantity cutoffs $\hat{Q}_j(m)$ that we index in increasing order so that $\hat{Q}_1(m) = \hat{Q}_L(m)$ and $\hat{Q}_k(m) = \hat{Q}_H(m)$. There is no wage dispersion for intervals with an upper quantity cutoff that corresponds to an even index and statements (i) and (ii) from Proposition 4 apply for intervals with an upper quantity cutoff that corresponds to an odd index.*

The cutoff quantities that define each of the intervals identified in Proposition 5 each correspond to an intersection of the functions γ and V. In this region, which is not included in Figure 3, equilibrium behaviour alternates between regions where statements (i) and (ii) from Proposition 3 apply and where there is no wage dispersion. Consequently, we alternate between regions where there is and where there is no wage dispersion and involuntary unemployment. While employment continuously increases in the minimum wage over these intervals, at the point where one transitions from a region without involuntary unemployment and wage dispersion into one with involuntary unemployment, both involuntary unemployment and wage dispersion increase discontinuously. As the minimum wage increases further, both involuntary unemployment then decrease continuously and become zero at the end of the interval.

Figure 7 illustrates how a small increase in the minimum wage can lead to a discontinuous increase in involuntary unemployment, which is then followed by a continuous decrease in involuntary unemployment. The figure is plotted for a piecewise linear example in which Q^p

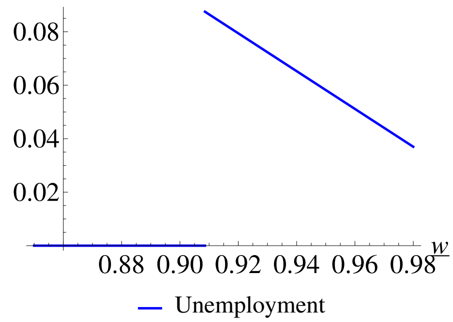

Figure 7: An example that exhibits a discontinuous increase in involuntary unemployment.

is part of the ironing range but Q^* is not.[24] This implies that V first crosses γ from below on (Q_1, Q_2), which in turn implies that for $\underline{w}$ close to but above $W(Q_1)$ there is no wage dispersion but for larger values of $\underline{w}$ there is both.

Implications for regulators We conclude this section by addressing the question of how a regulator who observes wages and whether there is involuntary unemployment at a given minimum wage can gauge whether marginally increasing the minimum will increase employment.

One implication of the above analysis is that the relationship between involuntary unemployment and minimum wages is non-monotone. If $Q^* \in (Q_1(m), Q_2(m))$, there is involuntary unemployment of size $Q_2(m) - Q^*$ without a minimum wage, or equivalently, for any minimum wage $\underline{w} \leq w_1(Q^*; m)$. For $\underline{w} \in (w_1(Q^*; m), W(\hat{Q}_L(m))$, involuntary unemployment decreases with $\underline{w}$ and becomes 0 at $\underline{w} = W(\hat{Q}_L(m))$. Whether it remains 0 or becomes positive again depends on whether or not $\hat{Q}_L(m) = \hat{Q}_H(m)$. In any case, employment is increasing in $\underline{w}$ for all $\underline{w} \in (w_1(Q^*; m), W(Q^p))$ and involuntary unemployment is 0 at $\underline{w} = W(Q^p)$. As $\underline{w}$ increases beyond $W(Q^p)$, there will be involuntary unemployment and employment decreases.

The richness and non-monotonicity of the aforementioned effects raise the question of whether a policymaker could assess when *marginally* increasing the minimum wage will decrease overall employment and increase or decrease involuntary unemployment. The answer to this question is affirmative and relates to wage dispersion. If at the present minimum wage there is involuntary unemployment and wage dispersion, increasing the minimum wage will increase employment and decrease involuntary unemployment. Likewise, if at the present minimum wage there is no wage dispersion and no involuntary unemployment, increasing the

[24]Specifically, the figure considers a parameterization such that $W(Q_1) = 0.67$ and $W(Q_2) = 1.11$ and assumes that $V(Q) = v$ with $v \in (W(Q_1)C'(Q_2))$ for $Q \leq 1/4$ and $V(Q) = 0$ otherwise. This V function can be written as the limit of a series of strictly decreasing V functions.

minimum wage will increase employment, provided $\underline{w} \neq W(Q^p)$. If increasing the minimum wage increases employment when there is no involuntary unemployment before the increase, the minimum wage increase may induce involuntary unemployment. This increase will be discontinuous. However, by the preceding argument, further increasing the minimum wage will eventually reduce involuntary unemployment, while continuing to increase employment. In sharp contrast, if at the current minimum wage, there is involuntary unemployment and no wage dispersion, then increasing the minimum wage will increase involuntary unemployment and decrease employment.

Putting all of this together yields the following theorem, which specifies the circumstances in which a regulator can expect a local increase in the minimum wage to increase employment. Because the theorem follows immediately from our previous observations, we do not provide a separate proof.

Theorem 1. *Whenever there is involuntary unemployment and wage dispersion at a given minimum wage, a sufficiently small increase in the minimum wage increases employment and decreases involuntary unemployment. If there is involuntary unemployment and no wage dispersion at a given minimum wage, increasing the minimum wage decreases employment and increases involuntary unemployment. Moreover, provided $\underline{w} \neq W(Q^p)$, if there is no involuntary unemployment at a given minimum wage, a sufficiently small increase in the minimum wage increases employment.*

Our analysis in this section also points to the possibility of conflicting interests among employed workers concerning the introduction of a minimum wage $\underline{w} \in (w_1(Q^*; m), W(Q_2(m)))$. While those employed at the low wage benefit from the imposition of the minimum wage, workers who earn the high wage in the absence of a minimum wage are harmed by a minimum wage such that $\underline{w} < W(Q_2(m))$. Whenever there is wage dispersion in equilibrium, the high wage decreases in $\underline{w}$. For $\underline{w} \in [W(\hat{Q}_H(m)), W(Q_2(m))]$ all workers earn the minimum wage, and hence the equilibrium wage increases in $\underline{w}$ but is evidently less than $W(Q_2(m))$ for $\underline{w} < W(Q_2(m))$. This effect is also illustrated in Figure 6(b), where the high wage decreases in $\underline{w}$, provided there is wage dispersion and $\underline{w}$ impacts employment and involuntary unemployment. If $Q^p < Q_2(m)$, as is the case in this example, the workers who earn the high wage absent wage regulation are still worse off with a minimum wage equal to $W(Q^p)$ since $W(Q^p) < W(Q_2(m))$.

5 Quantity competition

A natural question that the analysis in Sections 3 and 4 raises is to what extent the effects identified generalize to (imperfectly) competitive environments. To address this question, we now extend the model to allow for quantity competition between firms. We first introduce the setup, derive the equilibrium and discuss its properties. We then analyze the effects of minimum wages.

5.1 Setup

Suppose now that there are n firms procuring labor. We index these firms by i. For each firm i, the marginal value for procuring the y_i-th unit of labor is given by a continuously decreasing function $V(y_i)$ satisfying $V(0) > W(0)$ and $V(0) < W(Q)$ for Q sufficiently large, where we use y_i to distinguish individual firms' quantities from the quantities q_1 and q_2 that were introduced in the previous section. The firms compete in quantities as follows. They simultaneously submit quantities y_i to a Walrasian auctioneer as in standard oligopoly and oligopsony models with quantity competition. However, rather than procuring the $Q := \sum_{i=1}^{n} y_i$ units at the market-clearing wage $W(Q)$, which is the standard assumption in Cournot models and leads to a procurement cost function of C, we assume that the auctioneer can use the optimal procurement mechanism and thus procures the Q units at minimal total cost $\underline{C}(Q)$. Firm i who employs y_i units has to pay the cost $\frac{y_i}{Q}\underline{C}(Q)$. Modulo replacing the cost function C with $\underline{C}$, this is the same as in standard Cournot models since $\frac{y_i}{Q}\underline{C}(Q) = y_i W(Q)$ for $Q \notin (Q_1(m), Q_2(m))$ for any $m \in \mathcal{M}$. The efficient quantity for a given n is denoted by Q_n^p and such that

$$V\left(\frac{Q_n^p}{n}\right) = W(Q_n^p).$$

This is the quantity that would emerge if the firms were price-takers.

5.2 Equilibrium

The analysis of the previous section then extends to this model, insofar as we will have involuntary unemployment and efficiency wages whenever $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$.

Let Q_n^* denote the aggregate quantity in a symmetric equilibrium under quantity competition. In models in which market-clearing wages are imposed, this quantity satisfies

$$V\left(\frac{Q_n^*}{n}\right) = W(Q_n^*) + \frac{Q_n^*}{n} W'(Q_n^*), \tag{7}$$

provided a symmetric equilibrium exists. Since $W' > 0$, we have $Q_n^* < Q_n^p$. That is, the equilibrium quantity is inefficiently small.

Proposition 6. *The quantity setting game has a unique equilibrium, and this equilibrium is symmetric. The aggregate equilibrium quantity Q_n^* is increasing in n. If $Q_n^p \leq Q_n^*$, then $n > 1$ and $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. As $n \to \infty$, we have $Q_n^* \to Q^e$ if $C(Q^e) = \underline{C}(Q^e)$ and otherwise, we have $Q_n^* \to \tilde{Q}$, where $\tilde{Q}$ satisfies $Q^e < \tilde{Q} < Q_2(m_e)$.*

As Proposition 6 shows, in our model of quantity competition the equilibrium is always unique and symmetric. However, for n sufficiently large, $Q_n^p < Q_n^*$ is possible. That is, the equilibrium quantity can be excessively large. To develop an understanding of how such a reversal can occur, consider the first-order condition under symmetry,

$$V\left(\frac{Q}{n}\right) = \frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q),$$

whose right-hand side we denote by $h(Q, n)$. If $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then $h(Q, n)$ is increasing and concave in Q and for any finite n satisfies $h(Q_i(m), n) > W(Q_i(m))$. Moreover, $h(Q, n)$ decreases in n and satisfies $h(Q, 1) > W(Q)$ for all $Q \in (Q_1(m), Q_2(m)))$. In contrast, for n sufficiently large, there exists at least one interval $(a_n, b_n) \subset (Q_1(m), Q_2(m))$ such that $h(Q, n) < W(Q)$ for all $Q \in (a_n, b_n)$, where a_n decreases in n and b_n increases in n.[25] Consequently, if $V(Q/n) = h(Q, n)$ for $Q \in (a_n, b_n)$, then $Q_n^* \in (a_n, b_n)$ and $Q_n^p < Q_n^*$. Figure 8 illustrates the relation between W and $h(\cdot, n)$ as a function of n for our leading example given by (2). Intuitively, the first-order condition implies that a firm's perceived marginal cost $h(Q, n)$ of procuring the quantity Q is a convex combination of $\underline{C}'(Q)$ (which is larger than $W(Q)$) and $\underline{C}(Q)/Q$ (which is less than $W(Q)$ for $Q \in (Q_1(m), Q_2(m)))$. As n increases, the weight on $\underline{C}(Q)/Q$ increases, eventually leading to $h(Q, n) < W(Q)$ for some values of Q.

As n approaches infinity, Q_n^p converges to the efficient (or Walrasian) quantity Q^e, which in turn satisfies $V(0) = W(Q^e)$. Consequently, the last statement of Proposition 6 distinguishes the cases where there is no $m \in \mathcal{M}$ such that $Q^e \in (C(Q_1(m), Q_2(m)))$ and where

[25] If there are multiple subintervals over which $h(Q, n) < W(Q)$ for some n, index these by k. Then for each k, a_n^k is decreasing in n and b_n^k is increasing in n because h decreases in n. Of course, eventually two or more of these subintervals may collapse into one, that is if $b_n^k < a_n^{k+1}$, we may have $b_{n'}^k \geq a_{n'}^{k+1}$ for some $n' > n$. But this does not invalidate the point that the set of $Q \in (Q_1(m), Q_2(m))$ for which $h(Q, n) < W(Q)$ increases in n in the set inclusion sense.

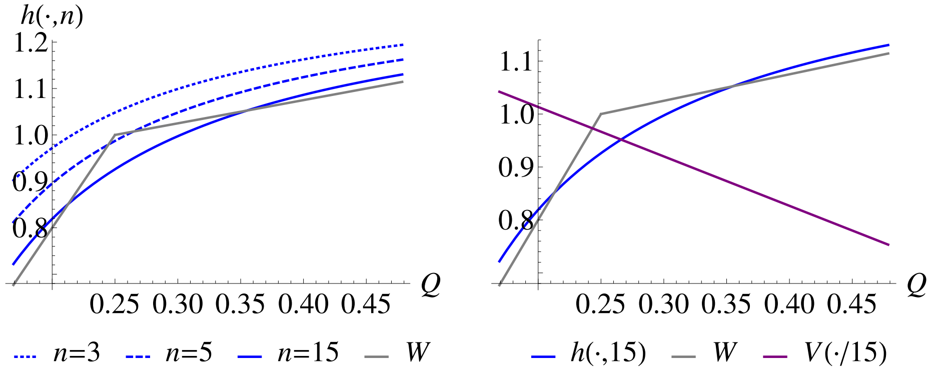

Figure 8: The left-hand panel displays W and $h(\cdot, n)$ for $n = 3$, $n = 5$ and $n = 15$ for our leading example (2). The right-hand panel focuses on the case where $n = 15$ and shows that $Q_n^p < Q_n^*$ for $V(Q/n) = 1.2 - 14Q/n$.

there exists a $m_e \in \mathcal{M}$ such that $Q^e \in (Q_1(m_e), Q_2(m_e)))$. Observe that in the latter case

$$\underline{C}'(Q^e) = C'(Q_2(m_e)) > W(Q_2(m_e)) > W(Q^e).$$

That is, $\underline{C}'(Q^e) > V(0)$.

Proposition 6 implies that key features of the monopsony model —efficiency wages, involuntary unemployment—extend to quantity competition. Moreover, there is not a monotone relationship between competition and involuntary unemployment because increasing competition can bring the equilibrium quantity into or out of an ironing interval $(Q_1(m), Q_2(m))$. Within such an interval, competition decreases wage dispersion and involuntary unemployment and increases $w_1(Q_n^*; m)$ and employment, while leaving the efficiency wage $W(Q_2(m))$ fixed. If $\underline{C}(Q^e) < C(Q^e)$ holds, there is involuntary unemployment and an efficiency wage even in the limit as $n \to \infty$. This yields a "natural" unemployment rate associated with perfect competition of $(Q_2(m_e) - \tilde{Q})/Q_2(m_e)$. In contrast to the usual notion of a natural unemployment rate, this unemployment is a result of inefficient resource allocation in the form of both random allocation and excessive economic activity (since $\tilde{Q} > Q^e$). In other words, there is the possibility of inefficient perfect competition. Figures 13 and 14 in Appendix B.2 illustrate these effects for our leading example.

5.3 Minimum wage effects and competition

In models with quantity competition and market-clearing wages, setting a minimum wage above $W(Q_n^*)$ (the market-clearing wage for the equilibrium quantity Q_n^* absent wage regulation) and below $W(Q_n^p)$ (the competitive wage) has a positive effect on total employment and, accordingly, workers' pay. To see this, recall that the competitive quantity Q_n^p is such that $V\left(\frac{Q_n^p}{n}\right) = W(Q_n^p)$ while the equilibrium quantity satisfies (7). Together with $W' > 0$, this implies that $Q_n^* < Q_n^p$. Any minimum wage $\underline{w} \in (W(Q_n^*), W(Q_n^p)]$ then has a positive employment effect. Since $\lim_{n\to\infty} Q_n^p = Q^e = \lim_{n\to\infty} Q_n^*$, the scope for this kind of quantity and social-surplus increasing minimum wage regulation vanishes in the limit as $n \to \infty$.[26]

Even if the symmetric equilibrium in the model with market-clearing wages is the unique equilibrium absent a minimum wage, a binding minimum wage $\underline{w} \in (W(Q_n^*), W(Q_n^p))$ inevitably gives rise to a continuum of equilibria. To see this, denote by $r_i(Q_{-i})$ the best response function of an arbitrary firm i to the aggregate quantity $Q_{-i} = \sum_{j\neq i} y_j$ demanded by its rivals. If the best response function is given by the first-order condition $V(r_i) - W(Q_{-i} + r_i) - r_i W'(Q_{-i} + r_i) = 0$, the equilibrium is unique and symmetric.[27] Denoting by $r_{\underline{w},i}(Q_{-i})$ the best response function given minimum wage $\underline{w} \in (W(Q_n^*), W(Q_n^p))$, we have

$$r_{\underline{w},i}(Q_{-i}) = \max\{r_i(Q_{-i}), \min\{S(\underline{w}) - Q_{-i}, V^{-1}(\underline{w})\}\},$$

where the term $\min\{S(\underline{w}) - Q_{-i}, V^{-1}(\underline{w})\}$ accounts for the possibility that even though the firm could procure the quantity $S(\underline{w}) - Q_{-i}$ at the minimum wage $\underline{w}$ it only wants to do so if this quantity is small enough and its willingness to pay is greater than $\underline{w}$. This means that it will not procure more than $V^{-1}(\underline{w})$.

Since $Q_n^* < S(\underline{w}) < Q_n^p$, we have

$$r'_{\underline{w},i}(Q_{-i})\,|_{Q_{-i}=\frac{n-1}{n}S(\underline{w})} = -1.$$

This implies that in the neighborhood of the symmetric equilibrium in which each firm chooses $S(\underline{w})/n$ there is a also a continuum of necessarily asymmetric equilibria as illustrated in Figure 9. Given that V is decreasing, the symmetric equilibrium is the one that maximizes social surplus and is therefore a natural selection.

In the analysis of minimum wage effects that follows, we will maintain the focus on the

[26]Whether the differences $W(Q_n^p) - W(Q_n^*)$ and $Q_n^p - Q_n^*$ monotonically decrease in n—and the scope for this kind of minimum wage of regulation—depends on the specifics of the model. If W and V are both linear, then both $W(Q_n^p) - W(Q_n^*)$ and $Q_n^p - Q_n^*$ decrease in n.

[27]To see this, totally differentiate the first-order condition to obtain $r_i' = -\frac{W' + r_i W''}{W' + r_i W'' + W' - V'}$, which satisfies $-1 < r_i' < 0$, where we drop arguments for ease of notation. The aggregate quantity Q given Q_{-i} and i's best response satisfies $Q = Q_{-i} + r_i(Q_{-i})$. The right-hand side is increasing in Q_{-i} and hence invertible. Following Anderson et al. (2020), we can thus write $Q_{-i} = f_i(Q)$ as a function of Q, where f_i is increasing. This allows us to construct what Anderson et al. call the *inclusive-best response* function $\tilde{r}_i(Q) := r_i(f_i(Q))$, which gives the optimal quantity that i would choose if the aggregate quantity is Q, which includes its own quantity. We have $\tilde{r}_i' = \frac{r_i'}{1+r_i'} < 0$. The aggregate quantity Q is an equilibrium quantity if and only if $\sum_{i=1}^n \tilde{r}_i(Q) = Q$. Because the left-hand side decreases and the right-hand side increases in Q, it follows that the Q satisfying this equality is unique. Moreover, because the firms are symmetric, we have $\tilde{r}_i = \tilde{r}_j$ for all $i, j \in \{1, \ldots, n\}$. Hence, the unique equilibrium is symmetric.

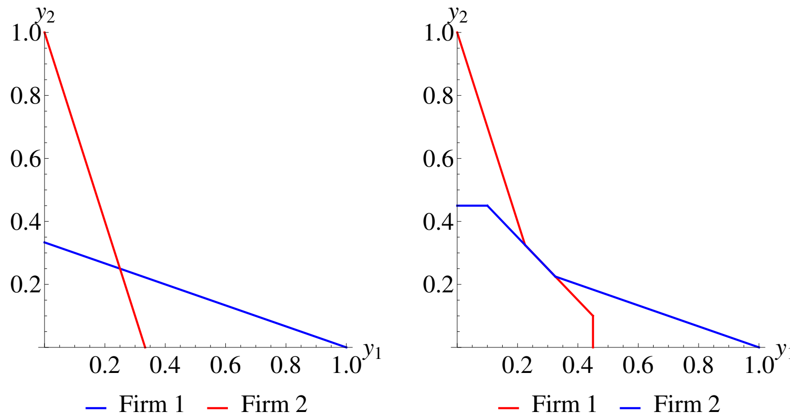

Figure 9: Standard quantity competition without a minimum wage (left panel) and with a minimum wage of $\underline{w} = 0.55$ (right panel). The minimum wage generates a continuum of equilibria.The figures assumes $V(y_i) = 1 - y_i$ and $W(Q) = Q$, which implies that $Q_n^* = 1/2$ and $Q_n^p = 2/3$.

symmetric equilibrium and study its comparative statics.[28] In analogy to the model without a minimum wage, let

$$h(Q, n, \underline{w}) := \frac{n-1}{n}\frac{\underline{C}(Q, \underline{w})}{Q} + \frac{1}{n}\underline{C}'(Q, \underline{w})$$

be the firm-level marginal cost of procurement under symmetry in the model with quantity competition given the minimum wage $\underline{w}$. Observe that for $Q \leq S(\underline{w})$, or equivalently, $\underline{w} \geq W(Q)$, $h(Q, n, \underline{w}) = \underline{w}$ because $\underline{C}(Q, \underline{w}) = \underline{w}Q$ and thus $\underline{C}'(Q, \underline{w}) = \underline{w} = \frac{\underline{C}(Q, \underline{w})}{Q}$. For $Q > S(\underline{w})$, $h(Q, n, \underline{w})$ is larger than $\underline{w}$ and strictly increasing in Q. Moreover, $h(Q, n, \underline{w})$ is continuous in Q everywhere except possibly at $Q = S(\underline{w})$. It is continuous at $Q = S(\underline{w})$ if and only if $\underline{C}'(Q, \underline{w})$ is continuous at that point ($\underline{C}(Q, \underline{w})$ is continuous and hence so is $\underline{C}(Q, \underline{w})/Q$). Finally, for $\underline{w} < w_1(Q; m)$, or equivalently, $Q > w_1^{-1}(\underline{w}; m)$, we have

$$h(Q, n, \underline{w}) = \frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q) = h(Q, n)$$

because $\underline{C}(Q, \underline{w}) = \underline{C}(Q)$ and hence $\underline{C}'(Q, \underline{w}) = \underline{C}'(Q)$ for $\underline{w} < w_1(Q; m)$. Hence, in the model with quantity competition the minimum wage binds in exactly the same instances as in the monopsony model.

[28] As stated in Proposition 6, without wage regulation, the symmetric equilibrium is the unique equilibrium. Whether given a minimum wage $\underline{w}$ the symmetric equilibrium is the socially optimal equilibrium when the equilibrium involves wage dispersion and involuntary unemployment is an open question. Of course, if the aggregate quantity is the same in a symmetric equilibrium and an asymmetric equilibrium, social surplus is larger in the symmetric equilibrium.

Because $V(Q/n)$ is decreasing in Q and because $h(Q, n, \underline{w})$ has the same curvature properties as $h(Q, n)$, it follows that if a Q exists that satisfies

$$V\,(Q/n) = h(Q, n, \underline{w}), \tag{8}$$

then Q/n is the symmetric equilibrium of the model with quantity competition given the minimum wage $\underline{w}$. If no such quantity exists, $h(Q, n, \underline{w})$ must be discontinuous at Q, which implies $Q = S(\underline{w})$. In this case, the symmetric equilibrium quantity is $S(\underline{w})/n$. Summarizing, we have:

Lemma 6. *The model with quantity-setting firms given minimum wage $\underline{w}$ has a symmetric equilibrium. In this equilibrium, each firm chooses the quantity Q/n with Q satisfying* (8) *if such a Q exists and $S(\underline{w})/n$ otherwise.*

The characterization of the symmetric equilibrium in the quantity setting game with a minimum wage mirrors the characterization of the optimal quantity in the monopsony model with a minimum wage. In particular, the aggregate quantity in the symmetric equilibrium is either given by equating marginal benefits and marginal costs as in (8) just like in Lemma 3 for the optimal monopsony quantity, provided a quantity that equates these marginal benefits and costs exist, or the quantity supplied at the minimum wage, $S(\underline{w})$. As we will show next, a difference arises for the comparative statics effects of increasing the minimum wage when the the equilibrium quantity is characterized by (8) and inside some $(Q_1(m), Q_2(m))$-interval. Recall that in the monopsony model, a marginal increase in $\underline{w}$ increases the equilibrium quantity and decreases the equilibrium level of involuntary unemployment because $\underline{C}'(Q, \underline{w})$ decreases in $\underline{w}$. In contrast, with $n \geq 2$, $h(Q, n, \underline{w})$ is a convex combination of $\underline{C}'(Q, \underline{w})$, which decreases in $\underline{w}$, and $\underline{C}(Q, \underline{w})/Q$, which increases in $\underline{w}$ because $\underline{C}(Q, \underline{w})$ increases in $\underline{w}$. Thus, with quantity competition, the effects of marginally increasing the minimum wage when there is wage dispersion and involuntary unemployment will not necessarily be monotone. This is illustrated in Figure 10 for our leading example given in (2) and a linear marginal benefit function V for $n = 5$ with $\underline{w} = 0.9$ (dotted), $\underline{w} = 0.95$ (dashed) and $\underline{w} = 1$ (solid). From $\underline{w} = 0.9$ to $\underline{w} = 0.95$, the equilibrium quantity increases, and from $\underline{w} = 0.95$ to $\underline{w} = 1$, it decreases.

However, as the following proposition shows, the marginal effect of increasing the minimum wage when the minimum wage is equal to the lower of the two wages absent wage regulation, that is at $\underline{w} = w_1(Q_n^*; m)$, on the equilibrium employment level $Q_n^*(\underline{w})$ is positive:

Proposition 7. *Suppose $n < \infty$ and $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Then at $\underline{w} = w_1(Q_n^*; m)$, the marginal effect of increasing the minimum wage on the equilibrium quantity $Q_n^*(\underline{w})$ is positive, that is, $\frac{dQ_n^*(\underline{w})}{d\underline{w}}|_{\underline{w}=w_1(Q_n^*;m)} > 0$.*

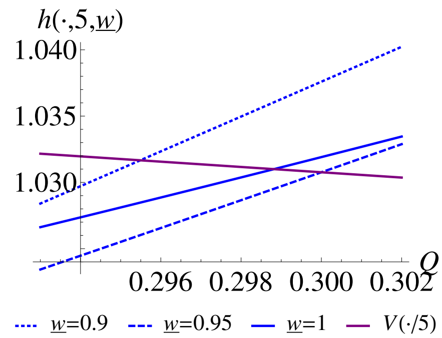

Figure 10: Illustration of non-monotone minimum wage effects with quantity competition.

Proposition 7 shows that a minimum wage close to but above $w_1(Q_n^*; m)$ increases the equilibrium quantity if $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. This resonates with an insight from the monopsony model (where any marginal increase in the minimum wage increases employment if there is wage dispersion and involuntary unemployment). However, in the model with quantity competition increasing the equilibrium quantity is not necessarily a move in the right direction because of the possibility of excessively high employment, that is, $Q_n^* > Q_n^p$. More generally, the following theorem describes the effects of imposing a binding minimum wage when $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$.

In the proof of the following theorem, we show that for $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}; m))$,

$$h_\gamma(Q, n; m) := \frac{n-1}{n} W(Q) + \frac{1}{n}\gamma(Q; m) \tag{9}$$

is the limit of $h(Q, n, \underline{w})$ as $\underline{w}$ approaches $W(Q)$ from below. This function is continuous in Q and its role and properties are analogous to those of $\gamma(Q; m)$ in the monopsony model. Assuming $Q_n^* \in (Q_1(m), Q_2(m))$, let $\hat{Q}_{H,n}(m)$ denote the largest value of Q such that $V(Q/n) = h_\gamma(Q, n; m)$.

Theorem 2. *Whenever there is involuntary unemployment and wage dispersion at a given minimum wage in the model with quantity competition, increasing the minimum wage to $\underline{w} = W(\hat{Q}_{H,n}(m))$ increases employment and eliminates involuntary unemployment. If there is involuntary unemployment and no wage dispersion at a given minimum wage, increasing the minimum wage decreases employment and increases involuntary unemployment. Moreover, provided $\underline{w} \neq W(Q_n^p)$, if there is no involuntary unemployment at a given minimum wage, a sufficiently small increase in the minimum wage increases employment.*

Note also that because $h_\gamma(Q, n; m) \geq W(Q)$, the aggregate equilibrium quantity in the presence of a minimum wage $\underline{w} = W(Q_n^*)$ is never larger than Q_n^p. Therefore, when $Q_n^* > Q_n^p$, one effect of imposing a minimum wage equal to the market-clearing wage for the equilibrium quantity absent wage regulation is that it prevents excessively high levels of employment. Since the ordering in (18) does not depend on the ordering of Q_n^* and Q_n^p, this also implies that even when $Q_n^* > Q_n^p$ holds absent wage regulation, total employment increases in $\underline{w}$ for $\underline{w} \in [W(\hat{Q}_{H,n}(m)), W(Q_n^p)]$ without inducing involuntary unemployment. Since we know from Proposition 7 that increasing the minimum wage at $w_1(Q_n^*; m)$ increases employment, if $Q_n^* > Q_n^p$, then the effects of the minimum wage on total employment must be non-monotone on $[w_1(Q_n^*; m), W(\hat{Q}_{H,n}(m))]$. Furthermore, if the Walrasian quantity Q^e is inside some ironing interval (i.e. if $Q^e \in (Q_1(m_e), Q_2(m_e))$ for some $m_e \in \mathcal{M}$) then there is scope for social-surplus increasing minimum wage regulation even in the perfectly competitive limit. Setting $\underline{w} = W(\hat{Q}_{H,n}(m_e))$ will eliminate involuntary unemployment and we have $\underline{w} \to W(Q^e)$ as $n \to \infty$ because $\lim_{n\to\infty} \hat{Q}_{H,n}(m) = Q^e$.

6 Extensions and discussion

We now provide extensions of the model to allow for differentiated jobs and discussion of the effects of policies that prohibit or permit wage discrimination. Then we discuss the effects of unemployment insurance and conclude the section with the analysis of fixed costs of migration or labor market participation.

6.1 Horizontally differentiated jobs

We first study a monopsony problem with horizontally differentiated jobs and workers.

Setup Consider a variant of the Hotelling model in which a monopsony with jobs at locations 0 and 1 has a willingness to pay of $V(Q_\ell)$ for the Q_ℓ-th worker employed at a given location $\ell \in \{0, 1\}$. As before, $V(Q_\ell)$ is assumed to be continuously decreasing. There is a continuum of workers with linear transportation costs whose locations are uniformly distributed between 0 and 1 and private information of each worker. The total mass of workers is 1. The value of the outside option of each worker is normalized to 0.[29] The payoff of a worker at location x that works at 0 for a wage of w is $w - x$, while this worker's payoff of working at 1 for a wage of w is $w-(1-x)$. Observe that this implies that the market-clearing wage to hire Q_ℓ workers at a given location is $W(Q_\ell) = Q_\ell$, which in turn means that the cost of procurement at each location under market-clearing wages is $C(Q_\ell) = Q_\ell^2$. Of course, the monopsony can hire Q_ℓ workers at $\ell = 0, 1$ if and only if $\sum_\ell Q_\ell \leq 1$.

[29] This is without loss of generality within the domain of problems in which the value of the outside option and the willingness to pay per worker are independent of the workers' locations since all that matters for these problems is the difference between the latter and the former.

Equilibrium We first derive the minimum cost to procure a quantity $Q_\ell \in [0, 1/2]$ at a given location, assuming that the same quantity is procured at the other location as well. To that end, notice first that the expected transportation cost of a worker at any location $x \in [0, 1]$ who is equally like to work at location 0 and at location 1, conditional on being employed, is 1/2. To satisfy the individual rationality constraints of workers employed under these terms and conditions, the wage they receive has to be no less than 1/2. Consequently, by paying a wage of 1/2 and leaving workers employed at this wage in the dark as to where they will work, or having them multi-task by having them spend half of their time at either location, the monopsony can procure any quantity $Q_\ell \in [0, 1/2]$ at both locations at a marginal procurement cost of 1/2. Since the marginal cost of procuring Q_ℓ at a market-clearing wage is $C'(Q_\ell) = 2Q_\ell$, it follows that the monopsony can procure the quantity $Q_\ell \in [0, 1/2]$ at each location at the cost

$$\underline{C}(Q_\ell) = \begin{cases} Q_\ell^2, & Q_\ell \in [0, 1/4] \\ Q_\ell/2 - 1/16, & Q_\ell \in (1/4, 1/2] \end{cases}$$

by offering a wage of 1/2 to attract "universalists"—workers who are willing to do either job—and a wage of 1/4 to attract "specialists," that is workers with locations no further away from 0 and 1 than 1/4 who will be guaranteed to do the job closest to their location. Notice that the individual rationality constraint will bind for all workers who are employed with locations $x \in (1/4, 3/4)$. Consequently, for the marginal worker at 1/4, the incentive compatibility constraint that this worker be indifferent between working as a specialist or as a universalist, coincides with this worker's individual rationality constraint.

The preceding arguments establish that this scheme with wage dispersion and random worker-job matchings results in smaller procurement costs than market-clearing wages for any $Q_\ell \in (1/4, 1/2]$. Arguments along the lines of those in Balestrieri et al. (2021) and Loertscher and Muir (2021b), who study optimal selling mechanisms on the Hotelling line, can be used to establish that $\underline{C}(Q_\ell)$ is indeed the minimal cost of procurement, subject to workers' incentive compatibility and individual rationality constraints.[30]

[30] An outline of the argument, adapted from the monopoly screening problem in Loertscher and Muir (2021b) to the procurement setting and assuming, for now, that all workers are employed, is as follows. Let $p_\ell(x)$ denote the probability that the worker who reports type $x \in [0, 1]$ works at location $\ell \in \{0, 1\}$. Incentive compatibility implies that $p_1(x) - p_0(x)$ be non-decreasing. Type $\hat{x}$ is the worst-off type if $p_1(\hat{x}) = p_0(\hat{x})$. Because all workers are employed, we have $p_0(x) + p_1(x) = 1$, implying $p(x) \equiv p_0(x)$ is sufficient, and incentive compatibility becomes equivalent to $p(x)$ being non-increasing, and $\hat{x}$ is worst-off if $p(\hat{x}) = 1/2$. Given any worst-off type $\hat{x} \in [0, 1]$, incentive compatibility yields the designer's objective in terms of virtual costs and values. Because its pointwise minimizer is not monotone, one needs to iron the virtual types. (Put differently, the cost of procurement is not convex in Q_0, the number of units procured at location 0.) The pointwise minimizer given the ironed virtual type function must assign a worker in the ironing interval

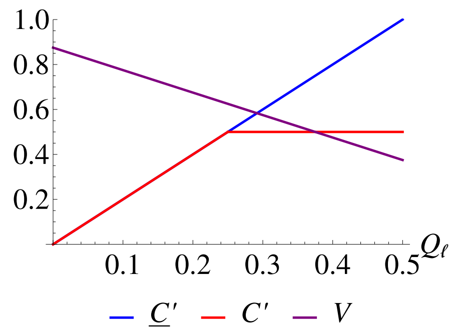

Figure 11: Illustration of Proposition 8 for $V(Q_\ell) = v - Q_\ell$ with $v = 7/8$.

The equilibrium level of employment Q_ℓ^* at each location is given by the unique number satisfying $V(Q_\ell^*) = \underline{C}'(Q_\ell^*)$. We say that the equilibrium involves *involuntary unemployment* if at the equilibrium wages there is a positive mass of workers who would be willing to work but are not employed, and we say that it involves *worker-job mismatching* if in equilibrium workers with $x < 1/2$ work at location 1 and workers with $x > 1/2$ at location 0.[31] The following proposition summarizes characteristics of the equilibrium. As it follows directly from the preceding arguments, we omit a proof.

Proposition 8. *If $V(1/4) \leq 1/2$, then $Q_\ell^* \leq 1/4$ and the equilibrium involves neither worker-job mismatchings nor involuntary unemployment. If $V(1/4) > 1/2 > V(1/2)$, then $Q_\ell^* \in (1/4, 1/2)$ and the equilibrium involves both worker-job mismatchings and involuntary unemployment. If $V(1/2) \geq 1/2$, then $Q_\ell^* = 1/2$ and the equilibrium involves worker-job mismatchings but no involuntary unemployment.*

Figure 11 illustrates the case $V(1/4) > 1/2 > V(1/2)$ in Proposition 8 for the linear specification $V(Q_\ell) = v - Q_\ell$ with $v = 7/8$. For this linear specification, $V(1/4) > 1/2 > V(1/2)$ is equivalent to $v \in (3/4, 1)$.

If a minimum wage of $\underline{w} = 1/2$ is imposed, the strict profitability of worker-jobs mismatching vanishes without any negative effects on the level of employment in equilibrium, provided $V(1/4) > 1/2$.

Effects of prohibiting wage discrimination The cost minimizing procurement mechanism involves wage dispersion or wage discrimination whenever the quantity procured at

with equal probability to jobs at 0 and 1. Consequently, the value of the ironed virtual type function over the ironing interval must be 0. Moreover, this also means that not employing some of these workers is also optimal. Thus, the assumption that all workers are employed can easily be relaxed.

[31] If worker-job mismatching is optimal, workers who work at the high wage of 1/2 are indifferent between working and not. Thus, those—if any—who are involuntarily unemployed are also indifferent between being unemployed and working.

each location is greater than 1/4. Wage discrimination is often perceived with suspicion in both public and academic debates and has led to pressure for pay transparency in a wide range of jurisdictions.[32] We now briefly investigate the effects of prohibiting wage discrimination on the equilibrium quantity, involuntary unemployment, social surplus, workers' total pay and workers' surplus.

Worker-job mismatching being optimal is equivalent to wage discrimination being optimal. From Proposition 8 we know that worker-job mismatching is optimal if and only if $Q^*_\ell > 1/4$. Consequently, prohibiting or permitting wage discrimination has no effect on equilibrium outcomes if and only if $Q^*_\ell \leq 1/4$, which is equivalent to $V(1/4) \leq 1/2$. In what follows, we therefore focus on the cases with $V(1/4) > 1/2$. The following effects of prohibiting wage discrimination hold in general (i.e. without additional assumptions on $V(Q_\ell)$):

Proposition 9. *Assume $V(1/4) > 1/2$. Then prohibiting wage discrimination: decreases the equilibrium quantity and strictly decreases it if $V(1/2) < 1$; decreases the monopsony's profit; and increases the surplus of all workers and strictly increases the surplus of all but the marginal workers who are employed when wage discrimination is prohibited.*

Proposition 9 implies that for $V(1/4) > 1/2$ and $V(1/2) < 1$, prohibiting wage discrimination decreases both the equilibrium level of employment and eliminates involuntary unemployment. This is similar to the effects observed in Section 5 that employment and involuntary unemployment can move in the same direction. The unambiguous effects of prohibiting wage discrimination on the surplus of individual workers contrast with the effects of minimum wages in Section 4, where, as discussed, high wage earners are typically harmed by minimum wages. These unambiguous effects arise here because the individual rationality constraint binds for all workers who are randomly matched to jobs, which also means that the individual rationality constraint binds for the marginal workers at locations 1/4 and 3/4 under wage discrimination. Hence, their net payoffs are 0. In contrast, if wage discrimination is prohibited, the marginal workers are further away from the extremes, which implies that all inframarginal workers enjoy larger information rents than they do with wage discrimination.

We conclude the analysis of prohibiting wage discrimination by studying the effects of wage discrimination on social surplus and total wage payments. Letting Q^d and Q^{nd} be the equilibrium quantities at each location with and without wage discrimination, the change in social surplus when wage discrimination is permitted compared to when it is not, denoted

[32] For a comprehensive list of recent references, see, for example, Cullen and Pakzad-Hurson (2021).

$\Delta SS(Q^d, Q^{nd})$, is

$$\Delta SS(Q^d, Q^{nd}) = \int_{Q^{nd}}^{Q^d} (V(x) - 1/2)dx - \int_{1/4}^{Q^{nd}} (1/2 - x)dx$$

while the change in total wage payments, denoted $\Delta C(Q^d, Q^{nd})$, is

$$\Delta C(Q^d, Q^{nd}) = \underline{C}(Q^d) - C(Q^{nd}) = \frac{1}{2}Q^d - \frac{1}{16} - (Q^{nd})^2.$$

The intuition for $\Delta SS(Q^d, Q^{nd})$ is simple. For all $x \in [Q^{nd}, Q^d]$, $V(x) - 1/2$ is the social benefit of the additional unit procured with wage discrimination, $V(x)$, minus the cost of production of $1/2$, while $\int_{1/4}^{Q^{nd}} (1/2 - x)dx$ is the additional cost of production on the inframarginal units between $1/4$ and Q^{nd} that are procured with and without wage discrimination.[33]

Recall first that permitting wage discrimination has a positive quantity effect if and only if $Q^{nd} \in (1/4, 1/2)$.[34] Notice next that $\Delta SS(Q^{nd}, Q^{nd}) = \frac{1}{2}Q^{nd}(Q^{nd} - 1) + \frac{3}{32} < 0$, which is to say that a positive quantity effect is necessary, that is, $Q^d > Q^{nd}$, for wage discrimination to increase social surplus, where the inequality follows from the fact that $\frac{1}{2}Q^{nd}(Q^{nd} - 1) < -\frac{3}{32}$ holds for any $Q^{nd} \in (1/4, 1/2)$. Similarly, $\Delta C(Q^{nd}, Q^{nd}) = \frac{1}{2}Q^{nd}(1 - 2Q^{nd}) - \frac{1}{16} < 0$, meaning that without a quantity effect, wage payments decrease with wage discrimination.[35] Moreover, we have

$$\frac{\partial \Delta SS(Q^d, Q^{nd})}{\partial Q^d} = V(Q^d) = \frac{1}{2} = \frac{\partial \Delta C(Q^d, Q^{nd})}{\partial Q^d}, \tag{10}$$

where the second equality makes use of the first-order condition $V(Q^d) = 1/2$. The proof of the following proposition makes use of these insights and provides the additional steps necessary to establish it.

Proposition 10. $\Delta C(Q^d, Q^{nd}) \leq 0$ *implies* $\Delta SS(Q^d, Q^{nd}) < 0$.

Proposition 10 does not say whether wage discrimination can increase social surplus but merely states that if it does increase it, it will also increase total wage payments. To see

[33]The expression for $\Delta C(Q^d, Q^{nd})$ follows straightforwardly by plugging in Q^d and Q^{nd} into $\underline{C}$ and C, respectively. To derive $\Delta SS(Q^d, Q^{nd})$, observe that social surplus with wage discrimination is $\int_0^{Q^d} V(x)dx - \int_0^{1/4} xdx - (Q^d - 1/2)\frac{1}{2}$ while social surplus without wage discrimination is $\int_0^{Q^{nd}} V(x)dx - \int_0^{Q^{nd}} xdx$. Subtracting the latter from the former yields $\Delta SS(Q^d, Q^{nd})$.

[34]For $Q^{nd} \leq 1/4$, permitting wage discrimination does not affect anything while for $Q^{nd} = 1/2$, the only effect of permitting wage discrimination is to decrease the procurement cost by mismatching workers to jobs.

[35]The function $\frac{1}{2}Q^{nd}(Q^{nd} - 1)$ is convex in Q^{nd} on $[1/4, 1/2]$, minimized at $Q^{nd} = 1/2$ and thus maximal at $Q^{nd} = 1/4$, at which point it is $-3/32$. To see that $\frac{1}{2}Q^{nd}(1 - 2Q^{nd}) - \frac{1}{16} < 0$, notice that $\frac{1}{2}Q^{nd}(1 - 2Q^{nd})$ is maximized at $Q^{nd} = 1/4$, at which point it equals $1/16$. Since $Q^{nd} > 1/4$, the inequality follows.

that it is indeed possible for permitting wage discrimination to increase social surplus, it is useful to consider the limiting case of a $V(Q)$ decreasing in which case $V(Q) = v$ for all $Q \in [1/4, 1/2]$. For $v \in (1/2, 1)$, this implies $Q^{nd} = v/2 \in (1/4, 1/2)$ and $Q^d = 1/2$. Here,

$$\Delta SS(Q^d, Q^{nd}) = (2Q^{nd} - 1/2)(Q^d - Q^{nd}) - \frac{1}{2}Q^{nd}(1 - Q^{nd}) + \frac{3}{32} = \frac{2Q^{nd} - 3(Q^{nd})^2}{2} - \frac{5}{32},$$

where the first equality uses $V(Q^{nd}) = 2Q^{nd}$ and the second equality follows from substituting $Q^d = 1/2$ and simplifying. Expressing $\Delta SS(Q^d, Q^{nd})$ in this way highlights the dual or countervailing role of Q^{nd}: If Q^{nd} is small, the additional costs due to wage discrimination are small and the benefits $v - 1/2$ are enjoyed over a large domain, namely from Q^{nd} to $1/2$, but these benefits are themselves small because Q^{nd} being small means that v is small. Maximizing $\Delta SS(1/2, Q^{nd})$ over Q^{nd} yields $Q^{nd} = 1/3$, which corresponds to $v = 2/3$, and $\Delta SS(1/2, 1/3) = 5/18 - 5/32 > 0$.[36] With constant willingness to pay v, one can show that social surplus increases with wage discrimination if and only if $v \in (1/2, 5/6)$, for which Figure 12 provides an illustration.

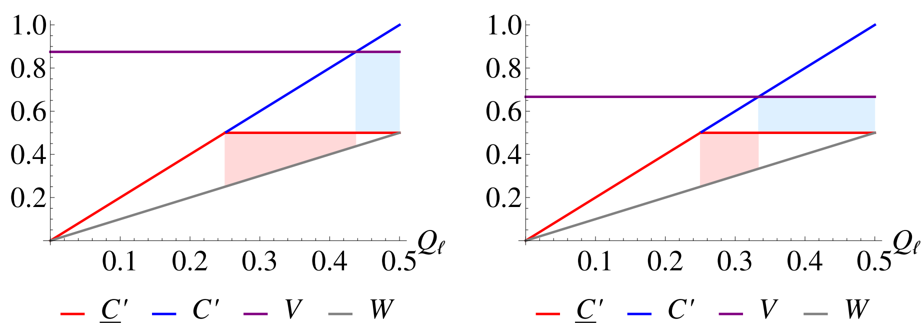

Figure 12: For $V(Q) = v$ and $W(Q) = Q$, permitting wage discrimination decreases social surplus when $v = 7/8$ (left-hand panel) and increases it when $v = 2/3$ (right-hand panel).

Implementation of monopsony outcome via labor market intermediary Suppose now that the plants operated at $\ell \in \{0, 1\}$ are independently owned and operated by firms that each have the same marginal benefit function $V(Q_\ell)$. To see how the implementation of the optimal mechanism for the multi-jobs monopsony can be implemented via a third party, suppose that in addition to the two independent firms, there is a labor market intermediary who offers wages of $1/2$ to workers who will then be randomly matched to one of the two

[36] If $V(Q) = v - Q$ with $v \in (3/4, 1)$, which implies $Q^d \in (1/4, 1/2)$, then we have $\Delta C(Q^d, Q^{nd}) > 0 > \Delta SS(Q^d, Q^{nd})$, that is, prohibiting wage discrimination decreases total wage payments and increases social surplus.

firms. Each firm pays a fee of $1/2$ to the intermediary for each worker that is referred, possibly up to a quantity constraint. In addition, each independent firm $\ell \in \{0,1\}$ offers a wage of $w_\ell = 1/4$ to workers it hires directly. These wages are mutually best responses given the intermediary's behaviour. The intermediary makes zero profits and each of the independent firms receives half of the multi-jobs monopsony profit. Of course, because the scheme is strictly profitable, the intermediary could charge the firms a fixed payment for its services. For example, with a constant willingness to pay of v per worker with $v \in (1/2, 1)$, each firm earns $v^2/4$ without the intermediary and $v/2 - 3/16$ with the intermediary. Hence, any fixed fee $\phi \in [0, v/2 - 3/16 - v^2/4)$ will be acceptable for the firms since they are still strictly better off with the intermediary and its fee than without it.

6.2 Discussion

The analysis of and discussion related to the Hotelling model above raises the questions of whether the same or similar effects are also present in the model set up in Section 2 and analyzed in Section 3. We now address these questions, beginning with the effects of permitting or prohibiting wage discrimination in that model.

Effects of wage discrimination with homogeneous workers Prohibiting wage discrimination in the baseline model of Section 2 means that the firm will optimally procure a quantity Q satisfying $V(Q) = C'(Q)$. Since C' is not monotone, there can be multiple local maxima. Of course, the monopsony will choose the quantity that corresponds to the global profit maximum, but this quantity may be larger or smaller than the quantity Q^* that the monopsony procures under the optimal mechanism when $\underline{C}(Q^*) < C(Q^*)$. This means that there is, in general, no monotone quantity effect of prohibiting wage discrimination akin to the one in Proposition 9 for the Hotelling model. Of course, just like there, keeping the quantity fixed, allowing the monopsony to wage discriminate can only decrease total wage payments. Even so, workers who are employed at the high wage are better off with wage discrimination than without it, keeping the employment level fixed. Interestingly, because of the possibility of a positive quantity effect and because, in contrast to the Hotelling model, all but a measure zero of workers who are employed in equilibrium get a strictly positive surplus in the baseline model, it is possible that permitting wage discrimination increases worker surplus.[37] For example, for the piecewise linear specification (2) and $V(Q) = v$ for all $Q \leq 1/4$ and $V(Q) = 0$ otherwise with $v \in (C'(Q_1), C'(1/4))$, the global maximum when

[37] When wage discrimination is prohibited, worker surplus with an employment level of Q is $WS(Q) = C(Q) - \int_0^Q W(x)dx$. When wage discrimination is permitted and $Q \in (Q_1(m), Q_2(m))$, worker surplus is $\underline{WS}(Q) = \underline{C}(Q) - \int_0^{Q_1(m)} W(x)dx - \frac{Q - Q_1(m)}{Q_2(m) - Q_1(m)} \int_{Q_1(m)}^{Q_2(m)} W(x)dx$.

wage discrimination is prohibited is always given by a quantity smaller than 1/4. When wage discrimination is permitted, the optimal quantity is 1/4. For v sufficiently small, that is, less than 1.65, worker surplus is larger with wage discrimination than without it.[38]

Heterogeneous tasks and endogenous multi-tasking In the baseline model, we have assumed that the firm only hires workers to perform a single homogeneous task. In Appendix B.3 we generalize this by assuming that a monopsony firm has demand for n different tasks, indexed by i, with a maximal demand for task i of k_i. Each task $i \in \{1, \ldots, n\}$ is associated with a weight $\theta_i > 0$, so that when the firm assigns the Q-th unit of hired labor to task i, its marginal willingness to pay for that unit of labor is $V(Q)\theta_i$. Similarly, the cost of executing task i for the Q-th lowest cost worker is given by $W(Q)\theta_i$.

The problem faced by the profit-maximizing monopsony in the presence of heterogeneous tasks is to choose the total number of workers it wants to employ and how to allocate tasks across these workers. In Appendix B.3 we show how this problem can be solved by applying the analysis of Loertscher and Muir (2021a). When C is convex, the monopsony optimally assigns tasks to workers in a positive assortative fashion. In contrast, if $\underline{C}$ fails to be convex, the monopsony may perform a generalized ironing procedure in which different tasks are packaged into a single job and workers are asked to *multi-task*. This analysis therefore provides an alternative interpretation of multi-tasking in the sense of Holmström and Milgrom (1991). In our setting, it arises from cost minimization by a monopsony with heterogeneous tasks that faces a non-convex procurement cost function.

6.3 Unemployment insurance and unemployment

Like minimum wages, unemployment insurance is often perceived as a cause of unemployment. We now briefly analyze unemployment insurance and show that, in contrast to minimum wages, unemployment insurance has the effect of exacerbating unemployment and decreasing employment when the equilibrium without government intervention exhibits involuntary unemployment. To fix ideas, we focus on the model from Section 2 involving a monopsony employer, homogeneous workers and homogeneous tasks without minimum wages and stipulate that any worker who is willing to work but is not employed is entitled to an amount of unemployment insurance $I > 0$. (Workers who are not willing to work cannot get the insurance payment, so in order to obtain I one has to participate in the lottery and take

[38] See Loertscher and Muir (2021a, Proposition 5) for a more elaborate analysis of the related problem of facilitating or prohibiting resale on consumer surplus in a monopoly pricing problem in which the optimal selling mechanism involves rationing. The two problems are related because resale as modelled there induces an efficient allocation among the agents, which is what occurs here without wage discrimination.

the "risk" of being employed.) We assume that the equilibrium without insurance involves involuntary unemployment, that is, $Q^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ and that in the presence of insurance, the monopsony sets wages w_1 and w_2, hiring Q_1 workers at w_1 with certainty and the fraction $\alpha = \frac{Q-Q_1}{Q_2-Q_1}$ workers willing to work at w_2, with Q_1, Q and Q_2 satisfying $Q_1 \leq Q \leq Q_2$.[39] We assume that workers pay the opportunity cost of working if and only if they are actually employed.

In the following proposition, we consider a small change in the level of unemployment insurance in a situation when the equilibrium without government intervention involves involuntary unemployment and wage dispersion. The key impact of unemployment insurance is that it relaxes the participation constraint for the marginal worker, allowing the monoposny to set a lower efficiency wage $w_2 = W(Q_2) - \frac{1-\alpha}{\alpha} I$. Since workers with opportunity costs between $W(Q_2) - \frac{1-\alpha}{\alpha} I$ and $W(Q_2)$ only participate because of the prospect of obtaining the unemployment benefit, not all workers who are unemployed and receive the unemployment benefit are involuntarily unemployed. Consequently, we only use the term *unemployment* here, where it is understood that this does not refer to workers with opportunity costs above $W(Q_2)$, who do not participate or obtain the unemployment insurance benefit I.

Proposition 11. *Assume $Q^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ and consider a marginal increase in I for $I > 0$ close to 0. Then the equilibrium level of unemployment increases in I and the equilibrium level of employment decreases in I.*

Proposition 11 resonates with the perceived wisdom that unemployment insurance can exacerbate unemployment and adds the insight that it may also decrease total employment. As the proof shows, the key difference to a minimum wage is that insurance increases the optimized marginal cost of procurement whereas a minimum wage decreases it.

6.4 Efficiency wages, migration and unemployment

Efficiency wage theory is customarily associated with the so-called Five-Dollar Day introduced by the Ford Motor Company in 1914; see Footnote 2. A pervasive feature of that wage increase was that it caused workers to migrate to Detroit (see, for example, Sward, 1948, p.53). As we now show, when workers face a fixed cost of moving or participating in the labor market, this gives rise to a procurement cost function that is non-convex and consequently may make the use of an efficiency wage and involuntary unemployment optimal.

Specifically, consider a model with a monopsony firm that operates in a market in which the inverse labor supply function is W_A. We assume that this function is increasing and

[39] It is an open question whether in the presence of unemployment insurance the focus on two-price mechanisms is without loss of generality.

differential. For ease of exposition, we also assume that it is convex. This implies that absent any migration, the cost $QW_A(Q)$ of procuring Q units of labor is convex in Q, which in turn implies that without migration the firm optimally sets a market-clearing wage. To model migration, we assume that there is another pool of workers whose opportunity costs of working after migrating are described by the inverse supply function W_B, which we also assume to be convex, differentiable and increasing. Each worker in this pool has the same fixed cost $k > 0$ of moving. For $i \in \{A, B\}$, let $S_i(w) = W_i^{-1}(w)$ and, for $w > W_B(0) + k$, let $S_{AB}(w) = S_A(w) + S_B(w - k)$ denote the supply function that the firm faces. Moreover, for $Q > S_A(W_B(0) + k) =: \check{Q}$, we let $W_{AB}(Q) = S_{AB}^{-1}(Q)$. Then the inverse labor supply function the firm faces is $W(Q) = W_A(Q)$ for $Q \leq \check{Q}$ and $W(Q) = W_{AB}(Q)$ for $Q > \check{Q}$, yielding the cost of procurement function $C(Q) = W(Q)Q$ that accounts for migration.[40] The key implication of this is that $C(Q)$ is not convex. As shown in Appendix A.14, we have

$$\lim_{Q \uparrow \check{Q}} C'(Q) > \lim_{Q \downarrow \check{Q}} C'(Q), \tag{11}$$

and the marginal cost of procurement decreases at $\check{Q}$. Geographical migration is only one possible interpretation of problems involving fixed costs. One can also think of workers moving from one industry to another or as workers joining the labor force at some fixed cost.

This perspective resonates with the prevalent view that migration is a cause of unemployment in the region to which workers migrate. However, here involuntary unemployment occurs not because of frictions such as costly search or costly wage adjustment, but rather as a consequence of optimal pricing on the part of the firm. It also offers a novel interpretation of the episode at the Ford Motor Company in the mid 1910s. According to this interpretation, with high enough wages, workers were willing to bear the fixed cost of moving, making the cost of procurement non-convex in the short run and efficiency wages optimal: "the greatest cost cutting measure" according to the dictum often attributed to Henry Ford. As the demand for its cars and its demand for labor continued increasing, eventually it became optimal to set market-clearing wages again. More broadly, the model with fixed costs of migration or labor market participation and an optimal mechanism used by the firm offers a framework in which economic expansion may be a cause of involuntary unemployment.

[40] For example, for $W_A(Q) = 4Q$, $W_B(Q) = \frac{4}{7}Q + \frac{1}{2}$ and $k = 1/2$, we obtain the specification in (2). To see this, note that $W_A(Q) = 4Q$ and $W_B(Q) = 4Q/7 + 1/2$ imply $S_A(w) = w/4$ and $S_B(w) = 7(w - 1/2)/4$ and hence using $k = 1/2$ for $w \geq \hat{w}$ we have $S_{AB}(w) = S_A(w) + S_B(w - k) = 2w - 7/4$. Inverting S_{AB} yields $W_{AB}(Q) = Q/2 + 7/8$, which is the second line in (2). It remains to verify that $\hat{Q} = 1/4$, which is the case since $S_A(W_B(0) + k) = (1/2 + 1/2)/4 = 1/4$.

7 Conclusions

Minimum wage legislation is at the forefront of public policy debates. We provide a model in which an appropriately chosen minimum wage increases total employment and decreases involuntary unemployment, possibly to the point of eliminating it. The model merely assumes that a monopsony firm minimizes the cost of procuring labor, subject to respecting workers' incentive compatibility and individual rationality constraints, and that the procurement cost under a market-clearing wage is not convex at the optimal level of employment. Extending the model to allow for quantity competition among firms, we show that there is no monotone relationship between competition and involuntary unemployment. The latter point is perhaps most starkly illustrated by the fact that it is possible to have involuntary unemployment and inefficient allocation under perfect competition.

In the mechanism design approach taken in this paper, randomization in the form of efficiency wages and involuntary unemployment (or mismatching of workers and jobs) occurs because it is optimal for the employer, and not as the result of search or other frictions. An interesting and relevant avenue for further research would therefore be to assess the empirical magnitude of these different causes of randomization. The policy implications may differ substantively. If these inefficiencies are caused by frictions, then reducing these frictions will typically improve welfare. If they are by design, then external interventions designed to reduce the randomness in workers-job matching may prove ineffective. Another open question for future research would be to study the policy instruments required to implement Ramsey pricing when a monopoly's marginal revenue or, equivalently, a monopsony's marginal cost of procurement is not monotone. While a price cap or floor may not suffice, its possible that both a cap *and* a floor would.

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Appendix

A Proofs

A.1 Proof of Lemma 2

Proof. Part I: Proof that the minimal cost is $\underline{C}(Q, \underline{w})$ *as given in* (5)

The designer's problem is to determine the cost-minimizing mechanism for procuring a fixed quantity Q of labor, subject to the constraint that the menu of wages does not include a wage below the minimum wage of $\underline{w}$. We let G denote the cumulative distribution of the opportunity cost of working for the population of workers. We denote the support of this distribution by $[\underline{c}, \overline{c}] \subseteq \mathbb{R}_+$ and its density by g. The assumptions introduced in Section 2 ensure that the function G admits a density g and that g is strictly positive on $[\underline{c}, \overline{c}]$. We let $\Gamma(c) := c + \frac{G(c)}{g(c)}$ denote the virtual cost function. The value of the worker's outside option of not participating in the mechanism is 0.

By the revelation principal, without loss of generality we can focus on direct mechanisms. We let $\langle x, t \rangle$ represent an arbitrary direct mechanism, with $x(c)$ denoting the probability that the worker has to work when reporting to be of type c and $t(c)$ denoting the expected transfer the worker receives if reporting to be of type c. By the payoff equivalence theorem, it is also Without loss of generality to assume that workers are paid a transfer upon becoming employed. Consequently, the worker's payoff when of type c and reporting to be of type $\hat{c}$ takes the form

$$t(\hat{c}) - x(\hat{c})c.$$

Let $U(c) := t(c) - x(c)c$ denote the worker's payoff when reporting truthfully. Individual rationality requires $U(c) \geq 0$ for all c. Incentive compatibility implies that x is non-increasing and that $U'(c) = -x(c)$ holds almost everywhere. For any $c, \hat{c} \in [\underline{c}, \overline{c}]$ we then have

$$U(c) = U(\hat{c}) + \int_c^{\hat{c}} x(y)dy.$$

Setting this equal to $t(c) - x(c)c$ and solving for $t(c)$ gives

$$t(c) = U(\hat{c}) + x(c)c + \int_c^{\hat{c}} x(y)dy.$$

Observing that for $c < \hat{c}$, $U(c) \geq U(\hat{c})$ holds because $\int_c^{\hat{c}} x(y)dy \geq 0$, the individual rationality constraint is satisfied for all types if and only if $U(\overline{c}) \geq 0$. In an optimal mechanism satisfying incentive compatibility and individual rationality, we must have $U(\bar{c}) = 0$ because otherwise the designer leaves money on the table. Expressing $t(c)$ with $\hat{c} = \bar{c}$ and using $U(\bar{c}) = 0$, we thus obtain

$$t(c) = x(c)c + \int_c^{\bar{c}} x(y)dy.$$

Note that $t(c)$ is the expected transfer paid to workers of type c. In line with real-world practice, we assume that the minimum wage $\underline{w}$ represents a constraint on the wage payments made to hired workers. Since workers of type c are hired with probability $x(c)$ and workers are only paid a wage upon being hired, the constraints that the minimum wage $\underline{w}$ imposes on the transfers $t(c)$ are given by $\underline{w}x(c) \leq t(c)$.

The designer's procurement cost minimization problem, subject to the minimum wage constraint parameterized by $\underline{w}$, is thus given by

$$\min_x \int_{\underline{c}}^{\bar{c}} t(c)\, dG(c)$$

$$\text{s.t. } x \text{ is non-increasing}, \qquad \int_{\underline{c}}^{\bar{c}} x(c)\, dG(c) = Q, \quad \underline{w}x(c) \leq t(c) \text{ for all } c \in [\underline{c}, \bar{c}].$$

We have a continuum of constraints given by $\underline{w}x(c) \leq t(c)$ for all $c \in [\underline{c}, \bar{c}]$. Under ex post individual rationality (EIR), no worker can ever be paid a wage w that is less than its opportunity cost. This means that for worker types with costs $c > \underline{w}$, the constraint never binds under EIR.

Using the fact that the constraint $\underline{w}x(c) \leq t(c)$ is equivalent to $h(c) := \underline{w}x(c) - t(c) \leq 0$, we next show that $h(c)$ decreases in c on $[\underline{c}, \underline{w}]$. Specifically, letting $c_0, c_1 \in [\underline{c}, \underline{w}]$ with $c_0 < c_1$, we have

$$\begin{aligned} h(c_1) - h(c_0) &= \underline{w}(x(c_1) - x(c_0)) - (x(c_1)c_1 - x(c_0)c_0) + \int_{c_0}^{c_1} x(y)dy \\ &= (\underline{w} - c_1)(x(c_1) - x(c_0)) + \int_{c_0}^{c_1} x(y)dy - (c_1 - c_0)x(c_0) \leq 0, \end{aligned}$$

where the inequality is strict if x is not constant on $[c_0, c_1]$.[41] This shows that it suffices to impose the constraint associated with the minimum wage on the lowest type $c = \underline{c}$.

We let λ denote the Lagrange multiplier corresponding to the lowest type $c = \underline{c}$. Setting aside the quantity constraint for now and using $t(c) = x(c)c + \int_c^{\bar{c}} x(y)\, dy$, the Lagrangian is

[41] Since x is non-increasing, if x is not constant on $[c_0, c_1]$ we have $(c_1 - c_0)x(c_0) > \int_{c_0}^{c_1} x(y)dy$ and $(\underline{w} - c_1)(x(c_1) - x(c_0)) \leq 0$ with strict inequality if $c_1 < \underline{w}$.

then given by

$$\begin{aligned}\mathcal{L}(x,\lambda) &= \int_{\underline{c}}^{\overline{c}} t(c)\, dG(c) + \lambda(\underline{w}x(\underline{c}) - t(\underline{c})) \\ &= \int_{\underline{c}}^{\overline{c}} \left(x(c)c + \int_{c}^{\overline{c}} x(y)\, dy \right) dF(v) + \lambda x(\underline{c})(\underline{w} - \underline{c}) - \lambda \int_{\underline{c}}^{\overline{c}} x(c)\, dc.\end{aligned}$$

Using

$$\int_{\underline{c}}^{\overline{c}} \int_{c}^{\overline{c}} g(c)x(y)\, dy\, dc = \int_{\underline{c}}^{\overline{c}} \int_{\underline{c}}^{y} g(c)x(y)\, dc\, dy = \int_{\underline{c}}^{\overline{c}} G(y)x(y)\, dy.$$

we have

$$\begin{aligned}\mathcal{L}(x,\lambda) &= \int_{\underline{c}}^{\overline{c}} \Gamma(c)x(c)\, dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}) - \lambda \int_{\underline{c}}^{\overline{c}} x(c)\, dc \\ &= \int_{\underline{c}}^{\overline{c}} \left(\Gamma(c) - \frac{\lambda}{g(c)} \right) x(c)\, dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}).\end{aligned}$$

Letting $H(x) = \mathbf{1}(x \geq 0)$ denote the Heaviside step function and using the the probability measure $G_\lambda(c) = \frac{\lambda}{1+\lambda} H(\underline{c} - c) + \frac{1}{1+\lambda} G(c)$, we can rewrite the Lagrangian as

$$\mathcal{L}(x,\lambda) = (1+\lambda) \int_{\underline{c}}^{\overline{c}} \left[\left(\Gamma(c) - \frac{\lambda}{g(c)} \right) \mathbf{1}(c > \underline{c}) + (\underline{w} - \underline{c})\, \mathbf{1}\,(c = \underline{c}) \right] x(v)\, dG_\lambda(c).$$

We can therefore derive the optimal allocation rule x^* by ironing the function

$$\Psi(c,\lambda) = \begin{cases} \Gamma(c) - \frac{\lambda}{g(c)}, & c \in (\underline{c}, \overline{c}] \\ \underline{w} - \underline{c}, & c = \underline{c} \end{cases}$$

with respect to the probability measure G_λ. Note that if the function Ψ discontinuously decreases at $x = \underline{c}$ this implies that the ironed function $\overline{\Psi}$ contains an ironing interval with an endpoint at $c = \underline{c}$. This ironing interval precisely corresponds to the region identified in Section 4.2 where the optimal mechanism does not involve rationing at the minimum wage $\underline{w}$. Any additional ironing regions correspond to two-price mechanisms with no randomization at the top and rationing at the minimum wage $\underline{w}$. This shows that our restriction to two-price mechanisms when $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}; m))$ is without loss of generality as required.

Part II: Proof of the stated properties of $\underline{C}(Q, \underline{w})$

Since $\underline{w} \in (Q_1(m), Q_2(m)]$, m is fixed and for the remainder of this proof we omit the dependence of $Q_i(m)$, $w_1^{-1}(\underline{w}; m)$ and $w_1(Q; m)$ on m and simply write Q_i, $\hat{Q}$ and $w_1(Q)$. The monopsony solves

$$\min_{q_1 \in [0,Q), q_2 > Q} (1-\alpha)C(q_1) + \alpha C(q_2),$$

where $\alpha = \frac{Q-q_1}{q_2-q_1}$, subject to the constraint $(1-\alpha)W(q_1) + \alpha W(q_2) \geq \underline{w}$. The corresponding Lagrangian is

$$\mathcal{L}(q_1, q_2, \lambda) = (1-\alpha)C(q_1) + \alpha C(q_2) - \lambda[(1-\alpha)W(q_1) + \alpha W(q_2) - \underline{w}],$$

where λ is the Lagrange multiplier associated with the minimum wage constraint. For $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ the constraint will bind (i.e. hold with equality at an optimum). Otherwise, the solution would be $q_i = Q_i$ and involve a low wage of $w_1(Q) < \underline{w}$, which violates the minimum wage constraint.

Using $C_\lambda(Q) := W(Q)(Q - \lambda)$, the Lagrangian can equivalently be written as

$$\mathcal{L}(q_1, q_2, \lambda) = (1-\alpha)C_\lambda(q_1) + \alpha C_\lambda(q_2) + \lambda \underline{w}.$$

Using the facts that

$$\frac{\partial \alpha}{\partial q_1} = -\frac{1-\alpha}{q_2-q_1} \quad \text{and} \quad \frac{\partial \alpha}{\partial q_2} = -\frac{\alpha}{q_2-q_1},$$

the first-order conditions with respect to q_1 and q_2 are those captured in

$$C_\lambda'(q_1) = \frac{C_\lambda(q_2) - C_\lambda(q_1)}{q_2-q_1} = C_\lambda'(q_2) \tag{12}$$

while the first-order condition with respect to λ is and

$$(1-\alpha)W(q_1) + \alpha W(q_2) = \underline{w}. \tag{13}$$

Letting

$$H(q_2, q_1, \lambda) = \frac{C_\lambda(q_2) - C_\lambda(q_1)}{q_2-q_1}$$

and using subscripts to denote partial derivatives, we have

$$H_1(q_2, q_1, \lambda) = \frac{1}{q_2-q_1}\left[C_\lambda'(q_2) - H(q_2, q_1, \lambda)\right], \; H_2(q_2, q_1, \lambda) = \frac{1}{q_2-q_1}\left[H(q_2, q_1, \lambda) - C_\lambda'(q_1)\right],$$

$$H_3(q_2, q_1, \lambda) = \frac{W(q_1) - W(q_2)}{q_2-q_1}.$$

Note that $H_3 < 0$ because by assumption we have $q_2 > q_1$ and W is an increasing function.

Observe also that (12) is equivalent to

$$C'_\lambda(q_1) = H(q_2, q_1, \lambda) = C'_\lambda(q_2). \tag{14}$$

Denote by $q_1^*(\lambda)$ and $q_2^*(\lambda)$ the values of q_1 and q_2 that satisfy (14). Evaluated at these values, we have

$$H_1(q_2^*(\lambda), q_1^*(\lambda), \lambda) = 0 = H_2(q_2^*(\lambda), q_1^*(\lambda), \lambda).$$

This implies that the second partials of $\mathcal{L}(q_1, q_2, \lambda)$ with respect to q_1 and q_2, evaluated at $q_i = q_i^*$ are

$$\frac{\partial^2 \mathcal{L}(q_1^*, q_2^*, \lambda^*)}{\partial q_1^2} = (1-\alpha)C''_\lambda(q_1^*) \quad \text{and} \quad \frac{\partial^2 \mathcal{L}(q_1^*, q_2^*, \lambda^*)}{\partial q_2^2} = \alpha C''_\lambda(q_2^*)$$

and

$$\frac{\partial^2 \mathcal{L}(q_1^*, q_2^*, \lambda^*)}{\partial q_1 \partial q_2} = 0.$$

The matrix of second partials is thus

$$\begin{pmatrix} (1-\alpha)C''_\lambda(q_1^*) & 0 \\ 0 & \alpha C''_\lambda(q_2^*) \end{pmatrix}.$$

This is positive definite if and only if $(1-\alpha)C''_\lambda(q_1^*) > 0$ and $\alpha C''_\lambda(q_2^*) > 0$. Thus, for each $i \in \{1, 2\}$, at the optimum we have

$$C''_\lambda(q_i^*) > 0.$$

Totally differentiating $C'_\lambda(q_i^*) = H(q_2^*, q_1^*, \lambda)$ with respect to λ and using $H_1(q_2^*, q_1^*, \lambda) = 0 = H_2(q_2^*, q_1^*, \lambda)$ yields

$$\frac{dq_i^*}{d\lambda} = \frac{H_3(q_2^*, q_1^*, \lambda) + W'(q_i^*)}{C''_\lambda(q_i^*)}.$$

Because $C''_\lambda(q_i^*) > 0$, it follows that $\frac{dq_i^*}{d\lambda}$ has the same sign as

$$H_3(q_2^*, q_1^*, \lambda) + W'(q_i^*) = \frac{W(q_1^*) - W(q_2^*)}{q_2^* - q_1^*} + W'(q_i^*).$$

We next show that this expression is positive for $i = 1$ and negative for $i = 2$.

To that end, we first notice that for $q_1^* < q_2^*$ and $Q \in (q_1^*, q_2^*)$, $C_\lambda(Q)$ is not convex. That is, for all $Q \in (q_1^*, q_2^*)$ we have $\underline{C}_\lambda(Q) < C_\lambda(Q)$. Otherwise, there would be no need to convexify $C_\lambda(Q)$. We now show that this implies that $W(Q)$ is not convex on $[q_1^*, q_2^*]$ by showing that convexity of W implies convexity of C_λ.

To see this, for $a \in [0,1]$ and x_0 and x_1 satisfying $q_1^* \leq x_0 < x_1 \leq q_2^*$, define $x^a := ax_0 + (1-a)x_1$. Convexity of W on $[q_1^*, q_2^*]$ means that

$$W(x^a) \leq aW(x_0) + (1-a)W(x_1).$$

Now by definition of C_λ, we have $C_\lambda(x^a) = C(x^a)(x^a - \lambda)$. Convexity of W then implies that

$$\begin{aligned}
C_\lambda(x^a) &\leq (aW(x_0) + (1-a)W(x_1))\,(ax_0 + (1-a)x_1 - \lambda) \\
&= (aW(x_0) + (1-a)W(x_1))\,(a(x_0 - \lambda) + (1-a)(x_1 - \lambda) \\
&= a\,(aW(x_0) + (1-a)W(x_1))\,(x_0 - \lambda) + (1-a)\,(aW(x_0) + (1-a)W(x_1))\,(x_1 - \lambda)) \\
&= aW(x_0)(x_0 - \lambda) + (1-a)W(x_1)(x_1 - \lambda) + a(1-a)(W(x_1) - W(x_0))(x_0 - x_1) \\
&= aC_\lambda(x_0) + (1-a)C_\lambda(x_1) + a(1-a)(W(x_1) - W(x_0))(x_0 - x_1) \\
&\leq aC_\lambda(x_0) + (1-a)C_\lambda(x_1).
\end{aligned}$$

Here, the second inequality follows because $W(x_1) - W(x_0) > 0$ and $x_0 - x_1 < 0$ (which also implies that the inequality is strict if $a \in (0,1)$.) Thus, C_λ is convex if W is convex. Because C_λ is not convex on $[q_1^*, q_2^*]$, this implies that $W(Q)$ is not convex on $[q_1^*, q_2^*]$. That is, for all $Q \in (q_1^*, q_2^*)$,

$$W(Q) > W(q_1^*) + (Q - q_1^*)\frac{W(q_2^*) - W(q_1^*)}{q_2^* - q_1^*}.$$

Finally, because $W(Q)$ intersects with the linear function $W(q_1^*) + (Q - q_1^*)\frac{W(q_2^*) - W(q_1^*)}{q_2^* - q_1^*}$ at $Q = q_2^*$ from above, it follows that the slope of W at that point is smaller than $\frac{W(q_2^*) - W(q_1^*)}{q_2^* - q_1^*}$. Consequently, we have $W'(q_2^*) < \frac{W(q_2^*) - W(q_1^*)}{q_2^* - q_1^*}$, which is equivalent to

$$\frac{W(q_1^*) - W(q_2^*)}{q_2^* - q_1^*} + W'(q_2^*) = H_3(q_2^*, q_1^*, \lambda) + W'(q_2^*) < 0.$$

This implies

$$\frac{\partial q_2^*(\lambda)}{d\lambda} < 0.$$

By the same token, $W(Q)$ intersects with the linear function $W(q_1^*) + (Q - q_1^*)\frac{W(q_2^*) - W(q_1^*)}{q_2^* - q_1^*}$ at $Q = q_1^*$ from below. This implies that $W(q_1^*) + (q_2^* - q_1^*)W'(q_1^*) > W(q_2^*)$, which is equivalent to

$$\frac{W(q_1^*) - W(q_2^*)}{q_2^* - q_1^*} + W'(q_1^*) = H_3(q_2^*, q_1^*, \lambda) + W'(q_1^*) > 0,$$

implying that

$$\frac{\partial q_1^*(\lambda)}{d\lambda} > 0.$$

Once we have established the comparative static properties of the solution value $\lambda^*(Q, \underline{w})$ with respect to Q and $\underline{w}$, the comparatives static properties of $q_i^*(Q, \underline{w})$ with respect to these parameters will follow from the definition of $q_i^*(Q, \underline{w})$ via $q_i^*(Q, \underline{w}) = q_i^*(\lambda^*(Q, \underline{w}))$ and the facts $\frac{\partial q_1^*(\lambda)}{d\lambda} > 0 > \frac{\partial q_2^*(\lambda)}{d\lambda}$. Using (13) and totally differentiating $(1-\alpha^*)W(q_1^*)+\alpha^* W(q_2^*) = \underline{w}$ with respect to $\underline{w}$, where $\alpha^* = \frac{Q-q_1^*}{q_2^*-q_1^*}$ and we have dropped dependence on λ^* for notational ease, yields

$$\left\{(1-\alpha^*)\frac{dq_1^*}{d\lambda}\left(W'(q_1^*(\lambda)) + H_3\right) + \alpha^* \frac{dq_2^*}{d\lambda}\left(W'(q_2^*(\lambda)) + H_3\right)\right\}\frac{d\lambda^*}{d\underline{w}} = 1.$$

Thus, $\frac{d\lambda^*}{d\underline{w}}$ is positive if the term in brackets is positive, which is the case if both summands are positive. To see that the second summand is positive, recall that $\frac{dq_2^*}{d\lambda} < 0$ and $W'(q_2^*(\lambda)) + \frac{W(q_1^*)-W(q_2^*)}{q_2^*-q_1^*} < 0$. To see that the first summand is positive, it suffices to recall that $\frac{dq_1^*}{d\lambda} > 0$ and that $W'(q_1^*) + \frac{W(q_1^*)-W(q_2^*)}{q_2^*-q_1^*} > 0$. Because $\frac{dq_i^*(Q,\underline{w})}{d\underline{w}} = \frac{dq_i^*(\lambda)}{d\lambda}\frac{d\lambda^*(Q,\underline{w})}{d\underline{w}}$, it follows that

$$\frac{dq_1^*(Q,\underline{w})}{d\underline{w}} > 0 > \frac{dq_2^*(Q,\underline{w})}{d\underline{w}}.$$

Similarly, totally differentiating $(1-\alpha^*)W(q_1^*) + \alpha^* W(q_2^*) = \underline{w}$ with respect to Q yields

$$\left\{(1-\alpha^*)\frac{dq_1^*}{d\lambda}\left(W'(q_1^*(\lambda)) + H_3\right) + \alpha^* \frac{dq_2^*}{d\lambda}\left(W'(q_2^*(\lambda)) + H_3\right)\right\}\frac{d\lambda^*}{dQ} = H_3.$$

Since the right-hand side is negative and the term in brackets on the left-hand side is, as just shown, positive, it follows that $\frac{d\lambda^*}{dQ} < 0$, implying

$$\frac{dq_1^*(Q,\underline{w})}{dQ} < 0 < \frac{dq_2^*(Q,\underline{w})}{dQ}.$$

It is useful to note that

$$\frac{d\lambda^*}{dQ} = H_3 \frac{d\lambda^*}{d\underline{w}}.$$

We next show that $\lambda^*(Q, \underline{w}) \downarrow 0$ as $Q \uparrow w_1^{-1}(\underline{w})$, or equivalently that $\lambda^*(Q, \underline{w}) \downarrow 0$ as $Q\underline{w} \downarrow w_1(Q)$. To see this, notice that $q_i^*(0) = Q_i$, in which case (13) is satisfied if $\underline{w} = w_1(Q)$. Hence, $q_i^*(Q, \underline{w}) \to Q_i$ as $Q \to w_1^{-1}(\underline{w})$ (and equivalently, as $\underline{w} \downarrow w_1(Q)$) follows.

We are left to establish the stated properties of $\mathcal{L}^*(Q, \underline{w})$. By construction, we have

$$\mathcal{L}^*(Q, \underline{w}) = (1 - \alpha^*) C_{\lambda^*}(q_1^*) + \alpha^* C_{\lambda^*}(q_2^*) + \lambda^* \underline{w},$$

where $\lambda^* = \lambda^*(Q, \underline{w})$, $q_i^* = q_i^*(Q, \underline{w})$ and $\alpha^* = \frac{Q - q_1^*}{q_2^* - q_1^*}$. Because $\lambda^* = 0$ at $\underline{w} = w_1(Q)$ (and equivalently at $Q = w_1^{-1}(\underline{w})$), $\mathcal{L}^*(Q, \underline{w}) = \underline{C}(Q)$ at $\underline{w} = w_1(Q)$ follows. Likewise, $\lambda^* = Q$ at $\underline{w} = W(Q)$ implies that $C_{\lambda^*}(Q) = 0$ and $\mathcal{L}^*(Q, \underline{w}) = Q\underline{w}$.

By the envelope theorem we have, for $\underline{w} > w_1(Q)$ and $Q < w_1^{-1}(\underline{w})$, respectively,

$$\frac{\partial \mathcal{L}^*(Q, \underline{w})}{\partial \underline{w}} = \lambda^* > 0 \quad \text{and} \quad \frac{\partial \mathcal{L}^*(Q, \underline{w})}{\partial Q} = H(q_2^*, q_1^*, \lambda^*) > 0,$$

establishing the required monotonicity properties. Finally, taking the derivative with respect to Q once more yields

$$\frac{\partial^2 \mathcal{L}^*(Q, \underline{w})}{\partial \underline{w} \partial Q} = \frac{\partial \lambda^*}{\partial Q} = H_3(q_2^*, q_1^*, \lambda^*) \frac{\partial \lambda^*}{\partial \underline{w}} < 0 \quad \text{and} \quad \frac{\partial^2 \mathcal{L}^*(Q, \underline{w})}{\partial Q^2} = H_3(q_2^*, q_1^*, \lambda^*) \frac{\partial \lambda^*}{\partial Q} > 0,$$

where the inequalities follows because $\frac{\partial \lambda^*}{\partial \underline{w}} > 0 > H_3(q_2^*, q_1^*, \lambda^*)$. Thus, $\mathcal{L}^*(Q, \underline{w})$ is convex in Q, and increases in $\underline{w}$ decrease the marginal cost of procurement $\frac{\partial \mathcal{L}^*(Q, \underline{w})}{\partial Q}$. □

A.2 Proof of Lemma 3

Proof. Given a minimum wage $\underline{w}$, $\underline{C}(Q, \underline{w})$ is the minimal cost of procuring the quantity Q and this cost is convex in Q. Moreover, $V(Q)$ is the marginal benefit of procuring the quantity Q. Putting these facts together, $Q^*(\underline{w})$ must then satisfy $V(Q^*(\underline{w})) = \underline{C}'(Q^*(\underline{w}), \underline{w})$, provided that a Q such that $V(Q) = \underline{C}'(Q, \underline{w})$ exists. When no such Q exists, the optimal quantity procured is $S(\underline{w})$ because $\lim_{Q \downarrow S(\underline{w})} \underline{C}'(Q, \underline{w}) > V(S(\underline{w}))$.

The optimal procurement mechanism involves wage dispersion if and only if $Q^*(\underline{w}) > S(\underline{w})$. Otherwise, we have $\underline{C}(Q, \underline{w}) = \underline{w}Q$ and the optimal mechanism involves procuring the Q workers at the minimum wage $\underline{w}$.

Whenever there is wage dispersion, the optimal mechanism involves involuntary unemployment. Similarly, when $Q^*(\underline{w}) < S(\underline{w})$, there is excess supply (and consequently involuntary unemployment) at the minimum wage. □

A.3 Proof of Lemma 4

Proof. For $\underline{w} \in (W(Q_1(m)), W(Q_2(m))$ and $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}; m))$, $\partial \underline{C}(Q, \underline{w}) / \partial Q$ is continuous in Q and $\underline{w}$. Hence, $\gamma(Q; m)$ is continuous.

Next we show that

$$\gamma(Q_1(m); m) = C'(Q_2(m)) = \gamma(Q_2(m); m).$$

To see this, notice that the constraint $\underline{w} = W(Q_1(m))$ does not bind at $Q = Q_1(m)$. That $\gamma(Q_1(m); m) = C'(Q_2(m))$ then follows from the definitions of $Q_1(m)$ and $Q_2(m)$. At $Q = Q_2(m)$, we have $\underline{C}(Q, \underline{w}) = \underline{w}Q$ for all $Q \leq Q_2(m)$ and $\underline{C}(Q, \underline{w}) = \underline{C}(Q) = C(Q)$ for $Q \in (Q_2(m), Q_2(m) + \delta)$, where $\delta > 0$ is sufficiently small. This implies that $\underline{C}'(Q, \underline{w}) = C'(Q)$ for all $Q \in (Q_2(m), Q_2(m) + \delta)$ and setting $\underline{w} = W(Q_2(m))$ we see that $\gamma(Q_2(m); m) = C'(Q_2(m))$ must also hold.

From Lemma 2 we know that $\underline{C}'(Q, \underline{w})$ is increasing in Q for all $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}; m))$. This implies that $\gamma(Q; m) < \underline{C}'(Q_2(m))$ holds for $Q \in (Q_1(m), Q_2(m))$. Moreover, $V(Q)$ is continuous and decreasing, and $\gamma(Q; m)$ is continuous and less than $\underline{C}'(Q_2(m))$ for $Q \in (Q_1(m), Q_2(m))$ and equal to $\underline{C}'(Q_2(m))$ for $Q = Q_2(m)$. By assumption, we also have $V(Q^*) = \underline{C}'(Q_2(m))$ with $Q^* \in (Q_1(m), Q_2(m))$. Putting all of this together, it follows that a smallest and a largest point of intersection of V and γ on $(Q_1(m), Q_2(m))$ exist and that the smallest point of intersection is strictly larger than Q^*. This establishes that $Q^* < \hat{Q}_L(m)$.

Since $\underline{C}(Q, \underline{w})$ is convex and $\underline{C}(Q, \underline{w}) = \underline{w}Q$ holds for $Q \leq S(\underline{w})$, $\underline{C}'(Q, \underline{w}) \geq \underline{w}$ holds for $Q > S(\underline{w})$. This implies that $\gamma(Q; m) \geq W(Q)$, which in turn implies that $\hat{Q}_H(m) \leq Q^p$, with equality if and only if $\gamma(Q^p; m) = W(Q^p)$. □

A.4 Proof of Proposition 3

Proof. By construction, for $\underline{w} \in (w_1(Q^*; m), W(\hat{Q}_L(m)))$, the point of intersection between $V(Q)$ and $\underline{C}'(Q, \underline{w})$, $Q^*(\underline{w})$, is larger than $S(\underline{w})$. By Lemma 3, this implies that there is wage dispersion and involuntary unemployment, which establishes (i).

We are left to prove (ii). That the equilibrium quantity increases follows from the fact that $V(Q)$ is downward sloping in Q and that marginal cost of procurement is decreasing in $\underline{w}$ stated in Lemma 2. Formally, $Q^*(\underline{w})$ satisfies

$$V'(Q^*(\underline{w})) = H(q_2^*, q_1^*, \lambda^*),$$

where $H(q_2^*, q_1^*, \lambda^*)$ is the marginal cost of procurement derived in the proof of Lemma 2. Totally differentiating yields

$$\frac{dQ^*(\underline{w})}{d\underline{w}} = \frac{H_3}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}} \frac{\partial \lambda^*}{\partial \underline{w}} > 0,$$

where the inequality holds because $V' < 0$, $H_3 < 0$ and $\frac{d\lambda^*}{dQ} < 0 < \frac{d\lambda^*}{d\underline{w}}$.

The lower of the two wages paid in equilibrium is $\underline{w}$, which trivially increases in $\underline{w}$. We are going to show that the higher of the two wages decreases in $\underline{w}$ by showing that $q_2^*(Q^*(\underline{w}), \underline{w})$ decreases in $\underline{w}$.

Using the definition of $q_2^*(Q, \underline{w}) = q_2^*(\lambda^*(Q, \underline{w}))$ and totally differentiation $q_2^*(\lambda^*(Q^*(\underline{w}, \underline{w}))$ with respect to $\underline{w}$ yields

$$\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} = \frac{\partial q_2^*}{\partial \lambda}\left[\frac{\partial \lambda^*}{\partial Q}\frac{\partial Q^*(\underline{w})}{\partial \underline{w}} + \frac{\partial \lambda^*}{\partial \underline{w}}\right] = \frac{\partial q_2^*}{\partial \lambda}\frac{\partial \lambda^*}{\partial \underline{w}}\left[H_3 \frac{\partial Q^*(\underline{w})}{\partial \underline{w}} + 1\right].$$

Here, the second equality follows from $\frac{\partial \lambda^*}{\partial Q} = H_3 \frac{\partial \lambda^*}{\partial \underline{w}}$. Substituting

$$\frac{dQ^*(\underline{w})}{d\underline{w}} = \frac{H_3 \frac{\partial \lambda^*}{\partial \underline{w}}}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}}$$

into this last expression yields

$$\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} = \frac{\partial q_2^*}{\partial \lambda}\frac{\partial \lambda^*}{\partial \underline{w}}\left[\frac{(H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}}}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}} + 1\right] = \frac{\partial q_2^*}{\partial \lambda}\frac{\partial \lambda^*}{\partial \underline{w}}\left[\frac{(H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}} + V' - H_3 \frac{\partial \lambda^*}{\partial Q}}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}}\right].$$

Since $\frac{\partial q_2^*}{\partial \lambda} < 0 < \frac{\partial \lambda^*}{\partial \underline{w}}$, $\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} < 0$ holds if the term in brackets is positive. To see that this is the case, we can again substitute $\frac{\partial \lambda^*}{\partial Q} = H_3 \frac{\partial \lambda^*}{\partial \underline{w}}$ to obtain

$$\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} = \frac{\partial q_2^*}{\partial \lambda}\frac{\partial \lambda^*}{\partial \underline{w}}\left[\frac{V'}{V' - (H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}}}\right].$$

Since $V' < 0$ and $V' - (H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}} < 0$, we have

$$\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} < 0$$

as desired.

That the minimum wage increase decreases involuntary unemployment is now an implication of the fact that $Q^*(\underline{w})$ increases and q_2^* decreases in $\underline{w}$. □

A.5 Proof of Lemma 5

Proof. The statement has been established in the proof of Lemma 2. □

A.6 Proof of Proposition 4

Proof. The definition of the quantity $\hat{Q}_H(m)$ implies that, by construction, there will be no wage dispersion for $\underline{w} \in W(\hat{Q}_H(m), W(Q_2(m)]$. Consequently, all the stated effects follow from standard monopsony pricing with market-clearing wages in the face of a minimum wage. $\square$

A.7 Proof of Proposition 5

Proof. Over intervals with an even index, we have $V(Q) < \gamma(Q;m)$, implying that there is no wage dispersion and no involuntary unemployment. Over intervals with an odd index, we have $V(Q) > \gamma(Q;m)$, implying that there is wage dispersion and involuntary unemployment. $\square$

A.8 Proof of Proposition 6

Proof. Firm i's first-order condition is

$$V(y_i) = \frac{Q - y_i}{Q^2}\underline{C}(Q) + \frac{y_i}{Q}\underline{C}'(Q).$$

The left-hand side is decreasing in y_i. The partial derivative of the right-hand side with respect to y_i is $-\frac{1}{Q^2}(\underline{C}(Q) - Q\underline{C}'(Q))$, which is positive because $\underline{C}$ is convex. This implies that for any aggregate quantity Q there is a unique y_i that satisfies the first-order condition. This y_i must thus be the same for all i. Hence, any equilibrium is symmetric. Given this, we can write the first-order condition as

$$V\left(\frac{Q}{n}\right) = \frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q). \tag{15}$$

The left-hand side is decreasing in Q. The derivative of the right-hand side with respect to Q is

$$-\frac{n-1}{nQ^2}\left(\underline{C}(Q) - Q\underline{C}'(Q)\right) + \frac{1}{n}\underline{C}''(Q) \geq 0. \tag{16}$$

Here the inequality follows from the fact that $\underline{C}$ is convex, which in turn implies that $\underline{C}'' \geq 0$ and $Q\underline{C}'(Q) \geq \underline{C}(Q)$. Because at $Q = 0$, the left-hand side is larger than the right-hand side, there is a unique Q that satisfies (15). This proves that the equilibrium is unique and symmetric.

To see that Q_n^* is increasing in n, suppose to the contrary that it is not and we have

$Q_n^* \geq Q_{n+1}^*$ for some n. This implies $\frac{Q_n^*}{n} > \frac{Q_{n+1}^*}{n+1}$ and therefore

$$\begin{aligned} V\left(\frac{Q_{n+1}^*}{n+1}\right) > V\left(\frac{Q_n^*}{n}\right) &= \frac{n-1}{n}\frac{\underline{C}(Q_n^*)}{Q_n^*} + \frac{1}{n}\underline{C}'(Q_n^*) \\ &\geq \frac{n-1}{n}\frac{\underline{C}(Q_{n+1}^*)}{Q_{n+1}^*} + \frac{1}{n}\underline{C}'(Q_{n+1}^*) \\ &\geq \frac{n}{n+1}\frac{\underline{C}(Q_{n+1}^*)}{Q_{n+1}^*} + \frac{1}{n+1}\underline{C}'(Q_{n+1}^*). \end{aligned}$$

Here, the first weak inequality is due to (16) and the second follows from the fact that the derivative of $\frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q)$ with respect to n is

$$\frac{1}{n^2 Q}\left[\underline{C}(Q) - Q\underline{C}'(Q)\right] \leq 0,$$

where the inequality holds because $\underline{C}(Q)$ is convex. Since in equilibrium

$$V\left(\frac{Q_{n+1}^*}{n+1}\right) = \frac{n}{n+1}\frac{\underline{C}(Q_{n+1}^*)}{Q_{n+1}^*} + \frac{1}{n+1}\underline{C}'(Q_{n+1}^*),$$

we have the desired contradiction.

That $Q_n^p < Q_n^*$ holds for n sufficiently small follows from the discussion after the proposition by choosing $n = 1$ since $h(Q,1) > W(Q)$ for all $Q \in (Q_1(m), Q_2(m))$. Moreover, $Q_n^p \leq Q_n^*$ requires $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ since otherwise $h(Q,n) = W(Q) + \frac{Q}{n}W'(Q)$, which implies $Q_n^* < Q_n^p$. The arguments after the proposition imply that $h(Q,n) < W(Q)$ for some $Q \in (Q_1(m), Q_2(m))$ can only occur if n is sufficiently large.

Assume now that $\underline{C}(Q^e) = C(Q^e)$ and let $Q_\infty := \lim_{n\to\infty} Q_n^*$. Taking limits of both sides of (15) yields

$$V(0) = \frac{\underline{C}(Q_\infty)}{Q_\infty}. \tag{17}$$

The definition of Q^e then implies that $V(0) = \frac{\underline{C}(Q_\infty)}{Q_\infty} = W(Q^e) = \frac{\underline{C}(Q^e)}{Q^e}$. Using

$$\frac{d}{dQ}\left(\frac{\underline{C}(Q)}{Q}\right) = \frac{Q\underline{C}'(Q) - \underline{C}(Q)}{Q^2} \geq 0,$$

where the inequality holds because $\underline{C}$ is convex, we have that the solution to the equation $V(0) = \frac{\underline{C}(Q_\infty)}{Q_\infty}$ is unique. Since Q^e satisfies this equation we thus have $Q_\infty = Q^e$. Hence, if $Q^e \notin \cup_{m\in\mathcal{M}}(Q_1(m)), Q_2(m))$ then Q^e is also the aggregate quantity in the limit as claimed.

Assume now that $Q^e \in (Q_1(m_e)), Q_2(m_e))$ for some $m_e \in \mathcal{M}$. For $Q \in (Q_1(m_e)), Q_2(m_e))$, $\underline{C}(Q)$ increases linearly from $C(Q_1(m_e))$ to $C(Q_2(m_e))$ with a slope that is greater than $V(0)$.

The latter follows from our observation that $\underline{C}'(Q^e) > V(0)$, which appeared immediately prior to the proposition statement. Because W is increasing we have

$$\frac{C(Q_1(m_e))}{Q_1(m_e)} = W(Q_1(m_e)) < W(Q^e) = V(0) < W(Q_2(m_e)) = \frac{C(Q_2(m_e))}{Q_2(m_e)}.$$

This implies there exists a unique number $\tilde{Q} \in (Q_1(m_e)), Q_2(m_e))$ such that $\frac{C(\tilde{Q})}{\tilde{Q}} = V(0)$. If $Q^e \in (Q_1(m_e)), Q_2(m_e))$ this is then the aggregate quantity in the limit as claimed.

We are left to show that $\tilde{Q} > Q^e$ holds whenever $Q^e \in (Q_1(m_e), Q_2(m_e))$. To see that this holds, rearrange (17) to

$$Q_\infty V(0) = \underline{C}(Q_\infty)$$

and recall that $Q^e V(0) = C(Q^e)$. Since $C(Q^e) > \underline{C}(Q^e)$, $\tilde{Q} = Q_\infty > Q^e$ follows. □

A.9 Proof of Proposition 7

Proof. Note first that $\underline{C}'(Q, \underline{w})$ is continuous at $\underline{w} = w_1(Q; m)$ because discontinuities in $\underline{C}'(Q, \underline{w})$ only occur at $\underline{w} = W(Q)$. The equilibrium condition is thus

$$V\left(\frac{Q_n^*(\underline{w})}{n}\right) = h(Q_n^*(\underline{w}), n, \underline{w}) = \frac{n-1}{n}\frac{\underline{C}(Q_n^*(\underline{w}), \underline{w})}{Q_n^*(\underline{w})} + \frac{1}{n}\underline{C}'(Q_n^*(\underline{w}), \underline{w}).$$

Totally differentiating with respect to $\underline{w}$, dropping arguments and writing $\underline{C}'$ and $\underline{C}''$ in lieu of $\frac{\partial \underline{C}}{\partial Q}$ and $\frac{\partial^2 \underline{C}}{\partial Q^2}$ yields

$$\left[V' - (n-1)\left[\frac{Q_n^*\underline{C}' - \underline{C}}{(Q_n^*)^2}\right] - \underline{C}''\right]\frac{dQ_n^*}{d\underline{w}} = (n-1)\frac{\partial \underline{C}}{\partial \underline{w}}\frac{1}{Q_n^*} + \frac{\partial \underline{C}'}{\partial \underline{w}}.$$

Since the term in brackets on the left-hand side is negative, $\frac{dQ_n^*}{d\underline{w}}$ has the opposite sign of $(n-1)\frac{\partial \underline{C}}{\partial \underline{w}}\frac{1}{Q_n^*} + \frac{\partial \underline{C}'}{\partial \underline{w}}$. From the proof of Lemma 2 , we know that $\frac{\partial \underline{C}}{\partial \underline{w}} = \lambda^* \geq 0$ and $\frac{\partial \underline{C}'}{\partial \underline{w}} = \frac{\partial \lambda^*}{\partial Q} \leq 0$, where λ^* is the solution value of the Lagrange multiplier associated with the minimum wage constraint. At $\underline{w} = w_1(Q; m)$, we have $\lambda^* = 0$ and $\frac{\partial \lambda^*}{\partial Q} < 0$. We therefore have $\frac{dQ_n^*}{d\underline{w}}|_{\underline{w}=w_1(Q_n^*;m)} > 0$ as required. □

A.10 Proof of Theorem 2

Proof. As noted, the minimum wage only binds if $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}; m))$. Fixing Q, define

$$h_\gamma(Q, n; m) := \lim_{\underline{w} \uparrow W(Q)} h(Q, n, \underline{w}).$$

Since $\underline{C}(Q, \underline{w})$ is continuous, it satisfies $\underline{C}(Q, W(Q)) = W(Q)Q$. Hence, $\lim_{\underline{w} \uparrow W(Q)} \frac{\underline{C}(Q,\underline{w})}{Q} = W(Q)$. From the monopsony model, we know that $\lim_{\underline{w} \uparrow W(Q)} \underline{C}'(Q, \underline{w}) = \gamma(Q; m)$, which is continuous in Q. We thus obtain $h_\gamma(Q, n; m)$ as given in (9). Since $h_\gamma(Q, n; m)$ is continuous, V is continuously decreasing and $Q_n^* \in (Q_1(m), Q_2(m))$, smallest and largest values of Q such that

$$V(Q/n) = h_\gamma(Q, n; m)$$

exist. We denote these values of Q by $\hat{Q}_{L,n}(m)$ and $\hat{Q}_{H,n}(m)$, respectively. Since V is decreasing and $h_\gamma(Q, n; m) < C'(Q_2(m))$ holds for $Q \in (Q_1(m), Q_2(m))$, we have

$$Q_n^* < \hat{Q}_{L,n}(m) \quad \text{and} \quad \hat{Q}_{H,n}(m) \leq Q_2(m).$$

Moreover, since $h_\gamma(Q, n; m) > W(Q)$ holds unless $\underline{C}'(Q, \underline{w})$ is continuous at $\underline{w} = W(Q)$, we have

$$\hat{Q}_{H,n}(m) \leq Q_n^p. \tag{18}$$

This last inequality is strict unless $h_\gamma(\hat{Q}_{H,n}(m), n; m) = W(\hat{Q}_{H,n}(m))$. Because $h_\gamma(Q, n; \underline{w})$ converges to $W(Q)$ as $n \to \infty$, provided $Q^e \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, we have $\lim_{n\to\infty} \hat{Q}_{H,n}(m) = \lim_{n\to\infty} Q_n^p = Q^e$.

It follows that for $\underline{w} \leq W(\hat{Q}_{L,n}(m))$, the equilibrium given the minimum wage $\underline{w}$ involves wage dispersion and involuntary unemployment. Moreover, for $\underline{w} \in [W(\hat{Q}_{H,n}(m)), W(Q_2(m)]$, there is no wage dispersion in equilibrium. Minimum wages $\underline{w} \in [W(\hat{Q}_{H,n}(m)), W(Q_n^p)]$ correspond to the *pure Stigler oligopsony* region, where increases in $\underline{w}$ increase equilibrium employment without inducing involuntary unemployment. □

A.11 Proof of Proposition 9

Proof. We prove the proposition statement by statement.

When wage discrimination is prohibited, the monopsony optimally procures the quantity Q^{nd} at each location satisfying $V(Q^{nd}) = C'(Q^{nd})$, provided $Q^{nd} \leq 1/2$. Otherwise, we have $Q^{nd} = 1/2$. Since $V(1/2) < 1$ is equivalent to $Q^{nd} < 1/2$ and $Q^d > Q^{nd}$ holds whenever $V(1/4) > 1/4$ and $V(1/2) < 1$, the statement follows.

That the monopsony's profit decreases when wage discrimination is prohibited follows simply because, for $V(1/4) > 1/2$, wage discrimination is strictly optimal.

With wage discrimination, only workers with $x \in [0, 1/4)$ and $x \in (3/4, 1]$ enjoy a positive surplus. All other workers are indifferent between working and not working and hence have a surplus of 0. When wage discrimination is prohibited, all workers with $x \in [0, Q^{nd})$ and $x \in (1 - Q^{nd}, 1]$ enjoy a positive surplus and are paid a wage of Q^{nd}, which is larger than

1/4 since $V(1/4) > 1/2$ implies that $Q^{nd} > 1/4$. □

A.12 Proof of Proposition 10

Proof. Note first that $\Delta SS(Q^{nd}, Q^{nd}) < \Delta C(Q^{nd}, Q^{nd})$ is equivalent to $Q^{nd}(2 - Q^{nd}) > 5/16$, which is equivalent to $Q^{nd} \in (1/4, 5/12)$. Together with the slope condition (10), $\Delta SS(Q^{nd}, Q^{nd}) < \Delta C(Q^{nd}, Q^{nd})$ implies the statement in the proposition. Consequently, the proof is complete if we can show that, for $Q^{nd} > 5/12$, $\Delta SS(Q^{d}, Q^{nd}) < 0$. To see that this is the case, notice that because V is decreasing, for all $x > Q^{nd}$, we have $V(x) \leq V(Q^{nd}) = 2Q^{nd}$, where the equality uses the first-order condition for Q^{nd}. This implies the first inequality in the following display

$$\begin{aligned}\Delta SS(Q^{d}, Q^{nd}) &\leq (2Q^{nd} - 1/2)(Q^{d} - Q^{nd}) - \frac{1}{2}Q^{nd}(1 - Q^{nd}) + \frac{3}{32} \\ &\leq (2Q^{nd} - 1/2)(1/2 - Q^{nd}) - \frac{1}{2}Q^{nd}(1 - Q^{nd}) + \frac{3}{32}.\end{aligned}$$

The second inequality follows from the fact that the right-hand side in the first line increases in Q^{d} because $Q^{nd} > 1/4$. Observe that

$$(2Q^{nd} - 1/2)(1/2 - Q^{nd}) - \frac{1}{2}Q^{nd}(1 - Q^{nd}) + \frac{3}{32} = -\frac{5}{32} + \frac{2Q^{nd} - 3(Q^{nd})^2}{2}.$$

Moreover, the right-hand side of this expression is decreasing for $Q^{nd} > 1/3$ and negative at $Q^{nd} = 5/12$. Thus, for all $Q^{nd} \in (1/4, 5/12)$, we have $\Delta SS(Q^{nd}, Q^{nd}) < \Delta C(Q^{nd}, Q^{nd})$, which jointly with (10) implies $\Delta SS(Q^{d}, Q^{nd}) < \Delta C(Q^{d}, Q^{nd})$, and for all $Q^{nd} \geq 5/12$, we have $\Delta SS(Q^{d}, Q^{nd})$. □

A.13 Proof of Proposition 11

Proof. We drop the index m and write Q_i for $i = 1, 2$ to denote the parameters of the ironing interval given I, where it is understood that these depend on I.

The participation constraint for the marginal worker willing to participate, whose opportunity cost of working is $W(Q_2)$, then becomes $\alpha(w_2 - W(Q_2)) + (1 - \alpha)I = 0$. This is equivalent to

$$w_2 = W(Q_2) - \frac{1 - \alpha}{\alpha}I.$$

The incentive compatibility constraint for workers with opportunity cost $W(Q_1)$ becomes

$w_1 - W(Q_1) = \alpha(w_2 - W(Q_1)) + (1-\alpha)I$. Plugging in the expression for w_2 yields

$$w_1 = (1-\alpha)W(Q_1) + \alpha W(Q_2).$$

Notice that this last expression also arises in the model without unemployment insurance. The firm's cost of hiring Q workers given I, denoted $C_I(Q, Q_1, Q_2)$, is $Q_1 w_1 + (Q - Q_1)w_2$. Putting all of this together we then have that

$$C_I(Q, Q_1, Q_2) = (1-\alpha)C(Q_1) + \alpha C(Q_2) - (Q_2 - Q)I.$$

The first-order conditions that pin down the optimal mechanism parameters are then

$$\frac{\partial C_I(Q, Q_1, Q_2)}{\partial Q_1} = (1-\alpha)\left[C'(Q_1) - \frac{C(Q_2) - C(Q_1)}{Q_2 - Q_1}\right] = 0$$
$$\frac{\partial C_I(Q, Q_1, Q_2)}{\partial Q_2} = \alpha\left[C'(Q_2) - \frac{C(Q_2) - C(Q_1)}{Q_2 - Q_1}\right] - I = 0.$$

The second-order condition for Q_2, evaluated at the first-order condition, is

$$C''(Q_2) - \frac{2I}{Q_2 - Q_1} > 0.$$

Notice that, evaluated at the first-order conditions, we have

$$\frac{\partial \frac{C(Q_2)-C(Q_1)}{Q_2-Q_1}}{\partial Q_1} = \frac{-C'(Q_1) + \frac{C(Q_2)-C(Q_1)}{Q_2-Q_1}}{Q_2 - Q_1} = 0$$

and

$$\frac{\partial \frac{C(Q_2)-C(Q_1)}{Q_2-Q_1}}{\partial Q_2} = \frac{C'(Q_2) - \frac{C(Q_2)-C(Q_1)}{Q_2-Q_1}}{Q_2 - Q_1} = \frac{I}{\alpha} > 0.$$

Totally differentiating the first-order condition for Q_2 with respect to I yields $dQ_2/dI > 0$. The first-order condition for Q_1 does not directly depend on I. Hence dQ_1/dI has the same sign as dQ_1/dQ_2, which is positive because $C''(Q_1) > 0$ by the second-order condition (and $\frac{\partial \frac{C(Q_2)-C(Q_1)}{Q_2-Q_1}}{\partial Q_2} > 0$). Thus, both Q_1 and Q_2 increase in I.

The results stated in the proposition then follows if we can show that the equilibrium quantity decreases in I. To see that this is indeed the case, notice that the slope of the ironed marginal cost curve is equal to $C'(Q_1)$, which is increasing in I because C is convex in a neighbourhood of Q_1 and Q_1 is increasing in I. Thus, the marginal cost increases in I, implying that the equilibrium quantity decreases in I. Because Q_2 increases in I, this also implies that unemployment increases in I. □

A.14 Establishing inequality (11)

Letting $\check{w} = W_B(0) + k$ (which is the same as $W_A(\check{Q})$), we have

$$\lim_{Q \uparrow \check{Q}} C'(Q) = W_A(\check{Q}) + \check{Q}(S'_A)^{-1}(\check{Q}) > W_{AB}(\check{Q}) + \check{Q}(S'_{AB})^{-1}(\check{Q}) = \lim_{Q \downarrow \check{Q}} C'(Q).$$

Here, the inequality holds because $W_A(\check{Q}) = W_{AB}(\check{Q}) = \check{w}$ and, for $w \geq \check{w}$, $S_{AB}(w) = S_A(w) + S_B(w-k)$. This implies that $S'_{AB}(w) = S'_A(w) + S'_B(w-k) > S'_A(w)$, which in turn implies that $(S'_{AB})^{-1}(\check{Q}) = \frac{1}{S'_{AB}(\check{w})} < \frac{1}{S'_A(\hat{w})} = (S'_A)^{-1}(\hat{Q})$. Consequently, the function C is not convex as required.

B Supplementary material

B.1 Effects of minimum wages $\underline{w} > W(Q_2(m))$

Assuming that $Q^* \in (Q_1(m), Q_2(m))$, we now briefly discuss the effects of minimum wages above $W(Q_2(m))$. If $Q_2(m) \geq Q^p$ then statement (ii) from Proposition 4 still holds in this case. If $Q_2(m) < Q^p$ and there is no additional ironing range between $Q_2(m)$ and Q^p (that is, if $(Q_2(m), Q^p] \cap \bigcup_{m' \in \mathcal{M}} (Q_1(m'), Q_2(m')) = \emptyset$), then increasing the minimum wage $\underline{w}$ within the range $[W(\hat{Q}_H), W(Q^p))$ increases employment without inducing involuntary unemployment and wage dispersion. Increasing the minimum wage beyond $W(Q^p)$ will induce involuntary unemployment but no wage dispersion. To see this, notice that if $Q^p \in (Q_1(m'), Q_2(m'))$ for some $m' \in \mathcal{M}$, then we have $\hat{Q}_H(m') \leq Q^p$ by the same arguments as those underlying Lemma 4. Consequently, for any $Q \geq \hat{Q}_H(m')$, which corresponds to $\underline{w} \geq W(\hat{Q}_H(m'))$, there will be no wage dispersion and $Q^*(\underline{w})$ is such that $V(Q^*(\underline{w}))) = \underline{w}$. If $Q^p < Q_1(m')$, then no minimum wage $\underline{w} \in [W(Q_1(m')), W(Q_2(m'))]$ will induce wage dispersion because $\gamma(Q; m') > V(Q)$ for all $Q \in [Q_1(m'), Q_2(m')]$. This follows from the facts that (i) $V(Q^p) = W(Q^p)$, (ii) V is decreasing and W is increasing, and (iii) $\gamma(Q; m') \geq W(Q)$.

B.2 Quantity competition equilibrium

In Figures 13 and 14 the left-hand panels are plotted using $V(y_i) = 1.1 - 8y_i$ and the right-hand panels are plotted using $V(y_i) = 1.2 - 8y_i$. This implies that for the left-hand panels we have $Q^e = 0.45 \in (Q_1, Q_2) = (0.169, 0.478)$ and $\tilde{Q} = 0.4516$, while for the right-hand panels we have $Q^e = 0.65 > Q_2$.

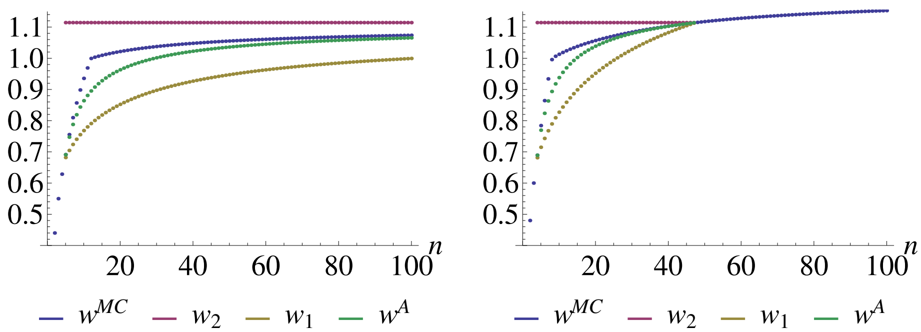

Figure 13: Equilibrium wages as a function on n, where w_1 denotes the lower equilibrium wage, w_2 denotes the higher equilibrium wage, $w^{MC} = W(Q_n^*)$ denotes the market-clearing wage and w^A the average wage $w^A = (w_1 + w_2)/2$. On the left, $W(Q^e) = 1.1 < 1.114 = w_2$ and on the right $W(Q^e) = 1.2 > w_2$.

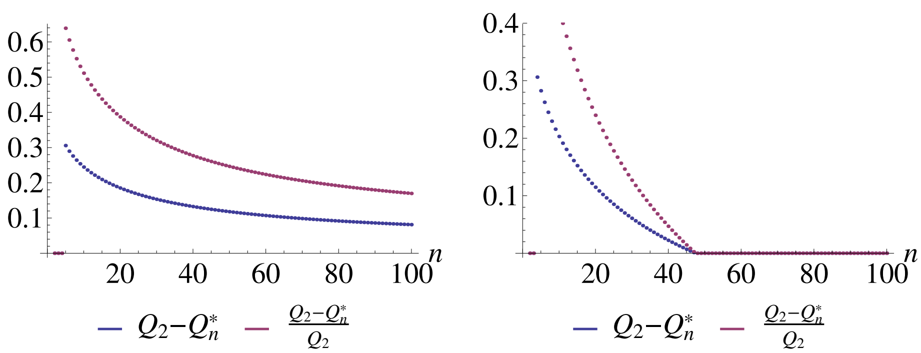

Figure 14: Involuntary unemployment and the unemployment rate as a function on n. On the left, there is involuntary unemployment of size $Q_2 - \tilde{Q} = 0.0269$ and an unemployment rate of 5.6% as $n \to \infty$.

B.3 Heterogeneous tasks

The problem faced by the monopsony firm in the presence of heterogeneous tasks is to choose the total number of workers it wants to employ, how to allocate the tasks across these workers, and how to match the executed tasks to its inverse demand function V. The solution to this last problem is simple. Because V is downward sloping, the optimal matching of executed tasks to its demand is positive assortative. That is, denoting by $q_i \le k_i$ the units of tasks i that are executed, profit is maximized by matching the best available tasks to the highest-value segment of demand, which generates a benefit of

$$\int_0^{q_1} V(x)dx + \theta_2 \int_{q_1}^{q_1+q_2} V(x)dx + \cdots + \theta_h \int_{\sum_{i=1}^{h-1} q_i}^{\sum_{i=1}^{h} q_i} V(x)dx,$$

where $h \leq n+1$ is the least productive task procured. If C is convex, then C' is increasing, which implies that the least costly way of having any collection of task $(q_1, \ldots, q_h)$ with $q_i \leq k_i$ executed is in a similar positive assortative fashion (having the lowest cost workers executing tasks 1, and so on). Provided that $K_n \geq Q^*$, it is then not hard to see that the total number of workers employed, Q^*, is given by equating marginal benefit and marginal cost, $V(Q^*) = C'(Q^*)$. If $K_n < Q^*$, then it is optimal to employ K_n workers. In either case, every worker executes exactly one task.

Optimal multi-tasking arises in equilibrium only if C is not convex. The optimal procurement mechanism with heterogeneous tasks can be derived by applying the analysis of Loertscher and Muir (2021a) to the procurement setting. First, without loss of generality, we introduce an arbitrarily large mass of job of intensity $\theta_{n+1} = 0$, and for convenience, we set $K_{(0)} = 0$ and $K_{(n+1)} = \infty$. We then identify the mass of jobs to be allocated within the interval $[0, \infty)$ by sorting them from most (θ_1) to least (θ_n) intensive, so that for $i \in \{1, \ldots, n\}$ the interval $[K_{(i-1)}, K_{(i)}]$ corresponds to the mass of jobs of intensity θ_i. Similar to the case where C is convex, these tasks are then assigned to the mass of Q workers in a positive assortative fashion so that the highest intensity tasks are allocated to the worker with the lowest cost of supplying labor. However, for each ironing interval $m \in \mathcal{M}$ of the function C, the corresponding mass of tasks that fall within the interval $[Q_1^*(m), Q_2^*(m)]$ are not assigned in a positive assortative fashion and are instead randomly assigned to the corresponding mass of workers. Alternatively, we can think of the firm as repackaging the tasks that fall within the interval $[Q_1^*(m), Q_2^*(m)]$ into a mass $Q_2^*(m) - Q_1^*(m)$ of homogeneous jobs and asking the corresponding mass of workers assigned to these jobs to multi-task. This analysis therefore provides an alternative interpretation of multi-tasking in the sense of Holmström and Milgrom (1991). In our setting, it arises from cost minimization by a monopsony with heterogeneous tasks that faces a non-convex procurement cost function.

Given the optimal mechanism for procuring the Q highest-value units of labor, it only remains to determine the precise mass of workers that are hired under the optimal mechanism. However, following Loertscher and Muir (2021a), this argument proceeds in precisely the same manner as for the case where C is convex, after we replace the cost function C with its concavification $\underline{C}$.