Wage dispersion, involuntary unemployment and minimum wages under monopsony and oligopsony*

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Abstract

Adopting a mechanism design approach to optimal monopsony pricing, we show that market power can cause involuntary unemployment and that introducing an appropriate minimum wage can eliminate it. Specifically, we characterize when using a procurement mechanism involving wage dispersion and involuntary unemployment is optimal for a monopsony. Setting a minimum wage equal to the equilibrium wage under price-taking behavior then maximizes total employment and social surplus, and eliminates involuntary unemployment. Even setting a minimum wage equal to the highest wage offered under the laissez-faire equilibrium increases total employment and workers' pay, and decreases (and possibly eliminates) involuntary unemployment. If a minimum wage does not induce involuntary unemployment or induces both involuntary unemployment and wage dispersion, then a marginal increase in the minimum wage generically increases employment and decreases involuntary unemployment and wage dispersion. Extensions show that our key insights generalize to quantity competition and horizontally differentiated workers and jobs.

Keywords: Market power, efficiency wage, regulated prices, quantity competition, employment
JEL-Classification: C72, D47, D82

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1 Introduction

Minimum wage legislation has been around for over a century.[1] So too have debates among economists and policy makers concerning the effects of minimum wages on total employment, involuntary unemployment and workers' pay. In models with price-taking firms and workers, minimum wages have either no effect or induce involuntary unemployment and inefficiently low employment. In contrast, as pointed out by Robinson (1933) and Stigler (1946), if employers exert monopsony power in labor markets, then appropriately chosen minimum wages can increase workers' pay and employment without creating involuntary unemployment. As is well known, the effects Robinson and Stigler identified are consistent with the empirical findings of Card and Krueger (1994).

In this paper, we offer a novel perspective on the effects of minimum wages that nests the aforementioned approaches. We analyze a model in which a monopsony employer faces a continuum of workers with privately known opportunity costs of supplying labor. The monopsony's optimal procurement mechanism minimizes the total procurement cost subject to workers' incentive compatibility and individual rationality constraints. Absent minimum wage regulation, this mechanism involves wage dispersion and induces involuntary unemployment whenever the procurement cost function—that is, the quantity procured multiplied by the market-clearing wage—lies above its convexification at the optimal level of employment. If this is the case, the optimal mechanism involves two wages—an efficiency wage and a low wage that are above and below the market-clearing wage, respectively. Workers with a relatively low opportunity cost of working are hired with certainty at the low wage, while the excess supply of labor associated with the efficiency wage is randomly rationed, generating involuntary unemployment.[2]

We then show that introducing any minimum wage between the lowest and the highest wage offered under the laissez-faire equilibrium increases total employment and decreases involuntary unemployment. In fact, it is always possible to eliminate involuntary unemployment and maximize total employment and social surplus by setting a minimum wage equal to the equilibrium wage under price-taking behavior. However, setting a minimum wage above the price-taking wage results in inefficiently low total employment and a positive level of involuntary unemployment. Figure 1 illustrates the effects of minimum wages for the textbook model that assumes price-taking behaviour; for Robinson's and Stigler's analysis of

[1]While precursors to minimum wage legislation date back to the Hammurabi Code (c. 1755–1750 BC), New Zealand became the first country to implement a minimum wage in 1894, followed by the Australian state of Victoria in 1896, and the United Kingdom in 1909 (Starr, 1981).

[2]That the optimal procurement mechanism may involve an efficiency wage and involuntary unemployment resonates with the dictum often attributed to Henry Ford that the Five-Dollar Day was "the best cost-cutting measure ever undertaken."

minimum wages under monopsony power with uniform wage setting; and for the case novel to this paper, in which the monoposony is allowed to use the optimal mechanism subject to workers' incentive compatibility and individual rationality constraints.

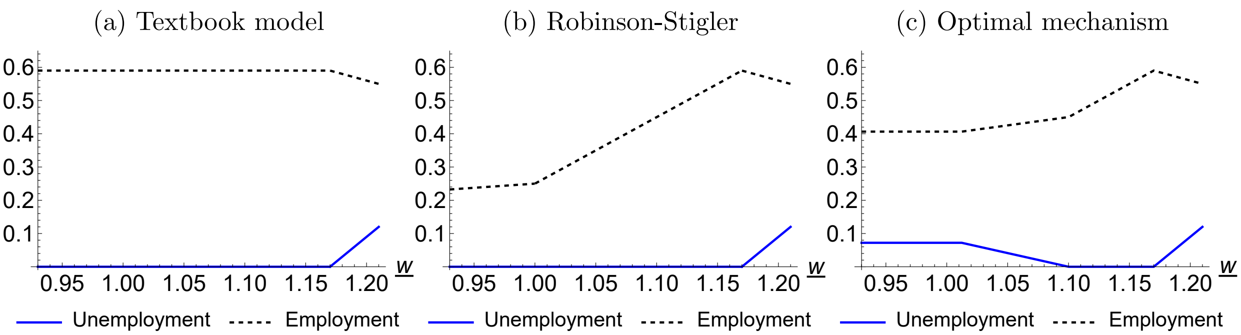

Figure 1: Illustration of how total employment and involuntary unemployment respond to changes in the minimum wage $\underline{w}$.

The rich and somewhat counter-intuitive nature of the minimum wage effects when a firm uses an optimal procurement mechanism raise the question of how a regulator could tell whether the problem at hand is such that a marginal increase in the minimum wage will increase employment and decrease involuntary unemployment, or have the opposite effects. As we show, the answer relates to whether or not there is involuntary unemployment and wage dispersion under the prevailing minimum wage. If there is no involuntary unemployment under a given minimum wage, then this indicates that the firm is not acting as a price-taker and is exerting market power in the labor market. A marginal increase in the minimum wage will then generically increase employment without generating involuntary unemployment.[3] Similarly, if there is both involuntary unemployment and wage dispersion under a given minimum wage, then this indicates that the firm is not acting as a price-taker and is exerting market power in the labor market by engaging in wage discrimination. Consequently, a marginal increase in the minimum wage will reduce its market power and lead to an increase in employment and a decrease in both involuntary unemployment and wage dispersion. However, if there is involuntary unemployment and no wage dispersion at a prevailing minimum wage, this indicates that the firm is a price-taker at that minimum wage and an increase in this wage will reduce employment and increase involuntary unemployment.

In extensions, we allow for quantity competition among firms and for horizontal differentiation of workers and jobs. For a model of quantity competition in which the aggregate quantity is procured at minimal cost, we show that total employment and involuntary un-

[3]The non-generic, knife-edge case arises when the minimum wage is exactly equal to the wage that would prevail under price-taking behaviour.

employment can move in the same direction as the number of firms increases and that there is no intrinsic relationship between the intensity of competition and the level of involuntary unemployment. The main insights from the monopsony model with regard to minimum wage effects carry over to the model with quantity competition. In particular, an appropriately chosen minimum wage still eliminates involuntary unemployment. With horizontally differentiated workers and jobs, optimal procurement may involve deliberate and inefficient mismatches of workers and jobs, in addition to involuntary unemployment.

Our paper relates to four strands of literature: efficiency wage theory; price regulation in monopsony models; mechanism design problems that do not satisfy Myerson's (1981) regularity condition; and models of quantity competition. That involuntary unemployment is beneficial for businesses and detrimental for workers is a popular idea whose origins date back at least to Friedrich Engels' and Karl Marx' notion of a *reserve army of labor*.[4] More recently, it appears in the guise of the efficiency-wage theory of involuntary unemployment. According to this theory firms deliberately offer wages that exceed their market-clearing level so that the resulting excess supply of labor (and corresponding level of involuntary unemployment) can be used to discipline their workforce. For example, firms may offer efficiency wages to increase workers' effort or reduce churn. The collection of essays in Akerlof and Yellen (1986) provides an overview of the early literature that formalized these ideas, while Krueger and Summers (1988) provide empirical evidence on industry wage structure. Notwithstanding their popular appeal, one major drawback of shirking and labor market turnover models of efficiency wages is that they rest on implicit or explicit restrictions on the contracting space. As Yellen (1984, p. 202) put it: "All these models suffer from a similar theoretical difficulty—that employment contracts more ingenious than the simple wage schemes considered, can reduce or eliminate involuntary unemployment." Our paper contributes to this literature by developing a model in which an efficiency wage that induces involuntary unemployment is optimal, subject only to individual rationality and incentive compatibility constraints. Because the mechanism design approach we use is free of institutional assumptions and does not restrict the contracting space, in our setting efficiency wages and involuntary unemployment arise from the primitives of the problem.

Robinson (1933) and Stigler (1946) first observed that equilibrium employment can be increased with a minimum wage in the presence of monopsony power.[5] As shown by Bhaskar, Manning, and To (2002), this basic logic also extends to imperfectly competitive markets.

[4] See Engels (1845) and Marx (1867).

[5] As mentioned, there is empirical evidence consistent with these effects, the classic reference being Card and Krueger (1994). More recently, Wiltshire (2021) provides an analysis of the labor market effects of Walmart supercenters and of the effects of minimum wages in the presence of monopsony power, as well as a comprehensive overview of this strand of literature.

Our paper shares the feature that minimum wages can increase employment. However, while these models can explain inefficiently low employment due to market power on the demand side, they cannot say anything about effects on involuntary unemployment because all unemployment is voluntary in models with market-clearing wages. By allowing the monopsony to use an optimal procurement mechanism, we obtain wage dispersion and involuntary unemployment absent wage regulation in equilibrium. Combining insights from the analysis of Robinson and Stigler and the mechanism design approach pioneered by Myerson, we show that appropriately tailored minimum wages can eliminate involuntary while increasing employment. Thereby, the paper contributes to recent advances in labor economics, where "a growing consensus is that firms have some wage-setting power" and that once this is accepted, "the analysis of wage setting becomes part of labor economics, just like the analysis of price setting is a part of IO" (Card, 2022b). With regards to minimum wage policies, our paper complements Lee and Saez (2012), who derive social surplus maximizing minimum wages in competitive labor markets, assuming that rationing is efficient. Specifically, we show that random rather than efficient rationing is in the interest of a firm with market power and that an appropriately chosen minimum wage eliminates involuntary unemployment and maximizes social surplus.

This paper also relates to the literature on monopoly pricing and mechanism design problems that fail to satisfy the regularity condition of Myerson (1981) and involve ironing. Dating back to Hotelling (1931) and with subsequent contributions by Mussa and Rosen (1978), Myerson (1981) and Bulow and Roberts (1989), there has been a recent upsurge of interest driven by the applications considered in Condorelli (2012), Dworczak, Kominers, and Akbarpour (2021), Loertscher and Muir (2022a) and Akbarpour, Dworczak, and Kominers (2020). This paper is most closely related to Loertscher and Muir (2022a) because it studies the monopsony variant of the non-regular monopoly pricing problem analyzed there. Motivated by the resale markets that often arise in practical applications when selling mechanisms involving rationing and randomization are used, Loertscher and Muir (2022a) derive optimal selling mechanisms in the presence of resale but do not consider price regulation. Their analysis rationalizes the "[puzzling] combination of low prices and rent seeking by speculators due to an active secondary market" (Budish and Bhave, 2022) consistently observed in ticket markets. In contrast, motivated by the central role minimum wages play in policy discussions related to labor markets and involuntary unemployment, this paper derives the optimal mechanism for a monopsony restricted by a minimum wage. It shows that the introduction of a minimum wage can *decrease* and, if appropriately chosen, *eliminate* involuntary unemployment. To the best of our knowledge, the connection between non-regular mechanism design problems, involuntary unemployment and minimum wage effects that are made in this paper have never been touched upon before. Put differently, using optimal pricing theory this paper shows that market power can be a cause of involuntary unemployment which can be eliminated by introducing an appropriate minimum wage. Methodologically, this requires incorporating pricing constraints into a mechanism design problem, which are fundamentally different from mechanism design problems involving quantity, moment or majorization constraints,[6] and deriving comparative statics of the corresponding optimal mechanisms. In light of the recent upsurge of interest in price regulation, in particular in the context of big tech (see, for example, Australian Competition and Consumer Commission, 2019; Crémer et al., 2019; Furman et al., 2019; Stigler Center, 2019), there are a range of applications for which this methodology could prove useful.

Our model of quantity competition is related to the literature on Cournot competition (Cournot, 1838) and heeds David Card's call to move "beyond the 'no strategic interactions' case" (Card, 2022b). This extension also shows that key aspects of the monopsony analysis extend directly to this model of imperfect competition. Our discussion of optimal mechanisms in the Hotelling model builds on Balestrieri, Izmalkov, and Leao (2021) and Loertscher and Muir (2022b).

The remainder of this paper is structured as follows. Section 2 introduces the baseline procurement setup and characterizes when a monopsony's optimal procurement mechanism to efficiency wages and involuntary unemployment. In Section 3, we analyze the effects of minimum wages. Section 4.1 extends the model to quantity competition and horizontal differentiation, respectively. The paper concludes with a discussion in Section 5.

2 Setup

In this section, we first introduce our model and the general class of procurement mechanisms that we use to solve it. We then state the conditions under which the monopsony optimally uses an efficiency wage and induces involuntary unemployment, and discuss our modelling assumptions.

2.1 Model

We consider the procurement problem of a monopsony whose marginal willingness to pay for $Q \in [0, 1]$ units of labor input is $V(Q)$. For ease of exposition we assume that the monop-

[6]See, for example, Kang (2021) and Kleiner, Moldovanu, and Strack (2021).

sony's *marginal value for labor* V is strictly decreasing and continuously differentiable.[7,8] Let W denote the *inverse supply function* faced by the monopsony. We assume that W is strictly increasing (so that the monopsony faces an upward sloping *labor supply schedule* $S := W^{-1}$) and continuously differentiable (for expositional convenience). The *cost function*

$$C(Q) := W(Q)Q$$

is then a strictly increasing and continuously differentiable function that specifies the cost of procuring $Q \in [0, 1]$ units at the market-clearing wage $W(Q)$. We assume that $V(0) > W(0)$ and $V(1) < W(1)$ so that under optimal procurement there are strictly positive masses of both employed and unemployed workers.

We microfound the inverse supply schedule W by assuming that the monopsony faces a continuum of workers of mass 1, each of whom supplies one unit of labor inelastically. Each worker has a *private* opportunity cost $c \in [\underline{c}, \overline{c}] := [W(0), W(1)]$ of supplying labor whose cumulative distribution function is denoted by G with density $g > 0$ on $[\underline{c}, \overline{c}]$. Consequently, $W(Q) = G^{-1}(Q)$ represents the opportunity cost of working for the worker with the Q-th lowest cost and $S(w) = G(w)$. We assume that workers are risk-neutral with quasi-linear utility. The interim expected payoff of a worker with cost c that is hired by the firm at a wage of $w \in \mathbb{R}_{\geq 0}$ with probability $x \in [0, 1]$ is therefore $x(w - c)$.

2.2 Mechanisms

A direct *procurement mechanism* $\langle \boldsymbol{x}, \boldsymbol{w} \rangle$ consists of an *allocation rule* $\boldsymbol{x} : [\underline{c}, \overline{c}] \to [0, 1]$ that maps worker reports to their probability of employment and a *wage schedule* $\boldsymbol{w} : [\underline{c}, \overline{c}] \to \mathbb{R}_{\geq 0}$ that maps worker reports to a wage that is paid conditional on employment. For all $c, \hat{c} \in [\underline{c}, \overline{c}]$, *incentive compatibility* requires that

$$x(c)(w(c) - c) \geq x(\hat{c})(w(\hat{c}) - c). \tag{IC}$$

[7]As will become clear, these assumptions allow us to adopt a first-order approach in order to uniquely characterize the optimal quantity Q^* of labor procured by the monopsony.

[8]If the firm uses Q units of input to generate a downstream profit of $\Pi(Q)$, where Π is concave and differentiable with $\Pi'(0) > 0$, then the firm's willingness to pay for the Q-th unit of input is given by $V(Q) = \Pi'(Q)$. For example, the firm could be a monopoly in the downstream market with access to a production technology that transforms one unit of labor into one unit of output or the firm could be a downstream price-taker using a production technology with a diminishing marginal product of labor.

Assuming that each worker receives the same payoff from not working and normalizing this payoff to 0, *individual rationality* requires that, for all $c \in [\underline{c}, \overline{c}]$,

$$w(c) - c \geq 0. \tag{IR}$$

By the revelation principle, the focus on direct mechanisms is without loss of generality. Furthermore, since there is no aggregate uncertainty, it is also without loss of generality to restrict attention to direct mechanisms that determine the employment probability and wage of a given worker independently of the reports of the other workers. As we will show, the optimal mechanism either pays all employed workers a market-clearing wage or employs workers at two different wages. Consequently, the optimal mechanism can be implemented as a Nash equilibrium by posting one or two wages.[9]

In Section 3 we analyze the case in which a regulator introduces a minimum wage of $\underline{w}$. Defining $w(c)$ as the wage a worker of type c is paid *conditional* on being employed allows us to account for this in the mechanism design problem by introducing the constraint that, for all $c \in [\underline{c}, \overline{c}]$, $w(c) \geq \underline{w}$.

This setting makes two departures from an otherwise standard monopsony pricing problem. First, we do not restrict the monopsony to setting the market-clearing wage $w = W(Q)$ when it procures the quantity Q. We say that a mechanism $\langle \boldsymbol{x}, \boldsymbol{w} \rangle$ under which in equilibrium a mass of Q workers is hired involves an *efficiency wage* w with $w > W(Q)$ if the set $\{c \in [\underline{c}, \overline{c}] : w(c) > W(Q)\}$ has positive measure. Such a mechanism necessarily induces *involuntary unemployment* since there is a positive mass of workers willing to supply labor at an efficiency wage that are nevertheless unemployed.[10] Second, we do not assume that the function C is convex. As we shall see, these assumptions go hand-in-hand: It is without loss of generality to restrict attention to market-clearing wages when the cost function C is convex. However, when C is not convex, the monopsony may strictly benefit from offering an efficiency wage and inducing involuntary unemployment.

2.3 Optimality of efficiency wages and involuntary unemployment

We begin by introducing a function that will play a central role in our analysis: the *convexification* $\underline{C}$ of the cost function C. This is the largest convex function that is everywhere less than C and is characterized by a countable set $\mathcal{M}$ and a set of disjoint open intervals

[9] In the latter case, there may be multiple Nash equilibria. As we will discuss, there is also a dynamic implementation that yields a unique equilibrium in dominant strategies.

[10] Defined this way, involuntary unemployment is observable, given the appropriate data; see, for example, Breza, Kaur, and Shamdasani (2021).

$\{(Q_1(m), Q_2(m))\}_{m\in\mathcal{M}}$ such that

$$\underline{C}(Q) = \begin{cases} C(Q_1(m)) + \frac{(Q-Q_1(m))(C(Q_2(m))-C(Q_1(m)))}{Q_2(m)-Q_1(m)}, & \exists\, m \in \mathcal{M} \text{ s.t. } Q \in (Q_1(m), Q_2(m)) \\ C(Q), & Q \notin \bigcup_{m\in\mathcal{M}}(Q_1(m), Q_2(m)). \end{cases}$$

Equivalently, for all $m \in \mathcal{M}$ and $Q \in (Q_1(m), Q_2(m))$, introducing $\alpha(Q) := \frac{Q-Q_1(m)}{Q_2(m)-Q_1(m)}$ we can write $\underline{C}(Q)$ as the convex combination

$$\underline{C}(Q) = (1-\alpha(Q))C(Q_1(m)) + \alpha(Q)C(Q_2(m)).$$

Since C is strictly increasing and continuously differentiable, $\underline{C}$ is strictly increasing and continuously differentiable with $\underline{C}(0) = C(0)$. For ease of exposition, we additionally assume that $\underline{C}(1) = C(1)$. For each $m \in \mathcal{M}$, $Q_1(m)$ and $Q_2(m)$ then satisfy the first-order condition

$$C'(Q_1(m)) = \frac{C(Q_2(m)) - C(Q_1(m))}{Q_2(m) - Q_1(m)} = C'(Q_2(m)). \tag{1}$$

The relevance of the convexification $\underline{C}$ becomes clear in the following proposition:

Proposition 1. *Under the cost-minimizing incentive compatible and individually rational mechanism for procuring the quantity Q, the cost of procurement is $\underline{C}(Q)$. If $\underline{C}(Q) < C(Q)$, then the minimum cost is achieved using a mechanism involving an efficiency wage that induces involuntary unemployment. Moreover, the monopsony optimally employs Q^* workers, where Q^* is the unique quantity such that $V(Q^*) = \underline{C}'(Q^*)$.*

To prove this proposition we start by explicitly constructing a mechanism that procures the quantity Q at a cost of $\underline{C}(Q)$, focusing on the non-trivial case where $\underline{C}(Q) < C(Q)$.[11] For the purpose of this proof and for the remainder of Section 2, we can assume without loss of generality that $\underline{C}$ is characterized by a single interval (Q_1, Q_2). Consider procuring $Q \in (Q_1, Q_2)$ workers using a *two-wage mechanism* involving wages w_1 and w_2 with $w_2 > w_1$ and random rationing at the higher wage. We construct this mechanism so that Q_2 is the equilibrium mass of worker who are willing to supply labor at the high wage, implying that $w_2 = W(Q_2)$. The quantity Q_1 is the equilibrium mass of worker who seek employment at the low wage w_1. At the high wage, the probability of employment is $\alpha(Q) = \frac{Q-Q_1}{Q_2-Q_1} < 1$, so the indifference condition for workers of type $W(Q_1)$ leads to $w_1 = (1-\alpha(Q))W(Q_1) + \alpha(Q)W(Q_2)$.[12] This yields a cost of $(1-\alpha(Q))C(Q_1) + \alpha(Q)C(Q_2) \equiv \underline{C}(Q)$ for the firm, as

[11] When $\underline{C}(Q) = C(Q)$, the monopsony can simply hire all Q workers at the market-clearing wage $W(Q)$.

[12] Note that rationing is random and independent of workers' types. The setup satisfies a single-crossing condition which ensures that all workers whose opportunity cost of supplying labor is less than $W(Q_1)$ prefer

required.

Following Loertscher and Muir (2022a) and applying the mechanism design approach and ironing procedure of Myerson (1981) to our monopsony setting shows that the two-wage mechanism we just constructed minimizes the procurement cost of the monopsony, subject to workers' incentive compatibility and individual rationality constraints. This immediately implies the final statement of the proposition, and shows that the monopsony will optimally use an efficiency wage and induce involuntary unemployment whenever $\underline{C}(Q^*) < C(Q^*)$. In such cases, the level of involuntary unemployment is given by $Q_2 - Q^*$, while the rate of involuntary unemployment is $(Q_2 - Q^*)/Q_2$. An equivalent interpretation of involuntary unemployment in this model is that the mass $Q_2 - Q_1$ of workers who want to work at the efficiency wage $W(Q_2)$ are all employed but only work part-time, having a fraction $\alpha(Q)$ of a full-time job. From this perspective, the high-wage workers are *underemployed*, while the low-wage workers are fully employed. For example, this is descriptive of the restaurant industry in France, where full-time waiters are paid hourly wages of 12 or 13 euros and part-time waiters (called extras) are paid 16 euros per hour.[13]

The intuition for why randomly rationing worker may be optimal when C is not convex is quite simple. Naturally, the monopsony always wants to hire workers at the lowest possible marginal cost. However, incentive compatibility dictates that higher cost workers are never hired with higher probability than lower cost workers.[14] If C is not convex, then C' is not monotone and the Q lowest marginal cost workers are not necessarily the Q workers with the lowest opportunity cost of working. Whenever this conflict arises, the best the monopsony can do is to randomly ration a subset of workers, hiring each with equal probability and thereby inducing involuntary unemployment. An illustration is provided in Figure 2. Note that all the figures included in the introduction, this section and Section 3 are plotted for a piecewise linear specification of W provided in Appendix B.1, as well as a linear V function. The exact parameterizations used to generate each of our figures are given in Appendix B.4.

2.4 Discussion

Before proceeding to the analysis of minimum wage effects, we briefly discuss various aspects related to the setup and the optimal mechanism, including robustness and implementation, as well as empirical background.

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working with certainty at the low wage w_1 and all workers whose opportunity cost is greater than $W(Q_1)$ prefer entering the lottery to working at the high wage w_2.

[13]See *Grille des salaires : Extracadabra sort son étude 2022* and *Extracadabra*, respectively.

[14]Otherwise, since worker costs are private, a lower cost worker could always profitably imitate a higher cost worker.

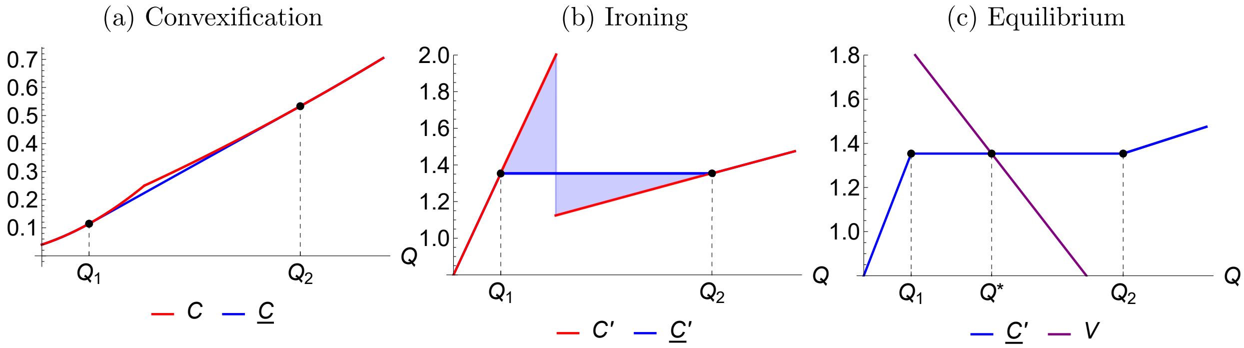

Figure 2: Panel (a) illustrates the convexification $\underline{C}$ of C for our leading example from Appendix B.1. Panel (b) illustrates the corresponding ironing procedure of Myerson (1981). Panel (c) illustrates an example where $Q^* \in (Q_1, Q_2)$.

Robustness Following Lee and Saez (2012), we assume risk-neutral agents with quasilinear utility and focus on the extensive margin in labor supply, which is the empirically relevant one (see Lee and Saez (2012) and references therein). The advantage of assuming risk neutrality and quasilinear utility is that under these assumptions the optimal procurement mechanism absent wage regulation is well-understood and this makes the more involved problem of deriving the optimal mechanism under a given minimum wage tractable.

That said, the two-wage mechanism that is optimal when $Q \in (Q_1, Q_2)$ is robust to the introduction of risk-averse workers in the following sense. Suppose all workers have the same initial wealth level—which without loss of generality can be normalized to zero—and the same, strictly concave utility function U. A worker with opportunity cost $W(Q)$ working at wage $w \geq W(Q)$ then has a utility of $u(w - W(Q))$, while an unemployed worker has a utility of $u(0)$. To replicate the risk-neutral equilibrium, the participation constraint for the marginal worker still requires $w_2 = W(Q_2)$. However, the wage $\hat{w}_1$ that makes workers with opportunity cost $W(Q_1)$ indifferent now satisfies $U(\hat{w}_1 - W(Q_1)) = \alpha(Q)u(W(Q_2) - W(Q_1)) + (1 - \alpha(Q))u(0)$. Since u is strictly concave, we have $w_1 > \hat{w}_1$.[15] Unsurprisingly, the insurance benefit associated with certain employment works in favor of the firm's scheme, reducing its procurement cost relative to the case with risk-neutral workers. However, with risk-averse agents the optimal mechanism does not necessarily involve at most two wages.

The superiority of using a two-wage mechanism over a market-clearing wage whenever $Q \in (Q_1, Q_2)$ is also robust to small errors in setting these wages. To see why, notice that an equilibrium with the same sorting structure can be induced using arbitrary quantities $\tilde{Q}_1$ and $\tilde{Q}_2$ satisfying $\tilde{Q}_1 \leq Q \leq \tilde{Q}_2$ and wages $\tilde{w}_1(Q) = (1 - \tilde{\alpha})W(\tilde{Q}_1) + \tilde{\alpha}W(\tilde{Q}_2)$ with $\tilde{\alpha} = (Q - \tilde{Q}_1)/(\tilde{Q}_2 - \tilde{Q}_1)$, yielding a cost of $(1 - \tilde{\alpha})C(\tilde{Q}_1) + \tilde{\alpha}C(\tilde{Q}_2)$. By construction, Q_1 and

[15] Moreover, the single-crossing condition (see footnote 12) is satisfied.

Q_2 are the respective minimizers of this cost over $\tilde{Q}_1$ and $\tilde{Q}_2$, but by continuity the two-wage mechanism with $\tilde{Q}_1$ and $\tilde{Q}_2$ sufficiently close to Q_1 and Q_2 yields a lower procurement cost than hiring workers at any market-clearing wage.

Implementation As mentioned, when $Q \in (Q_1, Q_2)$, a procurement cost of $\underline{C}(Q)$ can be achieved using two posted wages and having workers self-select into low- and high-wage openings, with the workers at the high wage being randomly rationed. Consequently, no reporting of types by workers is required to implement the optimal mechanism. Of course, there may be multiple Nash equilibria when hiring at the low and the high wage occurs simultaneously. However, as foreshadowed in footnote 9, there is also a dynamic implementation that induces a unique dominant strategy equilibrium: the monopsony first hires Q_1 workers at the low wage and starts to fill the $Q - Q_1$ vacancies at the high wage only after the Q_1 low-wage vacancies have been filled.

Randomization The randomization that occurs at the high wage can be achieved in a multitude of ways. For example, the low-wage workers can be thought of as permanently employed. High-wage workers can be thought of as casual staff, where $\alpha(Q)$ is the probability of being hired on a given day, or the fraction of time a casual worker is employed. Alternatively, workers may be randomly selected at the high wage if hiring occurs on a first-come-first-serve basis and the order of arrival is independent of workers' costs; if hiring occurs based on observable worker characteristics that are not correlated with their costs of working (and are in that sense irrelevant); or if hiring occurs literally via a lottery as was the case in the so-called "shape-up" that was commonly used for hiring dock workers.[16]

Moreover, while randomly rationing workers with costs between $W(Q_1)$ and $W(Q_2)$ is optimal for the monopsony, the superiority of a scheme involving involuntary unemployment induced by an efficiency wage does not hinge on the assumption that rationing is uniform. As mentioned, the only allocation rules consistent with incentive compatibility (and thus equilibrium) require workers with lower costs to be hired with weakly higher probability, so the only alternative rationing schemes are such that the allocation is more efficient than under uniform random rationing. If one parameterizes rationing schemes as convex combinations of the uniform random and the efficient allocation, the scheme with efficiency-wage induced involuntary unemployment remains optimal provided the weight on the efficient allocation is less than one.[17]

[16]This mechanism came to the general public's awareness through the movie "On the Waterfront" and the series of newspaper articles it was based on (see Johnson and Schulberg, 2005).

[17]Loertscher and Muir (2022a) formalize this in the context of a monopoly pricing problem in which an efficient resale market operates with some probability.

Monopsony power and non-regularity As mentioned in the literature review, there is growing evidence and recognition that employers exert market power in the labor market (Card, 2022a,b), and assuming a single employer provides a tractable mechanism design model capturing labor market power. Moreover, in Section 4.1 we extend the model to allow for quantity competition.

Allowing for C to be non-convex is of course less restrictive than requiring that it is convex, and we are not aware of any direct empirical evidence concerning the curvature properties of this function. However, a monopsony that faces a non-convex cost function is analogous to a monopoly that faces a non-concave revenue function. Both of these problems correspond to what are known as non-regular mechanism design problems (Myerson, 1981). When the assumption of concave revenue (or, equivalently, monotone marginal revenue) is tested empirically, it is frequently rejected; see, for example, Celis, Lewis, Mobius, and Nazerzadeh (2014), Appendix D in Larsen and Zhang (2018) and Section 5 in Larsen (2021), as well as the related discussion in Loertscher and Muir (2022a). It would be surprising if a property that is regularly rejected on output markets were to hold consistently on input markets. Moreover, while there is no theoretical reason as to why C should be convex, as we show in Appendix B.3, non-convex cost functions naturally arises when workers face a fixed cost of moving, changing occupation or participating in the labor market. Similarly, if two labor markets that differ with respect to the lowest opportunity cost of working are integrated, then the integrated labor market always exhibits a non-convex cost function, even when as standalone markets each market exhibits a convex cost function.[18]

3 Optimal procurement under a minimum wage

Minimum wages are commonly perceived as a cause of involuntary unemployment. However, as we just saw, monopsony power can also cause involuntary unemployment. In this section we will show that involuntary unemployment that arises as a result of monopsony power can actually be *eliminated* by introducing an appropriate minimum wage. Moreover, our analysis provides clear guidance as to how regulators and policy makers can distinguish involuntary unemployment caused by a prevailing minimum wage from involuntary unemployment caused by market power: In the former case the firm acts as a price-taker and sets a uniform wage, and in the latter case it engages in wage discrimination, which results in wage dispersion. We provide an overview of the policy-relevant results in Section 3.1, and the interested reader can find an overview of the technical details of the analysis in Section 3.2. The corresponding

[18]This is analogous to the observation made in Loertscher and Muir (2022a) that integrating output markets can render revenue non-concave.

proofs are in Appendix A.

3.1 Overview of policy-relevant results

Aside from the statement of Theorem 1 (which holds in general), our analysis and discussion throughout this subsection assumes that under the *laissez-faire equilibrium*—the equilibrium of the model without minimum wage regulation—the monopsony uses a procurement mechanism involving an efficiency wage and involuntary unemployment. That is, we assume that $Q^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. The perfectly competitive quantity $Q^p > Q^*$ will play an important role in the analysis. This quantity, which satisfies $V(Q^p) = W(Q^p)$, is the efficient employment level that would emerge under price-taking behaviour.[19]

Introducing a minimum wage We first consider a regulator that introduces a minimum wage starting from the laissez-faire equilibrium. We let $w_1(Q^*) := (1 - \alpha(Q^*))W(Q_1(m)) + \alpha(Q^*)W(Q_2(m))$ denote the low wage paid in the laissez-faire equilibrium. When we speak of surplus, this refers to the sum of the monopsony's profit and aggregate worker social surplus (defined as the total wages paid less the aggregate opportunity cost of the supplied labor).

Proposition 2. *Suppose that $Q^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Then introducing a minimum wage of $\underline{w} = W(Q^p)$ increases equilibrium employment to $Q^p > Q^*$, increases workers' total pay and eliminates both involuntary unemployment and wage dispersion. Moreover, such a minimum wage maximizes total employment and social surplus. Relative to the laissez-faire equilibrium, any minimum wage $\underline{w} \in (w_1(Q^*), W(Q_2(m))]$ increases total employment and workers' pay and decreases involuntary unemployment. Furthermore, a minimum wage of $\underline{w} = W(Q_2(m))$ eliminates involuntary unemployment if and only if $Q_2(m) \leq Q^p$. Any minimum wage that eliminates involuntary unemployment increases social surplus relative to the laissez-faire equilibrium.*

Proposition 2 shows that, relative to the laissez-faire equilibrium, setting $\underline{w} = W(Q^p)$ increases total employment and the total wage bill paid to workers, and eliminates involuntary unemployment. Such a minimum wage also maximizes both total employment and social surplus. In practice, it may be difficult for a regulator to observe or estimate $W(Q^p)$. However, as Proposition 2 also shows, even setting $\underline{w} = W(Q_2(m))$ (i.e. setting the minimum wage to the highest wage observed under the laissez-faire equilibrium) is guaranteed to increase employment and decrease involuntary unemployment, possibly to the point of eliminating it. If $\underline{w} = W(Q_2(m))$ eliminates involuntary unemployment, it also increases

[19] Since V is strictly decreasing and W is strictly increasing and these functions satisfy $V(0) > W(0)$ and $V(1) < W(1)$, Q^p exists and is unique. In the proof of Proposition 2 we show that $Q^p > Q^*$ always holds.

social surplus relative to the laissez-faire equilibrium because it brings total employment closer to the efficient level and eliminates the random, inefficient allocation associated with involuntary unemployment.

The intuition underlying Proposition 2 is simple. Given $\underline{w} = W(Q^p)$, the monopsony will optimally hire at least Q^p workers because the marginal benefit $V(Q) > V(Q^p)$ of hiring $Q < Q^p$ workers always exceeds the marginal cost $\underline{w} = V(Q^p)$. It will not hire any additional workers because—as we will later show in Theorem 2—the marginal cost of hiring $Q > Q^p$ workers under a minimum wage of $\underline{w} = W(Q^p)$ strictly exceeds $V(Q)$. This also implies that total employment is maximized under a minimum wage of $\underline{w} = W(Q^p)$ and that setting $\underline{w} > W(Q^p)$ will cause involuntary unemployment and result in inefficiently low employment. Consequently, the minimum wage $\underline{w} = W(Q^p)$ maximizes social surplus.

Similarly, under a minimum wage of $\underline{w} = W(Q_2(m))$ the monopsony is a price-taker on all units $Q \leq Q_2(m)$, and it will never hire more than $Q_2(m)$ workers. Since the monopsony now faces a strictly lower marginal cost of hiring any quantity $Q \in [Q^*, Q_2(m))$ of workers, it will always hire more than Q^* workers. Moreover, if $Q^p \geq Q_2(m)$ the monopsony will hire precisely $Q_2(m)$ workers and involuntary unemployment is eliminated (see Panel (a) of Figure 3). If $Q^p < Q_2(m)$ the monopsony will hire $V^{-1}(\underline{w}) \in (Q^*, Q^p)$ workers (see Panel (b) of Figure 3). In this case $\underline{w} > W(Q^p)$ and the minimum wage *causes* involuntary unemployment in the sense that setting a lower minimum wage $\underline{w} = W(Q^p)$ would have eliminated it. Even so, involuntary unemployment given $\underline{w} = W(Q_2(m))$ will be lower than under the laissez-faire equilibrium since in either case the total number of workers who participate is $Q_2(m)$ but given the minimum wage, $V^{-1}(\underline{w}) > Q^*$ workers are hired, implying that fewer are involuntarily unemployed.

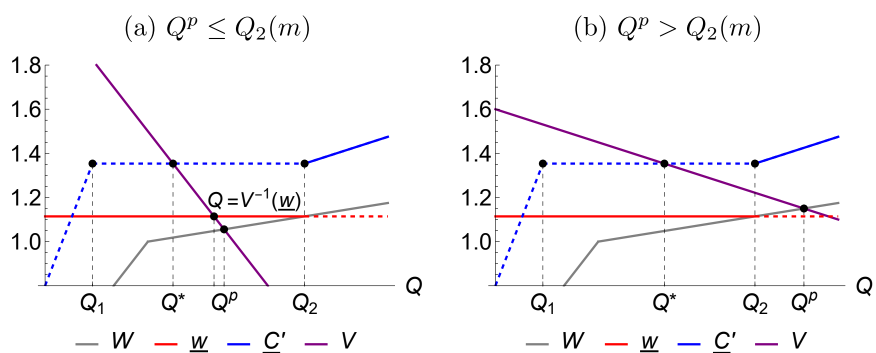

Figure 3: An illustration of the effects of imposing a minimum wage $\underline{w} = W(Q_2(m))$. In each panel the solid sections of the $\underline{w}$ (red) and $\underline{C}'$ (blue) curves indicate the marginal cost schedule associated with optimal procurement. In Panel (a), $Q^p \leq Q_2$ and $\underline{w}$ eliminates involuntary unemployment. In Panel (b), $Q^p > Q_2$ and $\underline{w}$ induces involuntary unemployment.

Marginal minimum wage effects Of course, real-world policy debates often pertain to the effects of increasing an existing minimum wage rather than the introduction of a minimum wage. Moreover, the primitive functions V and W and, consequently, the efficient level of employment Q^p are typically not observable to policy-makers. We now account for these constraints by assuming that regulators and policy-makers only observe whether or not a given minimum wage $\underline{w}$ results in equilibrium involuntary unemployment and wage dispersion. Theorem 1 below then answers the following question: What are the implications of a marginal increase in the minimum wage for total employment, involuntary unemployment and wage dispersion? As mentioned, our exposition here assumes $Q^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ but Theorem 1 is also valid if $Q^* \notin (Q_1(m), Q_2(m))$ for any $m \in \mathcal{M}$.

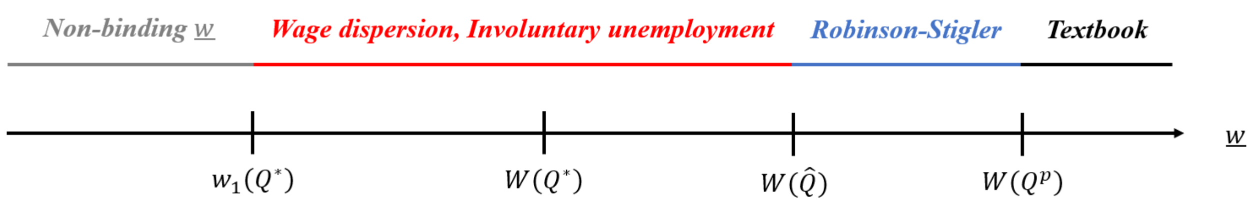

Figure 4: Schematic illustration of the implications of introducing a minimum wage $\underline{w}$ when the laissez-faire equilibrium involves involuntary unemployment.

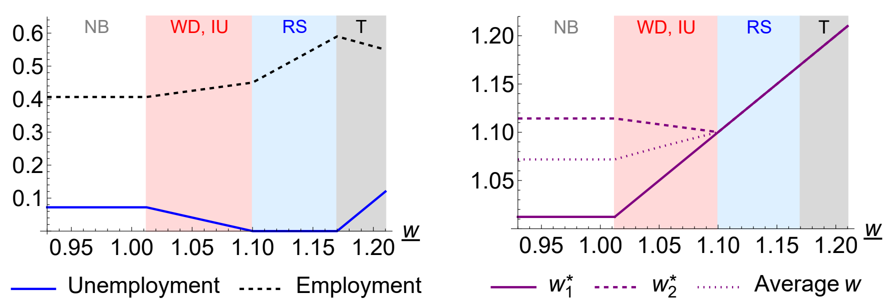

Figure 5: Equilibrium employment, involuntary unemployment and wages (including the lowest wage w_1^*, the highest wage w_2^* and the average wage) for an example with $Q^* \in (Q_1(m), Q_2(m))$ and $Q^p < Q_2(m)$. Each region of the schematic summary in Figure 4 is shaded and labeled.

Theorem 1.

1. *If there is involuntary unemployment and wage dispersion at a given minimum wage, then $\underline{w} < W(Q^p)$ and a marginal increase in the minimum wage increases employment and decreases involuntary unemployment and wage dispersion.*

2. *If $\underline{w} \neq W(Q^p)$ and there is no involuntary unemployment at a given minimum wage, then $\underline{w} < W(Q^p)$ and a marginal increase in the minimum wage increases employment.*

3. *If there is involuntary unemployment and no wage dispersion at a given minimum wage, then $\underline{w} > W(Q^p)$ and a marginal increase in the minimum wage decreases employment and increases involuntary unemployment without affecting wage dispersion.*

Figures 4 and 5 provide a schematic summary of the general minimum wage effects identified in Theorem 1.[20] Consider the thought experiment of continuously increasing the minimum wage from a starting wage of $\underline{w} = W(0)$, and thereby tracing out the effects on employment, involuntary unemployment and wages. Naturally, whenever $\underline{w} \leq w_1(Q^*)$, the minimum wage is not binding and has no effect on equilibrium outcomes. For $\underline{w} > w_1(Q^*)$, the effects of minimum wages on equilibrium outcomes can then be divided into three regions.

The first region is characterized by the presence of both involuntary unemployment and wage dispersion under a prevailing minimum wage, which indicates that the monopsony is engaging in wage discrimination and exerting market power on the input market. In Figure 4 this region is plotted in red and corresponds to $\underline{w} \in (w_1(Q^*), W(\hat{Q}))$.[21] Within this region, increasing the minimum wage decreases involuntary unemployment and wage dispersion, and increases employment.[22] Intuitively, when the monopsony faces a binding but relatively low minimum wage, this curtails its market power (if it were to hire $Q \leq S(\underline{w})$ workers, it would be a price-taker on these units) without entirely undermining the benefits it derives from engaging in wage discrimination.

The second region is characterized by the absence of both involuntary unemployment and wage dispersion under a prevailing minimum wage. In Figure 4 it corresponds to $\underline{w} \in [W(\hat{Q}), W(Q^p))$ and is plotted in blue. Here, the optimal procurement mechanism involves uniform wage-setting. This region, which we refer to as the *Robinson-Stigler* region, exhibits precisely the pro-competitive effects identified by Robinson (1933) and Stigler (1946): increasing the minimum wage increases employment without causing involuntary unemployment.[23] Intuitively, the fact that there is no involuntary unemployment under a prevailing minimum wage indicates that the monopsony is still exerting market power. However, the monopsony no longer benefits from using a procurement mechanism involving

[20] For the piecewise linear specification in Appendix B.1 with V linear and involuntary unemployment under the laissez-faire equilibrium, Figure 4 provides a precise illustration, aside form the fact that $W(\hat{Q}) = W(Q^p)$ may hold (see footnotes 21 and 23).

[21] The quantity $\hat{Q}$ is defined as the smallest quantity $Q \leq Q^p$ such that for all $\underline{w} \in [W(Q), W(Q^p)]$, the monopsony optimally hires $S(\underline{w})$ workers at the minimum wage $\underline{w}$.

[22] Clearly, the low wage offered by the monopsony within this region is given by the minimum wage. The non-trivial aspect associated with establishing that wage dispersion is decreasing in $\underline{w}$ in this region is showing that the efficiency wage decreases in $\underline{w}$ within this region.

[23] Note that we may have $\hat{Q} = Q^p$. In such cases the monopsony only hires $S(\underline{w})$ workers at the minimum wage $\underline{w}$ when $\underline{w} = W(Q^p)$ and the Robinson-Stigler region shown in blue in Figure 4 does not exist. However, for the piecewise linear specification in Appendix B.1, one can show that $\hat{Q} < Q^p$ holds unless $Q^p = \underline{q}$, where $\underline{q}$ is the "kink" of the function W.

an efficiency wage that induces involuntary unemployment. The summary in Figure 4 is "schematic" insofar as there may be additional regions inside the interval $(W(Q^*), W(Q^p))$ with and without involuntary unemployment and wage dispersion, and $W(\hat{Q})$ need not be strictly less than $W(Q^p)$ if $Q^p < Q_2(m)$.[24] Notwithstanding these complications, Theorem 1 shows that a regulator who only observes wage dispersion and involuntary unemployment can always identify which region they are in.

The third and final region, which corresponds to $\underline{w} > W(Q^p)$, is characterized by a positive level of involuntary unemployment without any accompanying wage dispersion. This combination indicates that the monopsony is now acting as a price-taking and the involuntary unemployment is caused by the minimum wage itself and not by market power. This region—which we call the *textbook* region—is plotted in black in Figure 4. Increasing the minimum wage within this region decreases employment and increases involuntary unemployment. These effects correspond to those found when a binding minimum wage is introduced to the textbook model of a perfectly competitive labor market.

To summarize, if a monopsony uses an optimal procurement mechanism, then minimum wages have rich and non-monotone effects on equilibrium employment, involuntary unemployment and wages. This is consistent with the controversial nature of debates concerning the effects of minimum wages. At the same time, Theorem 1 provides clear guidance concerning the effects of a marginal increase in the minimum wage based only on whether or not there is wage dispersion and involuntary unemployment under the prevailing minimum wage.

Redistribution and worker welfare A close look at Figure 5 shows that whenever there is involuntary unemployment and wage dispersion, both the average wage (the total wage payments divided by the level of employment) and the low wage increase in the minimum wage, while the efficiency wage is decreasing in the minimum wage. This points to a potential conflict of interest among workers since high-wage workers (those earning the efficiency wage) are made worse off by a marginal increase in the minimum wage, while low-wage workers (those earning the minimum wage) are made better off. Indeed, we have the following proposition:

Proposition 3. *If there is involuntary unemployment and wage dispersion under a given minimum wage $\underline{w}$, then a marginal increase in $\underline{w}$ increases the average wage and the lowest wage paid to workers, and decreases the efficiency wage. If there is no involuntary unemployment under a given minimum wage $\underline{w}$ and $\underline{w} \neq W(Q^p)$, then a marginal increase in $\underline{w}$ increases the wage of all employed workers.*

[24] A precise characterization is provided in Theorem 3 in Section 3.2.

The first and the last statement in Proposition 3 condition on the same regions as the first and last statement in Theorem 1. Thus, it is also the case that the equilibrium employment increases under these conditions. When there is no wage dispersion but involuntary unemployment at a given minimum wage, an increase in that minimum wage trivially increases the wage paid to all employed workers and the average wage since all of them are paid the minimum wage, but it decreases total employment.

3.2 Outline of the proof of Theorem 1

We now provide an overview of our formal analysis concerning the general effects of minimum wages on employment, wage dispersion, and involuntary unemployment.

3.2.1 Mechanism design problem

We start by characterizing the minimum cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity $Q \in [0,1]$ when the monopsony faces a minimum wage regulation $\underline{w} \in [W(0), W(1)]$, as well as the associated optimal mechanism. Our exposition here skips a lot of steps without mention, but the full details of the analysis can be found in the proof of Proposition 4. Formally, the cost minimization problem is now given by

$$\underline{C}_R(Q, \underline{w}) := \min_{\boldsymbol{x}, \boldsymbol{w}} \int_{\underline{c}}^{\overline{c}} w(c)x(c)\, dG(c),$$
$$\text{s.t.} \quad \text{(IC)}, \quad \text{(IR)}, \quad w(c) \geq \underline{w}\ \forall c \in [\underline{c}, \overline{c}], \quad \int_{\underline{c}}^{\overline{c}} x(c)\, dG(c) = Q.$$

Introducing the virtual cost function $\Gamma(c) = c + \frac{G(c)}{g(c)}$ and combining standard mechanism design arguments with the fact that it suffices to impose the minimum wage constraint on the type with the lowest opportunity cost of supplying labor, we can rewrite this problem as

$$\underline{C}_R(Q, \underline{w}) = \min_{\boldsymbol{x}} \int_{\underline{c}}^{\overline{c}} \Gamma(c)x(c)\, dG(c),$$
$$\text{s.t.} \quad x \text{ is non-increasing}, \quad x(\underline{c})\underline{c} + \int_{\underline{c}}^{\overline{c}} x(c)\, dc \geq \underline{w}x(\underline{c}), \quad \int_{\underline{c}}^{\overline{c}} x(c)\, dG(c) = Q.$$

Next, we let $\lambda \geq 0$ denote the Lagrange multiplier associated with the minimum wage constraint and consider the corresponding dual problem. Since strong duality holds, the primal problem is convex and solving the dual problem yields a solution that is also primal feasible, the solution to the dual problem also solves the primal problem. So from this point forward it is without loss of generality to focus on the dual problem:

$$\underline{C}_R(Q,\underline{w}) = \max_{\lambda\geq 0}\min_{\boldsymbol{x}} \int_{\underline{c}}^{\overline{c}} \left(\Gamma(c) - \frac{\lambda}{g(c)}\right) x(c)\, dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}), \tag{2}$$

$$\text{s.t.} \quad x \text{ is non-increasing}, \qquad \int_{\underline{c}}^{\overline{c}} x(c)\, dG(c) = Q.$$

Using the probability measure $G_\lambda(c) = \frac{\lambda}{1+\lambda}\mathbf{1}(\underline{c} = c) + \frac{1}{1+\lambda}G(c)$, we can rewrite the Lagrangian as

$$(1+\lambda)\int_{\underline{c}}^{\overline{c}} \left[\left(\Gamma(c) - \frac{\lambda}{g(c)}\right)\mathbf{1}(c > \underline{c}) + (\underline{w} - \underline{c})\,\mathbf{1}\,(c = \underline{c})\right] x(c)\, dG_\lambda(c).$$

Solving the dual problem then requires that we iron the function

$$\psi_\lambda(c) := \left(\Gamma(c) - \frac{\lambda}{g(c)}\right)\mathbf{1}(c > \underline{c}) + (\underline{w} - \underline{c})\,\mathbf{1}\,(c = \underline{c})$$

with respect to the probability measure G_λ. Denoting this ironed function by $\underline{\psi}_\lambda$ and imposing the quantity constraint, we have

$$\underline{C}_R(Q,\underline{w}) = \max_{\lambda\geq 0}\int_{\underline{c}}^{G^{-1}(Q)} \underline{\psi}_\lambda(c) x(c)\, dG_\lambda(c). \tag{3}$$

If $\lambda > 0$ and there is a binding minimum wage constraint, it is possible that $\underline{\psi}_\lambda(\underline{c}) > \lim_{c\downarrow\underline{c}}\underline{\psi}_\lambda(c)$. Consequently, $\underline{\psi}_\lambda$ may have an ironing interval at the origin (i.e. an ironing interval of the form (Q_1, Q_2), where $Q_1 = 0$ and $Q_2 > 0$). This leaves us with only three possibilities for the optimal mechanism: setting a market-clearing wage (corresponding to quantities outside an ironing interval), rationing workers at the minimum wage (corresponding to quantities within an ironing interval at the origin) or using a two-wage mechanism as introduced in Section 2.3 (corresponding to quantities within ironing intervals away from the origin).

Let

$$w_1(Q) := \begin{cases} (1-\alpha(Q))W(Q_1(m)) + \alpha(Q)W(Q_2(m)), & \exists\, m \in \mathcal{M} \text{ s.t. } Q \in (Q_1(m), Q_2(m)) \\ W(Q), & Q \notin \bigcup_{m\in\mathcal{M}}(Q_1(m), Q_2(m)) \end{cases}$$

denote the lowest wage paid under the optimal mechanism for procuring the quantity Q, absent wage regulation.[25] Since this function is increasing and continuous its inverse w_1^{-1} is

[25]Note that $w_1(0) = W(0)$ and $w_1 = W(1)$.

well defined. Note that we always have $S(\underline{w}) \leq w_1^{-1}(\underline{w})$ with $S(\underline{w}) < w_1^{-1}(\underline{w})$ if and only if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$. For a given value of $\underline{w}$, if $Q \geq w_1^{-1}(\underline{w})$, then the minimum wage constraint does not bind and $\lambda = 0$. Conversely, if $Q < w_1^{-1}(\underline{w})$, then the minimum wage constraint is binding and $\lambda > 0$. Here, the optimal procurement mechanism must involve either rationing workers at the minimum wage or a two-wage mechanism where the low wage is equal to the minimum wage.

If $Q \leq S(\underline{w})$, then $\underline{C}_R(Q, \underline{w}) = \underline{w}Q$ because this cost cannot be reduced by randomizing over wages that are at least as high as $\underline{w}$. For any $Q > S(\underline{w})$, it is not feasible to hire Q workers by rationing at the minimum wage. Consequently, if $S(\underline{w}) < w_1^{-1}(\underline{w})$, then the optimal mechanism is a two-wage mechanism for any $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$. Since these mechanisms do not randomize over the lowest-cost workers,[26] they can be computed using a convexification procedure that simplifies (3). Specifically, let $\underline{\Psi}$ denote the convexification the function $\Psi(\cdot, \lambda) := C(\cdot) - \lambda W(\cdot)$ with respect to its first argument. Then rewriting (3) with respect to the uniform measure and integrating by parts yields $\underline{C}_R(Q, \underline{w}) = \underline{\Psi}(Q, \lambda^*) + \lambda^* \underline{w}$, where λ^* is pinned down by the first-order condition $-\left.\frac{d\underline{\Psi}(Q,\lambda)}{d\lambda}\right|_{\lambda=\lambda^*} = \underline{w}$. Intuitively, we end up with an objective function of the form $C(\cdot) - \lambda W(\cdot)$ because, fixing an arbitrary mechanism, one can compute the procurement cost by taking an appropriate convex combination of the function C, and taking the corresponding convex combination of the function W yields the lowest wage paid under that mechanism.[27] Putting all of this together, we have

$$\underline{C}_R(Q, \underline{w}) = \begin{cases} \underline{w}Q, & Q \in [0, S(\underline{w})] \\ \mathcal{D}^*(Q, \underline{w}), & Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big) \\ \underline{C}(Q), & Q \geq w_1^{-1}(\underline{w}), \end{cases} \tag{4}$$

where $\mathcal{D}^*(Q, \underline{w})$ is the value of the dual problem

$$\mathcal{D}^*(Q, \underline{w}) := \max_{\lambda \geq 0} \min_{q_1 \in [0,Q],\, q_2 \geq Q} \{(1 - \beta(Q, q_1, q_2))\Psi(q_1, \lambda) + \beta(Q, q_1, q_2)\Psi(q_2, \lambda) + \lambda \underline{w}\} \tag{5}$$

with $\beta(Q, q_1, q_2) := \frac{Q - q_1}{q_2 - q_1}$.

[26] This is the same as the feature commonly known as "no randomization at the top" that arises in optimal selling mechanisms. Here, it means no randomization at the bottom (of the type space).

[27] In essence, the reason that two-wage mechanisms remain optimal under a minimum wage constraint is that the objective function and the constraint are of the same functional form. The second-best mechanisms studied by Myerson and Satterthwaite (1983) similarly exhibit this feature because there both social surplus and revenue have the same functional form. In contrast, moment or majorization constraints (see Kang (2021) and Kleiner, Moldovanu, and Strack (2021)) generally increase the number of prices that need to be considered.

3.2.2 Properties of cost-minimizing mechanisms

The following theorem formally summarizes the mechanism design analysis and establishes a number of useful properties of $\underline{C}_R$, as well as the corresponding marginal cost function $\underline{C}'_R(Q,\underline{w}) := \lim_{\epsilon\uparrow 0} \frac{\underline{C}_R(Q+\epsilon,\underline{w})-\underline{C}_R(Q,\underline{w})}{\epsilon}$, which is the *left derivative* of $\underline{C}_R$ with respect to Q.

Theorem 2. *The minimal cost $\underline{C}_R(Q,\underline{w})$ of procuring the quantity $Q \in [0,1]$ under the minimum wage $\underline{w} \in [W(0), W(1)]$ is given by (4). This function is convex (and hence continuous) in Q and increasing in both Q and $\underline{w}$. The marginal cost function $\underline{C}'_R$ is well-defined and continuous on $(Q,\omega) \in [0,1] \times [W(0), W(1)]$ with $Q \neq S(\underline{w})$. Moreover, for $Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big)$, $\underline{C}'_R$ is bounded and*

$$\frac{\partial \underline{C}'_R(Q,\underline{w})}{\partial Q} > 0 > \frac{\partial \underline{C}'_R(Q,\underline{w})}{\partial \underline{w}}.$$

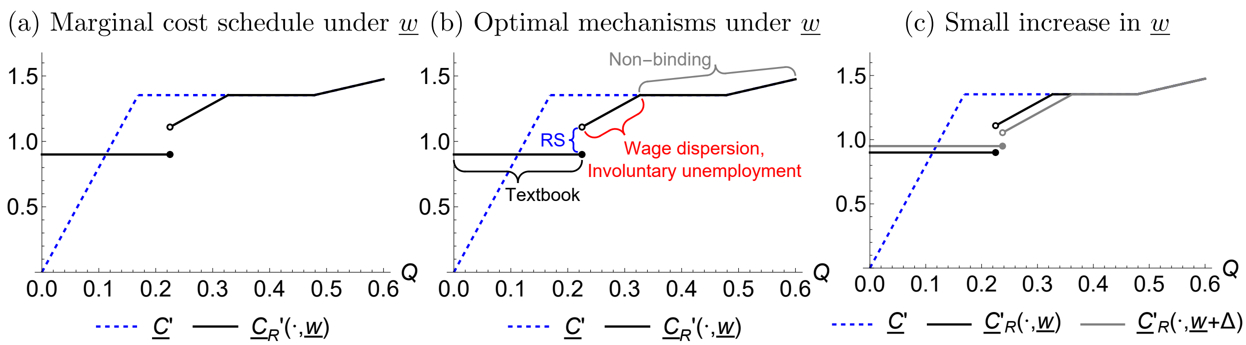

Figure 6: Panel (a) illustrates $\underline{C}'_R(\cdot,\underline{w})$ and $\underline{C}'$ for $\underline{w} = 0.9$. Panel (b) indicates the corresponding optimal mechanisms, using the terminology introduced in Section 3.1 (see Figure 4). Panel (c) illustrates how $\underline{C}'_R(\cdot,\underline{w})$ shifts in response to a small increase in $\underline{w}$ to $\underline{w} = 0.95$.

Panel (a) in Figure 6 provides a representative illustration of the marginal cost function $\underline{C}'_R(\cdot,\underline{w})$ for a given minimum wage $\underline{w}$. Panel (b) relates $\underline{C}'_R$ to the regions in the schematic summary in Figure 4. Panel (c) illustrates the implications of a marginal increase in $\underline{w}$ on $\underline{C}'_R(\cdot,\underline{w})$. For $Q \geq w_1^{-1}(\underline{w})$ the minimum wage constraint does not bind and $\underline{C}'_R(\cdot,\underline{w})$ simply coincides with $\underline{C}'$. A marginal increase in $\underline{w}$ decreases (in a set inclusion sense) the set of Q values such that this case applies. On the interval $Q \in [0, S(\underline{w})]$, where the optimal mechanism involves rationing workers at the minimum wage, $\underline{C}'_R(\cdot,\underline{w})$ is constant and equal to $\underline{w}$. $\underline{C}'_R(\cdot,\underline{w})$ may be discontinuous at the point $Q = S(\underline{w})$, where the optimal procurement mechanism involves posting a market-clearing wage of $\underline{w}$.[28] As Theorem 1 shows and Panel (c) in Figure 6 illustrates: An increase in $\underline{w}$ expands the interval $[0, S(\underline{w})]$—shifting any

[28]As we have established, given $\epsilon > 0$ sufficiently small, the firm optimally hires $S(\underline{w}) - \epsilon$ workers by rationing $S(\underline{w})$ workers at the minimum wage. It optimally hires $S(\underline{w}) + \epsilon$ using a two-wage mechanisms

discontinuity in $\underline{C}'_R(\cdot, \underline{w})$ at $Q = S(\underline{w})$ to the right—and increasing the value of $\underline{C}'_R(\cdot, \underline{w})$ on $[0, S(\underline{w})]$. Over the interval $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ the optimal mechanism is a two-wage mechanism but the "ironed" marginal cost function $\underline{C}'_R(\cdot, \underline{w})$ is strictly increasing.[29] A marginal increase in $\underline{w}$ decreases $\underline{C}'_R(\cdot, \underline{w})$ over this region.

Computing comparative statics in mechanism design problems involving constraints—where uniform pricing is not necessarily optimal—is challenging. Nevertheless, the mechanism design machinery developed in this paper can be used to derive comparative statics pertaining to the parameters of the optimal mechanism over the interval $Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big)$. In particular, we have the following lemma which is illustrated in the piecewise linear specification from Appendix B.1 in Figure 7.

Lemma 1. *Suppose that $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and that $Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big)$. For $i \in \{1, 2\}$, let $q_i^*(Q, \underline{w})$ denote the solution value of q_i in (5). Then $q_1^*(Q, \underline{w})$ increases in $\underline{w}$ and decreases in Q and $q_2^*(Q, \underline{w})$ decreases in $\underline{w}$ and increases in Q.*

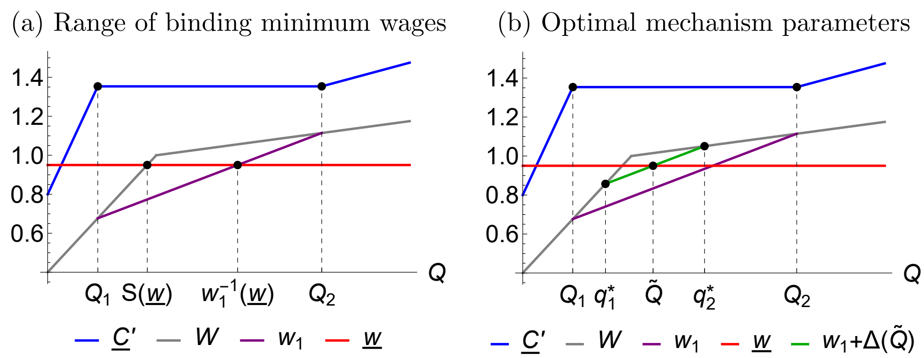

Figure 7: As Panel (a) illustrates, for $Q \in (Q_1, Q_2)$ the lowest wage offered under the laissez-faire equilibrium is represented by the function w_1, which is a linear combination of $W(Q_1)$ and $W(Q_2)$. A minimum wage $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ is binding for $\tilde{Q} < w_1^{-1}(\underline{w})$. For the piecewise linear specification of W and $\tilde{Q} \in (S(\underline{w}), w_1^{-1}(\underline{w}))$, the parameters of the optimal two-wage mechanism can be computed as illustrated in Panel (b), by taking a parallel shift of the w_1 function. A proof of this property is provided in Appendix B.1.

The fact stated in Lemma 1 that $q_1^*(Q, \underline{w})$ increases in $\underline{w}$ is a formalization of the intuition that, as the minimum wage increases, the monopsony will procure more units at the minimum wage since it is a price-taker on these units.

where some workers are hired with certainty at the minimum wage and others are rationed at an efficiency wage. The difference between the left-hand and right-hand mechanisms at $Q = S(\underline{w})$ explains why the marginal cost function $\underline{C}'_R(\cdot, \underline{w})$ is not necessarily continuous at $Q = S(\underline{w})$.

[29] In more standard problems—such as that considered in Section 2.3—ironed functions are constant over an ironing interval. Here, the slope of the function $\underline{C}_R(\cdot, \underline{w})$ varies with Q over the interval $Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big)$ because the Lagrange multiplier associated with the minimum wage constraint (i.e. the shadow price of that constraint) decreases as Q increases.

3.2.3 Comparative statics from Theorem 1

We are now in a position to analyze the equilibrium effects associated with changes in the minimum wage. Letting $Q^*(\underline{w})$ denote the optimal level of employment under the minimum wage $\underline{w}$, we have the following corollary to Theorem 2:

Corollary 1. *If there is a $Q^*(\underline{w})$ satisfying $V(Q^*(\underline{w})) = \underline{C}'(Q^*(\underline{w}), \underline{w})$, then $Q^*(\underline{w})$ characterizes the optimal level of employment under a given minimum wage $\underline{w}$. If there is no $Q^*(\underline{w})$ such that $V(Q^*(\underline{w})) = \underline{C}'(Q^*(\underline{w}), \underline{w})$, then the optimal level of employment given $\underline{w}$ is $Q^*(\underline{w}) = S(\underline{w})$.*

We now consider how a marginal increase in the minimum wage affects equilibrium employment, involuntary unemployment and wage dispersion. Let $U^*(\underline{w}) := q_2^*(Q^*(\underline{w}), \underline{w}) - Q^*(\underline{w})$ and $\Delta w^*(\underline{w})$ respectively denote the level of involuntary unemployment and the difference between the highest wage and the lowest wage paid under the optimal procurement mechanism for a given minimum wage $\underline{w}$. Combining the results of Theorem 2 and Lemma 1 and using the notation ∂_+ to represent taking right derivatives, then yields the following proposition.

Proposition 4.

0. *If $Q^*(\underline{w}) > w_1^{-1}(\underline{w})$, then $\partial_+ Q^*(\underline{w}) = \partial_+ U^*(\underline{w}) = \partial_+ \Delta w^*(\underline{w}) = 0$.*

1. *If $w_1^{-1}(\underline{w}) > S(\underline{w})$ and $Q^*(\underline{w}) \in (S(\underline{w}), w_1^{-1}(\underline{w})]$, then $\partial_+ Q^*(\underline{w}) > 0$, $\partial_+ U^*(\underline{w}) < 0$ and $\partial_+ \Delta w^*(\underline{w}) < 0$.*

2. *If $Q^*(\underline{w}) \neq Q^p$ and $Q^*(\underline{w}) = S(\underline{w})$, then $\partial_+ Q^*(\underline{w}) > 0$, $\partial_+ U^*(\underline{w}) \geq 0$ and $\partial_+ \Delta w^*(\underline{w}) \geq 0$.*

3. *If $Q^*(\underline{w}) = Q^p$ or $Q^*(\underline{w}) < S(\underline{w})$, then $\partial_+ Q^*(\underline{w}) < 0$, $\partial_+ U^*(\underline{w}) > 0$ and $\partial_+ \Delta w^*(\underline{w}) = 0$.*

This last proposition, which is illustrated in Figure 8, immediately implies the statement of Theorem 1. Case 0 corresponds to the interior of the region where the minimum wage is not binding and, consequently, a marginal increase in the minimum wage has no effect on equilibrium employment, involuntary unemployment and wage dispersion. All the other cases correspond to the respective cases in Theorem 1. Case 1 corresponds to the first region studied in and after Theorem 1, in which the monopsony uses a two-wage mechanism under a binding minimum wage.[30] The comparative statics proven in Theorem 2 immediately imply that in this region a marginal increase in the minimum wage increases equilibrium

[30] Strictly speaking, the minimum wage is not binding for the knife-edge case where $Q^*(\underline{w}) = w_1^{-1}(\underline{w}) > S(\underline{w})$. However, here the minimum wage will be binding after a marginal increase in the minimum wage.

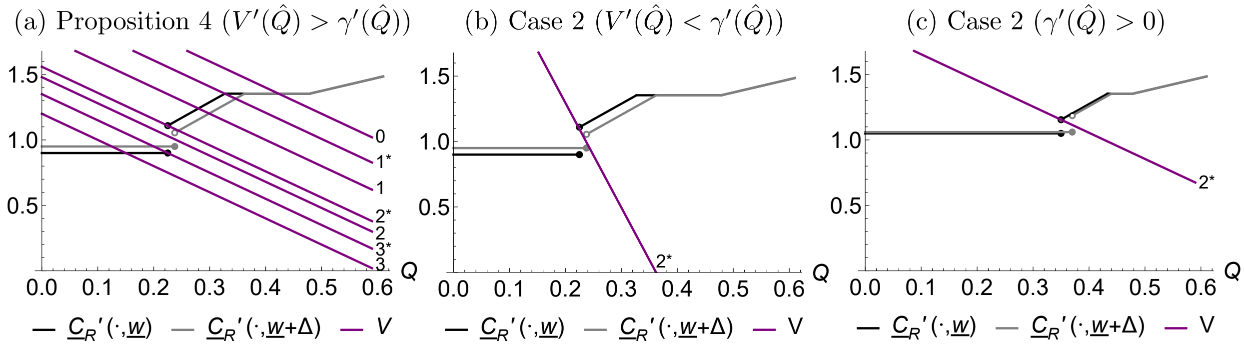

Figure 8: Panel (a) illustrates Proposition 4 using a range of V functions labelled according to the cases they represent, with the asterisks indicating knife-edge instances of each case. Panel (a) includes a knife-edge instance of Case 2 (corresponding to the V function labelled 2^*) where a marginal increase in the minimum wage results in a transition from the Robinson-Stigler region to a region with involuntary unemployment and wage dispersion. Panel (b) and Panel (c) illustrate two instances of this knife-edge case where a marginal increase in the minimum wage does not result in a transition out of the Robinson-Stigler region.

employment (see Figure 8). Showing that wage dispersion and involuntary unemployment also decrease in this region is the most cumbersome part of the proof of Proposition 4 due to the countervailing effects involved.[31] Case 2 corresponds to the Robinson-Stigler region from Section 3.1, where the monopsony sets a market-clearing wage equal to the minimum wage. The condition $Q^*(\underline{w}) = S(\underline{w})$ with $Q^*(\underline{w}) \neq Q^p$ ensures that, for $\epsilon > 0$ sufficiently small, $Q^*(\underline{w} + \epsilon) \geq S(\underline{w})$ (see Figure 8).[32] Consequently, a marginal increase in the minimum wage increases employment. If we are in the interior of the Robinson-Stigler region, involuntary unemployment or wage dispersion are also unaffected. However, in general it is possible for a marginal increase in the minimum wage to result in a transition from the Robinson-Stigler region to a region with involuntary unemployment and wage dispersion. Case 3 corresponds to the textbook region from Section 3.1, where the monopsony rations workers at the minimum wage. Here, the comparative statics proven in Theorem 2 again imply that a marginal increase in the minimum wage decreases employment and increases involuntary unemployment without having any effect on wage dispersion. Note that the knife-edge case $Q^* = Q^p$ is also covered here (see Figure 8).

As in Theorem 1, the cases covered in Proposition 4 condition on equilibrium outcomes. The final step in our analysis involves characterizing when each of these cases arise as a

[31] In particular, proving Proposition 4 requires showing that $q_2^*(Q^*(\underline{w}), \underline{w})$ decreases in $\underline{w}$ in this region. We know that $Q^*(\underline{w})$ increases in $\underline{w}$ in this region and by Lemma 1 we also know that $q_2^*(Q, \underline{w})$ is increasing in Q and decreasing in $\underline{w}$. Proving that $q_2^*(Q^*(\underline{w}), \underline{w})$ decreases in $\underline{w}$ therefore requires showing that the latter effect dominates the former.

[32] Note that if the value of $\underline{w}$ such that $Q^*(\underline{w}) = S(\underline{w})$ is such that the function $\underline{C}'_R(\cdot, \underline{w})$ is continuous, then there is no Robinson-Stigler region.

function of the minimum wage itself. This will formalize the schematic summary illustrated in Figure 4.

3.2.4 The details behind the schematic summary

The preceding analysis derived comparative statics results for equilibrium employment, wages and involuntary for each of the regions covered in Theorem 1 but did not analyze when transitions from one region to another occur. Of particular interest are transitions into and out of the region involving wage dispersion and involuntary unemployment.

We begin by delineating the case when there is no region with wage dispersion. In particular, if $\bigcup_{m\in\mathcal{M}}(Q_1(m), Q_2(m))\bigcap[Q^*, Q^p] = \emptyset$, then there can never be equilibrium wage dispersion, regardless of the value of $\underline{w}$. The minimum wage effects are then precisely those identified by standard monopsony pricing models in the tradition of Robinson (1933) and Stigler (1946), in which the monopsony always sets a uniform wage. For $\underline{w} < W(Q^*)$, the minimum wage does not bind and the monopsony hires Q^* workers at the wage $W(Q^*)$. For $\underline{w} \in [W(Q^*), W(Q^p)]$, the monopsony hires $S(\underline{w})$ workers at the minimum wage, and employment increases in $\underline{w}$. In either case, there is no involuntary unemployment. Finally, for $\underline{w} > W(Q^p)$, we are in the textbook region and the monopsony hires $V^{-1}(\underline{w})$ workers at the minimum wage. Here, involuntary unemployment $S(\underline{w}) - V^{-1}(\underline{w})$ increases in $\underline{w}$ and employment $V^{-1}(\underline{w})$ decreases in $\underline{w}$.

Now suppose that there exists $m \in \mathcal{M}$ such that $[Q^*, Q^p]\bigcap(Q_1(m), Q_2(m)) \neq \emptyset$. Given a binding minimum wage $\underline{w}$, there is then equilibrium wage dispersion if and only if $w_1^{-1}(\underline{w}) > S(\underline{w})$ and the function V intersects with $\underline{C}'_R(\cdot, \underline{w})$ on $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ (i.e. the region right of the discontinuity in $\underline{C}'_R(\cdot, \underline{w})$ where the minimum wage is binding and $\underline{C}'_R(\cdot, \underline{w})$ is strictly increasing). To trace out where the function $\underline{C}'_R(\cdot, \underline{w})$ starts to strictly increase in Q under a binding minimum wage, we define the function

$$\gamma(Q) := \lim_{\underline{w}\uparrow W(Q)} \underline{C}'_R(Q, \underline{w}),$$

which gives the marginal cost of procuring $Q \in [0, 1]$ as $\underline{w}$ approaches the market-clearing wage $W(Q)$ from below.[33] Note that γ is well-defined and continuous.[34] If Q is such that $\underline{C}(Q) = C(Q)$ then γ satisfies $\gamma(Q) = C'(Q)$. If Q is such that $\underline{C}(Q) < C(Q)$ then γ traces

[33] If $\underline{w}$ approaches $W(Q)$ from above, the marginal cost is simply $W(Q)$ (i.e. $\lim_{\underline{w}\downarrow W(Q)} \underline{C}'_R(Q, \underline{w}) = C'_R(Q, W(Q)) = W(Q)$).

[34] Consider the function $\underline{C}'^{+}_R(Q, \underline{w}) := \lim_{\epsilon\downarrow 0} \frac{\underline{C}_R(Q+\epsilon, \underline{w}) - \underline{C}_R(Q, \underline{w})}{\epsilon}$, which is the *right derivative* of $\underline{C}_R$ with respect to Q. Then $\underline{C}'^{+}_R$ is the continuous extension of $\underline{C}'_R$ on the closed set $\{(Q, \underline{w}) \in [0, 1] \times [W(0), W(1)] : \underline{w} \leq W(Q)\}$. The function γ is well-defined and continuous because $\gamma(Q) = \underline{C}'^{+}_R(Q, W(Q))$ holds for all $Q \in [0, 1]$ and, consequently, γ is simply the composition of two continuous functions.

out the right limit of $\underline{C}'_R(Q,\cdot)$ at the discontinuity at $\underline{w} = W(Q)$ that arises as a result of the transition in the optimal procurement mechanism from a two-wage mechanism to a mechanism involving a single wage of $\underline{w}$.

Figure 9 illustrates the region where $\underline{C}'_R(\cdot,\underline{w})$ is strictly increasing under a binding minimum wage and traces out the function γ for the piecewise linear specification with a single ironing interval (Q_1, Q_2) from Appendix B.1. Panel (c) in Figure 9 shows that, aside from the point $\underline{q}$ such that $\gamma(\underline{q}) = W(\underline{q})$, $\gamma(Q) > W(Q)$ for all $Q \in (Q_1, Q_2)$. This implies that $\underline{C}'_R(\cdot,\underline{w})$ is continuous at $Q = S(\underline{w})$ only if $\underline{w} = W(\underline{q})$ and is discontinuous at $Q = S(\underline{w})$ for all $\underline{w} \in (W(0), W(1))\backslash\{W(\underline{q})\}$. Panel (c) then illustrates how for $Q^* \in (Q_1, Q_2)$ and V linear, the piecewise linear specification of W generically exhibits the structure depicted in Figure 4. Specifically, letting $\hat{Q}$ denote the unique point of intersection between V and γ, we have equilibrium wage dispersion and involuntary unemployment for all $\underline{w} \in (w_1(Q^*), W(\hat{Q}))$. Moreover, for all $\underline{w} \in [W(\hat{Q}), W(Q^p))$, we are in the Robinson-Stigler region where the monopsony optimally hires $S(\underline{w})$ workers at the minimum wage $\underline{w}$. The non-generic case occurs if, as mentioned in footnote 23, $Q^p = \underline{q}$, in which case $\hat{Q} = Q^p$ and there is no Robinson-Stigler region.

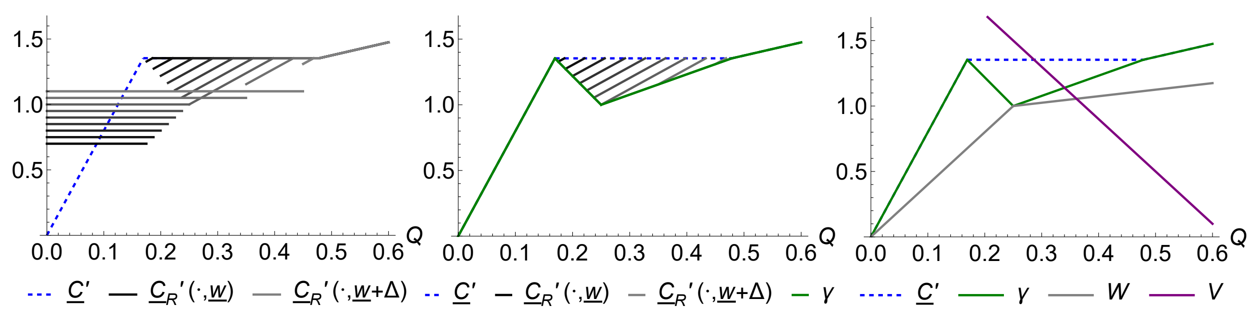

Figure 9: An illustration of how the contour of the strictly increasing region of $\underline{C}'_R(\cdot,\underline{w})$ (where this function is illustrated for a variety of $\underline{w}$ values) under a binding minimum wage defines the function γ.

More generally, we need to compare the functions V and γ to determine whether there is equilibrium wage dispersion and involuntary unemployment under a given minimum wage.[35] Consider the sets $\mathcal{W} := (w_1(Q^*), W(Q^p))$ and

$$\overline{\mathcal{W}} = \bigcup_{m\in\mathcal{M}} (W(Q_1(m)), W(Q_2(m))) \cap (w_1(Q^*), W(Q^p)).$$

[35]Transitions into and out of the Robinson-Stigler region occur at points $\hat{Q}$ where the functions V and γ intersect, and depend on the sign and relative slopes of these functions at their points of intersection. In Figure 8, Panel (a) illustrates a transition out of the Robinson-Stigler region, while Panel (b) and Panel (c) illustrate transitions into the Robinson-Stigler region.

The optimal mechanism involves wage dispersion if and only if $\underline{w} \in \overline{\mathcal{W}}$ and $V(Q) > \gamma(Q)$, where Q is such that $\underline{w} = W(Q)$. Equivalently, the optimal mechanism involves wage dispersion if and only if $\underline{w} \in \overline{\mathcal{W}}$ and $V(S(\underline{w})) > \gamma(S(\underline{w}))$. Let $\tilde{V}(\underline{w}) := V(S(\underline{w}))$ and $\tilde{\gamma}(\underline{w}) := \gamma(S(\underline{w}))$ and define the sets

$$\mathcal{T} := \left\{\underline{w} \in \overline{\mathcal{W}} : \tilde{V}(\underline{w}) > \tilde{\gamma}(\underline{w})\right\} \quad \text{and} \quad \mathcal{S} := \mathcal{W} \setminus \mathcal{T}.$$

A two-wage mechanism is then used under a binding minimum wage if and only if $\underline{w} \in \mathcal{T}$ and single-wage mechanism is used under a binding minimum wage if and only if $\underline{w} \in \mathcal{S}$. Since the functions $\tilde{V}$ and $\tilde{\gamma}$ are continuous the sets $\mathcal{T}$ and $\mathcal{S}$ can be written as a union of disjoint intervals and transitions from single-wage (two-wage) mechanisms to two-wage (single-wage) mechanisms occur as $\underline{w}$ transitions from $\mathcal{S}$ ($\mathcal{T}$) to $\mathcal{T}$ ($\mathcal{S}$). The following theorem then summarizes this analysis and formalizes the schematic summary illustrated in Figure 4.

Theorem 3. *For all $\underline{w} \in [W(0), w_1(Q^*)]$ the minimum wage constraint is not binding and for all $\underline{w} \in (W(Q^p), W(1)]$ we are in the textbook region. The Robinson-Stigler region is given by the set $\mathcal{S}$ and $\mathcal{T}$ is the set of binding minimum wages where the optimal procurement mechanism involves wage dispersion and involuntary unemployment.*

As previously discussed, the piecewise linear specification in Appendix B.1 with V linear generically exhibits the structure depicted in Figure 4. In contrast, Figure 10 illustrates a case in which there is no involuntary unemployment under the laissez-faire equilibrium but a sufficiently high minimum can induce involuntary unemployment and wage dispersion. In such cases, the transition from the Robinson-Stigler region to the region with involuntary unemployment and wage dispersion occurs results in a there is a discontinuous increase in these quantities.[36] This implies that V first crosses γ from below on (Q_1, Q_2), which in turn implies that for $\underline{w}$ close to but above $W(Q_1)$ there is no wage dispersion but for larger values of $\underline{w}$ there is both.

As pointed out in Proposition 2, appropriately chosen minimum wages increase social surplus when there is involuntary unemployment under the laissez-faire equilibrium. The discontinuous increase in involuntary unemployment associated with a transition from the Robinson-Stigler region to one with wage dispersion provides a less sanguine perspective. The positive employment effect associated with such a transition is second-order (as is the effect on workers' total pay since $\underline{C}_R$ is continuous) relative to the inefficiency associated with randomly rationing some workers at the efficiency wage. Thus, a marginal increase

[36] Panel (a) of Figure 8 also illustrates a transition from the Robinson-Stigler region to a region with involuntary unemployment and wage dispersion. This possibility is covered by Case 2 in Proposition 4.

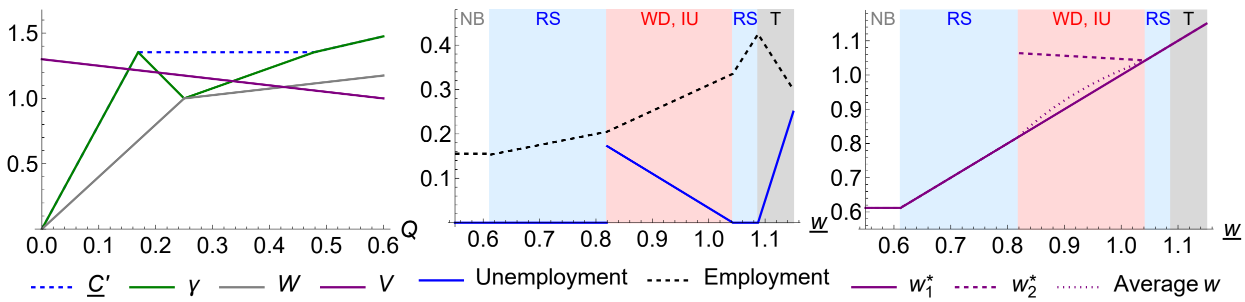

Figure 10: Transitioning from the Robinson-Stigler region to one with wage dispersion causes a discontinuous increase in involuntary unemployment.

in the minimum wage that causes a transition from the Robinson-Stigler region to a region with wage dispersion are associated with a reduction in social surplus. Total worker surplus also decreases at these points because the increase in total pay is outweighed by the harm associated with inefficient allocation.

4 Extensions

We now extend the analysis to accommodate, in turn, quantity competition among firms and horizontal differentiation of jobs and workers.

4.1 Quantity competition

A natural question is to what extent the effects identified above generalize to more competitive environments. To address this question, we now extend the model to allow for quantity competition between firms. This extension is not only in line with David Card's call for models of wage-setting with imperfect competition (Card, 2022b) but—since it relates to a Cournot-based setup—it also generalizes a framework that has proved productive for empirical analysis of market power in labor markets (Berger, Herkenhoff, and Mongey, 2022). We first introduce the setup, derive the equilibrium and discuss its properties. Then we analyze the effects of minimum wages.

4.1.1 Setup

Suppose now that there are n firms procuring labor. We index these firms by i. For each firm i, the marginal value for procuring the y_i-th unit of labor is given by a continuously decreasing function $V(y_i)$ satisfying $V(0) > W(0)$ and $V(1) < W(1)$, where we use y_i to distinguish individual firms' quantities from the quantities q_1 and q_2 that were introduced in the previous section. The firms compete in quantities as follows. They simultaneously submit quantities y_i to a Walrasian auctioneer as in standard oligopoly and oligopsony models with quantity competition. However, rather than procuring the $Q := \sum_{i=1}^n y_i$ units at the market-clearing wage $W(Q)$, which is the standard assumption in Cournot models and leads to a procurement cost function of C, we assume that the auctioneer can use the optimal procurement mechanism and thus procures the Q units at minimal total cost $\underline{C}(Q)$. Firm i who employs y_i units has to pay the cost $\frac{y_i}{Q}\underline{C}(Q)$. Modulo replacing the cost function C with $\underline{C}$, this is the same as in standard Cournot models since $\frac{y_i}{Q}\underline{C}(Q) = y_i W(Q)$ for all $Q \notin \bigcup_{m \in \mathcal{M}}(Q_1(m), Q_2(m))$. The efficient quantity for a given n is denoted by Q_n^p and is such that

$$V\left(\frac{Q_n^p}{n}\right) = W(Q_n^p).$$

This is the quantity that would emerge if the firms were price-takers.

4.1.2 Equilibrium

The analysis from Section 2.3 then extends to this model, insofar as we will have involuntary unemployment and efficiency wages whenever $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. In models in which market-clearing wages are imposed, the quantity in a symmetric equilibrium, denoted Q_n^C, satisfies

$$V\left(\frac{Q_n^C}{n}\right) = W(Q_n^C) + \frac{Q_n^C}{n} W'(Q_n^C), \tag{6}$$

provided a symmetric equilibrium exists. Since $W' > 0$, we have $Q_n^C < Q_n^p$. That is, with market-clearing wages the equilibrium quantity is inefficiently small.

Let Q_n^* denote the aggregate quantity in a symmetric equilibrium under quantity competition when the quantity is procured at minimal cost and denote by Q^e the equilibrium quantity under perfect competition and price-taking behavior, that is, $Q^e = S(V(0))$.

Proposition 5. *The quantity setting game has a unique equilibrium, and this equilibrium is symmetric. The aggregate equilibrium quantity Q_n^* is increasing in n. If $Q_n^p \leq Q_n^*$, then $n > 1$ and $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. As $n \to \infty$, if $C(Q^e) = \underline{C}(Q^e)$, then we have $Q_n^* \to Q^e$ and if $Q^e \in (Q_1(m_e), Q_2(m_e))$ for some $m_e \in \mathcal{M}$, then we have $Q_n^* \to \tilde{Q}$, where $\tilde{Q} \in (Q^e, Q_2(m_e))$.*

As Proposition 5 shows, in our model of quantity competition the equilibrium is always unique and symmetric and the equilibrium quantity is increasing in n. For n sufficiently large, $Q_n^p < Q_n^*$ is possible. That is, the equilibrium quantity can be excessively large. To develop an understanding of how such a reversal can occur, consider the first-order condition under symmetry,

$$V\left(\frac{Q}{n}\right) = \frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q) =: h(Q,n).$$

If $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then $h(Q,n)$ is increasing and concave in Q and, for all $n \in \mathbb{N}$, it satisfies $h(Q_i(m),n) > W(Q_i(m))$. Moreover, $h(Q,n)$ decreases in n and satisfies $h(Q,1) > W(Q)$ for all $Q \in (Q_1(m), Q_2(m)))$. In contrast, for n sufficiently large, there exists at least one interval $(a_n, b_n) \subset (Q_1(m), Q_2(m))$ such that $h(Q,n) < W(Q)$ for all $Q \in (a_n, b_n)$, where a_n decreases in n and b_n increases in n.[37] Consequently, if $V(Q/n) = h(Q,n)$ for $Q \in (a_n, b_n)$, then $Q_n^* \in (a_n, b_n)$ and $Q_n^p < Q_n^*$. Figure 11 illustrates the relationship between the functions W and h. Intuitively, the first-order condition implies that a firm's perceived marginal cost $h(Q,n)$ of procuring the quantity Q when it faces $n-1$ competitors is a convex combination of $\underline{C}'(Q)$ (which is larger than $W(Q)$) and $\underline{C}(Q)/Q$ (which is less than $W(Q)$ for $Q \in (Q_1(m), Q_2(m)))$. As n increases, the weight on $\underline{C}(Q)/Q$ increases, eventually leading to $h(Q,n) < W(Q)$ for some values of Q.

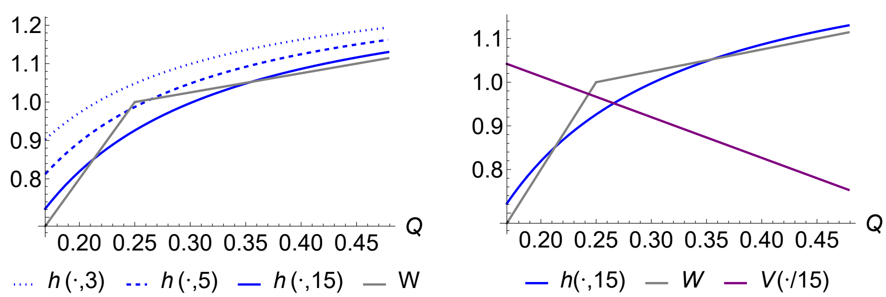

Figure 11: The left-hand panel displays W for the piecewise linear specification from Appendix B.1, and $h(\cdot,n)$ for $n=3$, $n=5$ and $n=15$. The right-hand panel focuses on the case where $n=15$ and shows that $Q_n^p < Q_n^*$ for $V(Q/n) = 1.2 - 14Q/n$.

As $n \to \infty$, Q_n^p converges to the efficient (or Walrasian) quantity Q^e, which in turn satisfies $V(0) = W(Q^e)$. Consequently, the last statement of Proposition 5 distinguishes the cases where there is no $m \in \mathcal{M}$ such that $Q^e \in (C(Q_1(m), Q_2(m)))$ and where there exists a $m_e \in \mathcal{M}$ such that $Q^e \in (Q_1(m_e), Q_2(m_e)))$. Observe that in the latter case

$$\underline{C}'(Q^e) = C'(Q_2(m_e)) > W(Q_2(m_e)) > W(Q^e).$$

That is, $\underline{C}'(Q^e) > V(0)$.

[37] If there are multiple subintervals over which $h(Q,n) < W(Q)$ for some n, index these by k. Then for each k, a_n^k is decreasing in n and b_n^k is increasing in n because h decreases in n. Of course, eventually two or more of these subintervals may collapse into one, that is if $b_n^k < a_n^{k+1}$, we may have $b_{n'}^k \geq a_{n'}^{k+1}$ for some $n' > n$. But this does not invalidate the point that the set of $Q \in (Q_1(m), Q_2(m))$ for which $h(Q,n) < W(Q)$ increases in n in the set inclusion sense.

Proposition 5 implies that key features of the monopsony model —efficiency wages, involuntary unemployment—extend to quantity competition. Moreover, the relationship between competition and involuntary unemployment is not monotone because increasing competition can bring the equilibrium quantity into or out of an ironing interval $(Q_1(m), Q_2(m))$. Within such an interval, competition decreases wage dispersion and involuntary unemployment and increases $w_1(Q_n^*)$ and employment, while leaving the efficiency wage $W(Q_2(m))$ fixed. If $\underline{C}(Q^e) < C(Q^e)$ holds, there is involuntary unemployment and an efficiency wage even in the limit as $n \to \infty$. This yields a "natural" unemployment rate associated with perfect competition of $(Q_2(m_e) - \tilde{Q})/Q_2(m_e)$. In contrast to the usual notion of a natural unemployment rate, this unemployment is a result of inefficient resource allocation in the form of both random allocation and excessive economic activity (since $\tilde{Q} > Q^e$). In other words, there is the possibility of inefficient perfect competition. Figures 15 and 16 in Appendix B.2 illustrate these effects for our leading example. Even though each firm's market share becomes infinitesimal as $n \to \infty$, market power is still exerted in equilibrium because the auctioneer procures the aggregate quantity at the minimal cost.

4.1.3 Minimum wage effects and competition

In models with quantity competition and market-clearing wages, setting a minimum wage above $W(Q_n^C)$ (the market-clearing wage for the equilibrium quantity Q_n^C absent wage regulation) and below $W(Q_n^p)$ (the competitive wage) has a positive effect on total employment and, accordingly, workers' pay. To see this, recall that the competitive quantity Q_n^p is such that $V\left(\frac{Q_n^p}{n}\right) = W(Q_n^p)$ while the equilibrium quantity satisfies (6). Together with $W' > 0$, this implies that $Q_n^C < Q_n^p$. Any minimum wage $\underline{w} \in (W(Q_n^C), W(Q_n^p)]$ then has a positive employment effect. Since $\lim_{n\to\infty} Q_n^p = Q^e = \lim_{n\to\infty} Q_n^C$, the scope for this kind of quantity and social-surplus increasing minimum wage regulation vanishes in the limit as $n \to \infty$.[38]

Even if the symmetric equilibrium in the model with market-clearing wages is the unique equilibrium absent a minimum wage, a binding minimum wage $\underline{w} \in (W(Q_n^C), W(Q_n^p))$ inevitably gives rise to a continuum of equilibria. To see this, denote by $r_i(Q_{-i})$ the best response function of an arbitrary firm i to the aggregate quantity $Q_{-i} = \sum_{j\neq i} y_j$ demanded by its rivals. If the best response function is given by the first-order condition $V(r_i) - W(Q_{-i} + r_i) - r_i W'(Q_{-i} + r_i) = 0$, the equilibrium is unique and symmetric.[39] De-

[38] Whether the differences $W(Q_n^p) - W(Q_n^C)$ and $Q_n^p - Q_n^C$ monotonically decrease in n—and the scope for this kind of minimum wage of regulation—depends on the specifics of the model. If W and V are both linear, then both $W(Q_n^p) - W(Q_n^C)$ and $Q_n^p - Q_n^C$ decrease in n.

[39] To see this, totally differentiate the first-order condition to obtain $r_i' = -\frac{W' + r_i W''}{W' + r_i W'' + W' - V'}$, which satisfies $-1 < r_i' < 0$, where we drop arguments for ease of notation. The aggregate quantity Q given Q_{-i} and i's best response satisfies $Q = Q_{-i} + r_i(Q_{-i})$. The right-hand side is increasing in Q_{-i} and hence invertible.

noting by $r_{\underline{w},i}(Q_{-i})$ the best response function given minimum wage $\underline{w} \in (W(Q_n^*), W(Q_n^p))$, we have

$$r_{\underline{w},i}(Q_{-i}) = \max\{r_i(Q_{-i}), \min\{S(\underline{w}) - Q_{-i}, V^{-1}(\underline{w})\}\},$$

where the term $\min\{S(\underline{w}) - Q_{-i}, V^{-1}(\underline{w})\}$ accounts for the possibility that even though the firm could procure the quantity $S(\underline{w}) - Q_{-i}$ at the minimum wage $\underline{w}$ it only wants to do so if this quantity is small enough and its willingness to pay is greater than $\underline{w}$. This means that it will not procure more than $V^{-1}(\underline{w})$. Since $Q_n^C < S(\underline{w}) < Q_n^p$, we have

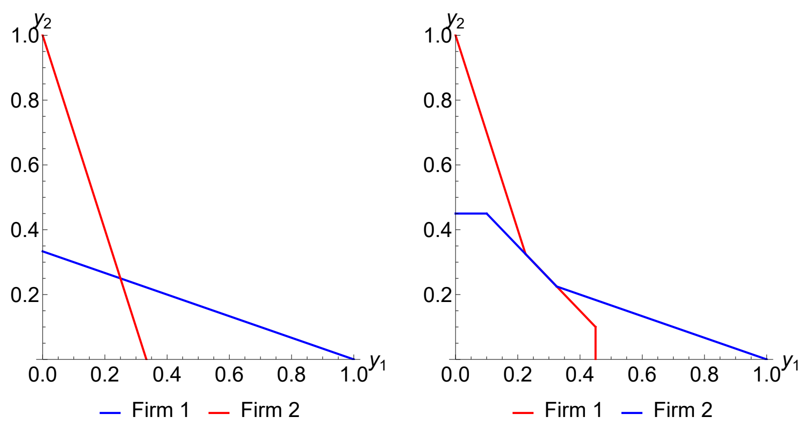

Figure 12: Standard quantity competition without a minimum wage (left panel) and with a minimum wage of $\underline{w} = 0.55$ (right panel). The minimum wage generates a continuum of equilibria. The figures assumes $V(y_i) = 1 - y_i$ and $W(Q) = Q$, which implies that $Q_n^* = 1/2$ and $Q_n^p = 2/3$.

$$r'_{\underline{w},i}\left(Q_{-i}\right)|_{Q_{-i}=\frac{n-1}{n}S(\underline{w})} = -1.$$

This implies that in the neighborhood of the symmetric equilibrium in which each firm chooses $S(\underline{w})/n$ there is a also a continuum of necessarily asymmetric equilibria as illustrated in Figure 12. Given that V is decreasing, the symmetric equilibrium is the one that maximizes social surplus and is therefore a natural selection.

To analyze the effects of introducing a minimum wage, we maintain focus on the symmetric equilibrium and study its comparative statics.[40] Similarly to the model without a

Following Anderson, Erkal, and Piccinin (2020), we can thus write $Q_{-i} = f_i(Q)$ as a function of Q, where f_i is increasing. This allows us to construct what Anderson, Erkal, and Piccinin call the *inclusive-best response* function $\tilde{r}_i(Q) := r_i(f_i(Q))$, which gives the optimal quantity that i would choose if the aggregate quantity is Q, which includes its own quantity. We have $\tilde{r}_i' = \frac{r_i'}{1+r_i'} < 0$. The aggregate quantity Q is an equilibrium quantity if and only if $\sum_{i=1}^n \tilde{r}_i(Q) = Q$. Because the left-hand side decreases and the right-hand side increases in Q, it follows that the Q satisfying this equality is unique. Moreover, because the firms are symmetric, we have $\tilde{r}_i = \tilde{r}_j$ for all $i, j \in \{1, \ldots, n\}$. Hence, the unique equilibrium is symmetric.

[40] As stated in Proposition 5, without wage regulation, the symmetric equilibrium is the unique equilibrium.

minimum wage, given a minimum wage $\underline{w}$, we let

$$h_R(Q,n,\underline{w}) := \frac{n-1}{n}\frac{\underline{C}_R(Q,\underline{w})}{Q} + \frac{1}{n}\underline{C}'_R(Q,\underline{w})$$

denote the firm-level marginal cost of procurement under symmetry in the model with quantity competition. Observe that for $Q \leq S(\underline{w})$ (equivalently, $\underline{w} \geq W(Q)$), we have $\underline{C}_R(Q,\underline{w}) = \underline{w}Q$ and $\underline{C}'_R(Q,\underline{w}) = \underline{w} = \frac{\underline{C}_R(Q,\underline{w})}{Q}$, which implies that $h_R(Q,n,\underline{w}) = \underline{w}$. For $Q > S(\underline{w})$, $h_R(Q,n,\underline{w})$ is larger than $\underline{w}$ and strictly increasing in Q. Moreover, $h_R(Q,n,\underline{w})$ is continuous in Q everywhere, except possibly at $Q = S(\underline{w})$, where it is continuous if and only if $\underline{C}'_R(Q,\underline{w})$ is at that point.[41] Finally, for $\underline{w} < w_1(Q)$ (equivalently, $Q > w_1^{-1}(\underline{w})$), we have

$$h_R(Q,n,\underline{w}) = \frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q) = h(Q,n)$$

because $\underline{C}_R(Q,\underline{w}) = \underline{C}(Q)$ and hence $\underline{C}'_R(Q,\underline{w}) = \underline{C}'(Q)$ for $\underline{w} < w_1(Q)$. Putting all of this together, in the model with quantity competition the minimum wage binds in exactly the same instances as in the monopsony model.

Since $V(Q/n)$ is decreasing in Q and $h_R(Q,n,\underline{w})$ has the same curvature properties as $h(Q,n)$, it follows that if there exists a Q satisfying

$$V\left(Q/n\right) = h_R(Q,n,\underline{w}), \tag{7}$$

then Q/n is the symmetric equilibrium of the model with quantity competition given the minimum wage $\underline{w}$. If no such quantity exists, $h_R(Q,n,\underline{w})$ must be discontinuous at Q, which implies $Q = S(\underline{w})$. In this case, the symmetric equilibrium quantity is $S(\underline{w})/n$. Summarizing, we have the following lemma:

Lemma 2. *The model with quantity-setting firms and a given minimum wage $\underline{w}$ has a symmetric equilibrium. In this equilibrium, each firm chooses the quantity Q/n with Q satisfying* (7) *if such a Q exists and $S(\underline{w})/n$ otherwise.*

The characterization of the symmetric equilibrium in the quantity setting game with a minimum wage mirrors the characterization of the optimal quantity in the monopsony model with a minimum wage. Similarly to Corollary 1, the aggregate quantity in the symmetric equilibrium is given by the quantity that satisfies (7) and equates the firm-level marginal

Whether given a minimum wage $\underline{w}$ the symmetric equilibrium is the socially optimal equilibrium when the equilibrium involves wage dispersion and involuntary unemployment is an open question. Of course, if the aggregate quantity is the same in a symmetric equilibrium and an asymmetric equilibrium, social surplus is larger in the symmetric equilibrium.

[41] This last observation follows from the fact that $\underline{C}_R(Q,\underline{w})$ and hence $\underline{C}_R(Q,\underline{w})/Q$ are continuous at $Q = S(\underline{w})$.

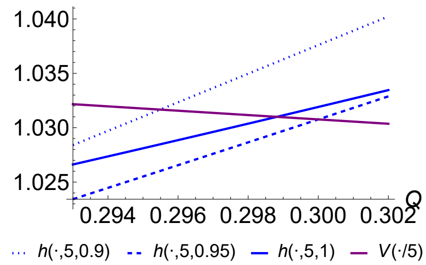

Figure 13: Illustration of non-monotone minimum wage effects with quantity competition.

benefits and marginal costs, whenever such a quantity exists, and is otherwise given by the quantity $S(\underline{w})$ supplied at the minimum wage. As we will show next, relative to the monopsony case, a difference arises for the comparative statics associated with an increase in the minimum wage when the equilibrium quantity is characterized by (7) and inside an ironing interval $(Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Recall that in the monopsony model, when there is equilibrium wage dispersion and involuntary unemployment a marginal increase in $\underline{w}$ increases the equilibrium quantity and decreases the equilibrium level of involuntary unemployment because $\underline{C}'_R(Q, \underline{w})$ decreases in $\underline{w}$. In contrast, with $n \geq 2$, $h_R(Q, n, \underline{w})$ is a convex combination of $\underline{C}'_R(Q, \underline{w})$, which decreases in $\underline{w}$, and $\underline{C}_R(Q, \underline{w})/Q$, which increases in $\underline{w}$. Thus, with quantity competition, the effect of a marginal increase in the minimum wage will not necessarily be monotone when there is wage dispersion and involuntary unemployment. This is illustrated in Figure 13 for the piecewise linear specification from Appendix B.1 and a linear marginal benefit function V for $n = 5$ with $\underline{w} = 0.9$ (dotted), $\underline{w} = 0.95$ (dashed) and $\underline{w} = 1$ (solid). From $\underline{w} = 0.9$ to $\underline{w} = 0.95$, the equilibrium quantity increases, and from $\underline{w} = 0.95$ to $\underline{w} = 1$, it decreases.

However, as the following proposition shows, the marginal effect of increasing the minimum wage when the minimum wage is equal to the lower of the two wages absent wage regulation, that is at $\underline{w} = w_1(Q_n^*)$, on the equilibrium employment level $Q_n^*(\underline{w})$ is positive:

Proposition 6. *Suppose $n \in \mathbb{N}$ and $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Then at $\underline{w} = w_1(Q_n^*)$, the marginal effect of increasing the minimum wage on the equilibrium quantity $Q_n^*(\underline{w})$ is positive, that is, $\frac{dQ_n^*(\underline{w})}{d\underline{w}}|_{\underline{w}=w_1(Q_n^*)} > 0$.*

Proposition 6 shows that a minimum wage close to but above $w_1(Q_n^*)$ increases the equilibrium quantity if $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. This resonates with an insight from the monopsony model, where a marginal increase in the minimum wage increases employment whenever there is wage dispersion and involuntary unemployment. However, in the model with quantity competition increasing the equilibrium quantity is not necessarily a move in the right direction because of the possibility of excessively high employment, that is, $Q_n^* > Q_n^p$. More generally, the following theorem describes the effects of imposing a binding minimum wage when $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. In its proof, we show that for $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$,

$$h_\gamma(Q,n) := \frac{n-1}{n} W(Q) + \frac{1}{n}\gamma(Q) \tag{8}$$

is the limit of $h_R(Q,n,\underline{w})$ as $\underline{w}$ approaches $W(Q)$ from below. This function is continuous in Q and its role and properties are analogous to those of γ in the monopsony model. Assuming $Q_n^* \in (Q_1(m), Q_2(m))$, we let $\hat{Q}_{H,n}$ denote the largest value of Q such that $V(Q/n) = h_\gamma(Q,n)$.

Theorem 4. *Whenever there is involuntary unemployment and wage dispersion under a given minimum wage in the model with quantity competition, increasing the minimum wage to $\underline{w} = W(\hat{Q}_{H,n}(m))$ increases employment and eliminates involuntary unemployment. If there is involuntary unemployment and no wage dispersion under a given minimum wage, increasing the minimum wage decreases employment and increases involuntary unemployment. Moreover, if $\underline{w} \neq W(Q_n^p)$ and there is no involuntary unemployment under a given minimum wage, a marginal increase in the minimum wage increases employment.*

Note also that because $h_\gamma(Q,n) \geq W(Q)$, the aggregate equilibrium quantity in the presence of a minimum wage $\underline{w} = W(Q_n^*)$ is never larger than Q_n^p. Therefore, when $Q_n^* > Q_n^p$, one effect of imposing a minimum wage equal to the market-clearing wage for the equilibrium quantity absent wage regulation is that it prevents excessively high levels of employment. Since the ordering $\hat{Q}_{H,n}(m) \leq Q_n^p$ (see (19) in Appendix A) does not depend on the ordering of Q_n^* and Q_n^p, this also implies that even when $Q_n^* > Q_n^p$ holds under the laissez-faire equilibrium, total employment increases in $\underline{w}$ for $\underline{w} \in [W(\hat{Q}_{H,n}(m)), W(Q_n^p)]$ without inducing involuntary unemployment. Since we know from Proposition 6 that increasing the minimum wage at $w_1(Q_n^*)$ increases employment, if $Q_n^* > Q_n^p$, then the effects of the minimum wage on total employment must be non-monotone on $[w_1(Q_n^*), W(\hat{Q}_{H,n}(m))]$. Furthermore, if the Walrasian quantity Q^e is inside some ironing interval (i.e. if $Q^e \in (Q_1(m_e), Q_2(m_e))$ for some $m_e \in \mathcal{M}$) then there is scope for social-surplus increasing minimum wage regulation even in the perfectly competitive limit. Setting $\underline{w} = W(\hat{Q}_{H,n}(m_e))$ will eliminate involuntary unemployment and we have $\underline{w} \to W(Q^e)$ as $n \to \infty$ because $\lim_{n\to\infty} \hat{Q}_{H,n}(m) = Q^e$.

4.2 Horizontally differentiated jobs

We now return to a monopsony setting but allow for horizontal differentiation of workers, with the monopsony offering horizontally differentiated jobs. For this setting, we show that in addition to involuntary unemployment, the optimal mechanism may involve inefficient matching of workers to jobs, both of which can be remedied by an appropriately chosen minimum wage.

Setup Consider a variant of the Hotelling model in which a monopsony with jobs at locations 0 and 1 has a willingness to pay of $V(Q_\ell)$ for the Q_ℓ-th worker employed at a given location $\ell \in \{0,1\}$. As before, $V(Q_\ell)$ is assumed to be continuous and strictly decreasing. There is a continuum of workers with linear transportation costs whose locations, which are the private information of each worker, are uniformly distributed between 0 and 1. The total mass of workers is 1. The value of the outside option of each worker is normalized to 0.[42] The payoff of a worker at location z that works at 0 for a wage of w is $w - z$, while this worker's payoff of working at 1 for a wage of w is $w - (1 - z)$. Observe that this implies that the market-clearing wage to hire Q_ℓ workers at a given location is $W(Q_\ell) = Q_\ell$, which in turn means that the cost of procurement at each location under market-clearing wages is $C(Q_\ell) = Q_\ell^2$. Of course, the monopsony can hire Q_ℓ workers at $\ell = 0, 1$ if and only if $\sum_\ell Q_\ell \leq 1$.

Equilibrium We first derive the minimum cost of procuring the quantity $Q_\ell \in [0, 1/2]$ at a given location, assuming that the same quantity is procured at the other location. First, notice that conditional on being employed, the expected transportation cost of a worker at any location $z \in [0,1]$, who is equally like to work at each location, is $1/2$. To satisfy the individual rationality constraint of such a worker, they must be paid a wage of at least $1/2$. Consequently, by offering a wage of $1/2$ to workers who agree to enter a lottery which allocates them to work at location 0 or location 1, each with probability $1/2$, or who multitask by spending half their time at each location, the monopsony can procure any quantity $Q_\ell \in [0, 1/2]$ at both locations at a marginal procurement cost of $1/2$. Since the marginal cost of procuring Q_ℓ workers at a market-clearing wage is $2Q_\ell$, the monopsony can procure the quantity $Q_\ell \in [0, 1/2]$ at each location at a cost of

$$\underline{C}_H(Q_\ell) := \begin{cases} Q_\ell^2, & Q_\ell \in [0, 1/4] \\ (Q_\ell - 1/4)/2 + 1/16, & Q_\ell \in (1/4, 1/2] \end{cases}$$

by offering a wage of $1/2$ to attract "universalists" (workers who are willing to do either job) and a wage of $1/4$ to attract "specialists" (workers with locations no further away from 0 and 1 than $1/4$, who are guaranteed the job closest to their location). Notice that the individual rationality constraint will bind for all employed workers with locations $z \in (1/4, 3/4)$.

[42] This is without loss of generality within the domain of problems in which the value of the outside option and the willingness to pay per worker are independent of the workers' locations since all that matters for these problems is the difference between the latter and the former.

Consequently, for the marginal workers at 1/4 and 3/4, the incentive compatibility constraints, which require that these workers are indifferent between working as a specialist or as a universalist, coincide with their individual rationality constraints.

The preceding arguments establish that this scheme with wage dispersion and random worker-job matches results in lower procurement costs, relative to market-clearing wages for any $Q_\ell \in (1/4, 1/2]$. Arguments along the lines of those in Balestrieri, Izmalkov, and Leao (2021) and Loertscher and Muir (2022b), who study optimal selling mechanisms on the Hotelling line, can be used to establish that $\underline{C}(Q_\ell)$ is in fact the minimal cost of procurement, subject to workers' incentive compatibility and individual rationality constraints.[43]

The equilibrium level of employment Q_ℓ^* at each location $\ell \in \{0,1\}$ is given by the unique number satisfying $V(Q_\ell^*) = \underline{C}'(Q_\ell^*)$. We say that the equilibrium involves *involuntary unemployment* if there is a positive mass of workers who would be willing to work but are not employed at the equilibrium wages, and we say that it involves *worker-job mismatches* if workers with $z < 1/2$ work at location 1 or workers with $z > 1/2$ work at location 0 in equilibrium.[44] The following proposition summarizes characteristics of the equilibrium. As it follows directly from the preceding arguments, we do not provide a separate proof.

Proposition 7. *If $V(1/4) \leq 1/2$, then $Q_\ell^* \leq 1/4$ and worker-job mismatches and involuntary unemployment do not occur in equilibrium. If $V(1/4) > 1/2 > V(1/2)$, then $Q_\ell^* \in (1/4, 1/2)$ and worker-job mismatches and involuntary unemployment do occur in equilibrium. If $V(1/2) \geq 1/2$, then $Q_\ell^* = 1/2$ and the equilibrium involves worker-job mismatches but no involuntary unemployment.*

Figure 14 illustrates the case $V(1/4) > 1/2 > V(1/2)$ in Proposition 7 for the linear specification $V(Q_\ell) = v - Q_\ell$ with $v = 7/8$. For this linear specification, $V(1/4) > 1/2 > V(1/2)$ is equivalent to $v \in (3/4, 1)$.

[43] An outline of the argument, adapted from the monopoly screening problem in Loertscher and Muir (2022b) to the procurement setting and assuming, for now, that all workers are employed, is as follows. Let $x_\ell(z)$ denote the probability that the worker who reports type $z \in [0,1]$ works at location $\ell \in \{0,1\}$. Incentive compatibility implies that $x_1(z) - x_0(z)$ is non-decreasing. Type $\hat{z}$ is the worst-off type if $x_1(\hat{z}) = x_0(\hat{z})$. Because all workers are employed, we have $x_0(z) + x_1(z) = 1$, implying $x(z) \equiv x_0(z)$ is sufficient, and incentive compatibility becomes equivalent to $x(z)$ being non-increasing, and $\hat{z}$ is worst-off if $x(\hat{z}) = 1/2$. Given any worst-off type $\hat{z} \in [0,1]$, incentive compatibility yields the designer's objective in terms of virtual costs and values. Because its pointwise minimizer is not monotone, one needs to iron the virtual types. (Put differently, the cost of procurement is not convex in Q_0, the number of units procured at location 0.) The pointwise minimizer given the ironed virtual type function must assign a worker in the ironing interval with equal probability to jobs at 0 and 1. Consequently, the value of the ironed virtual type function over the ironing interval must be 0. Moreover, this also means that not employing some of these workers is also optimal. Thus, the assumption that all workers are employed can easily be relaxed.

[44] If worker-job mismatches are optimal, workers who work at the high wage of 1/2 are indifferent between working and not. Thus, those—if any—who are involuntarily unemployed are also indifferent between being unemployed and working.

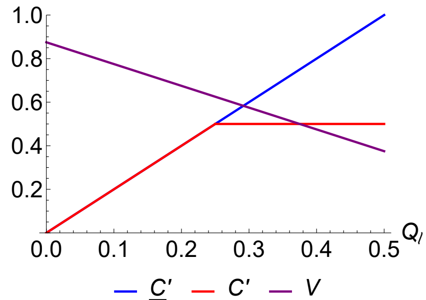

Figure 14: Illustration of Proposition 7 for $V(Q_\ell) = v - Q_\ell$ with $v = 7/8$.

Minimum wage effects If a minimum wage of $\underline{w} = 1/2$ is imposed, then provided $V(1/4) > 1/2$, the strict profitability of worker-job mismatches vanishes without any negative effects on the equilibrium level of employment. More generally, the minimum cost of procuring $Q_\ell \in [0, 1/2]$ at location $\ell \in \{0, 1\}$ given the regulated minimum wage $\underline{w} \in [0, 1/2]$, denoted $\underline{C}_{H,R}(Q_\ell, \underline{w})$, is

$$\underline{C}_{H,R}(Q_\ell, \underline{w}) = \begin{cases} \underline{w} Q_\ell, & Q_\ell \in [0, S_\ell(\underline{w})] \\ (Q_\ell - S_\ell(\underline{w}))/2 + 1/16, & Q_\ell \in (S_\ell(\underline{w}), 1/2], \end{cases}$$

where $S_\ell(w) = w$ is the labor supply function at location ℓ. Consequently, the marginal cost of procuring labor is $\underline{C}'_{H,R}(Q_\ell, \underline{w}) = \underline{w}$ for $Q_\ell \leq \underline{w}$ and $\underline{C}'_{H,R}(Q_\ell, \underline{w}) = 1/2$ for $Q_\ell \in (\underline{w}, 1/2]$.[45]

Denoting by Q_ℓ^p the quantity the monopsony would procure under price-taking behavior at location ℓ, which is the unique number satisfying $V(Q_\ell^p) = Q_\ell^p$ if $V(1/2) \leq 1/2$ (and otherwise $Q_\ell^p = 1/2$) and by Q_ℓ^* the equilibrium quantity employed at ℓ under the laissez-faire equilibrium, the effects of minimum wages $\underline{w} \in [0, 1/2]$ are as follows.

If $Q_\ell^* \leq 1/4$, then for $\underline{w} \in (Q_\ell^*, Q_\ell^p)$, a marginal increase in $\underline{w}$ increases the equilibrium quantity employed and workers' pay without inducing involuntary unemployment. This corresponds to a standard Robinson-Stigler region. For $Q_\ell^* \in (1/4, 1/2)$, minimum wages $\underline{w} < 1/4$ have no effect. Under a minimum wage $\underline{w} \in (1/4, Q_\ell^*)$, the monopsony hires $\underline{w}$ workers at the minimum wage and the $Q_\ell^* - \underline{w}$ workers at a wage of $1/2$. A marginal increase in the minimum wage leaves total employment and involuntary unemployment unaffected and increases the number of workers employed at the minimum wage. Social surplus increases because the number of worker-job mismatches decreases. All workers are weakly better off with the minimum wage increase. For $\underline{w} \in (Q_\ell^*, Q_\ell^p)$, employment is $\underline{w}$ and all workers who are employed are paid the minimum wage. There is no involuntary unemployment. A marginal increase in the minimum wage increases employment and social surplus. All workers are weakly better off with the minimum wage increase. For $\underline{w} > Q_\ell^p$ with $Q_\ell^p < 1/2$,

[45]Note that there is no scope for ironing because $\underline{C}'_{H,R}(Q_\ell, \underline{w})$ is already monotone.

a marginal increase in $\underline{w}$ reduces the equilibrium quantity employed and increases involuntary unemployment, corresponding to the textbook effects of minimum wages.

The guidance for policy makers contemplating a marginal increase in $\underline{w}$, presented in the following proposition, is similar to that provided in Theorem 1. As it follows immediately from the preceeding discusion and reults, we do not provide a separate proof.

Proposition 8. *If there is no involuntary unemployment under a given minimum wage $\underline{w} < 1/2$ and $\underline{w} \neq Q^p_\ell$, a marginal increase in $\underline{w}$ increases total employment without inducing involuntary unemployment. If there is involuntary unemployment and wage dispersion under a given minimum wage $\underline{w} < 1/2$, then a marginal increase in $\underline{w}$ increases social surplus and the surplus of all workers (with a strict increase for workers employed at the minimum wage after the increase in $\underline{w}$), without affecting total employment. If there is involuntary unemployment and no wage dispersion, then a marginal increase in $\underline{w}$ decreases total employment and increases involuntary unemployment.*

5 Conclusions

This paper studies the effects of minimum wage in the presence of monopsony power, without restricting the set of contracts the monopsony can offer. Our analysis highlights how monopsony power can be a cause of involuntary unemployment and how introducing an appropriate minimum wage can be a cure for it.

We conclude with a short discussion of avenues for future research. First, beyond minimum wages, our model also provides scope for analyzing effects of prohibiting wage discrimination.[46] In the model with horizontal differentiation, if there is wage dispersion and involuntary unemployment under the laissez-faire equilibrium, the effects of such a policy on workers' surplus, employment and involuntary unemployment are clear: Total employment will decrease, involuntary unemployment will be eliminated, more workers will work at the equilibrium wage than worked at the low wage under the laissez-faire equilibrium, and their wage will be higher. An open question is whether social surplus is larger with or without wage discrimination and to what extent these effects carry over to a model with homogeneous workers. Second, one could extend the baseline model to allow for vertically differentiated tasks. If the cost function the firm faces is not convex, this gives rise to a model of multitasking based on price theory. The effects of task-specific minimum wages are not known to

[46]In concurrent policy debates, there is pressure for greater transparency concerning the wages paid by employers, and policies imposing wage transparency may have similar effects to prohibiting wage discrimination. For empirical evidence concerning the effects of requiring wage transparency and a comprehensive list of recent references, see Cullen and Pakzad-Hurson (2021).

date. Third, one could analyze the effects of introducing unemployment insurance in models in which there is involuntary unemployment under the laissez-faire equilibrium.

Of interest for Industrial Organization more generally may be the possibility—arising from the model with horizontal differentiation—of synergies of mergers to monopoly or monopsony that do not hinge on contractual restrictions or technological efficiencies. The analysis of this paper also naturally raises the question of what form optimal price regulation takes more generally, when a monopoly or monopsony faces a non-regular mechanism design problem. For example, a regulator may have a Ramsey objective consisting of a weighted sum social surplus and the firm's profit.

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Appendix

A Omitted proofs

A.1 Proof of Proposition 2

Proof. We begin this proof by showing that $Q^p > Q^*$ holds. Note that for all $Q > 0$, we have $C'(Q) = W'(Q)Q + W(Q) > W(Q)$. Consequently, whenever $C(Q) = \underline{C}(Q)$, we have $\underline{C}(Q) > W(Q)$. Moreover, for all $m \in \mathcal{M}$ and $Q \in (Q_1(m), Q_2(m))$, we also have $\underline{C}'(Q) = C'(Q_2(m)) > W(Q_2(m)) > W(Q)$. Combining $\underline{C}(Q) > W(Q)$ with the optimality condition $V(Q^*) = \underline{C}'(Q^*)$ shows that $V(Q^*) > W(Q^*)$. Since V is strictly decreasing, W is strictly increasing and Q^p satisfies $V(Q^p) = W(Q^p)$, $Q^p > Q^*$ follows, as required.

Now consider introducing a minimum wage of $\underline{w} = W(Q^p)$. The monopsony will then optimally hire at least Q^p workers because the marginal benefit $V(Q) > V(Q^p)$ of hiring $Q < Q^p$ workers always exceeds the marginal cost $\underline{w} = V(Q^p)$.[47] Theorem 2 establishes that the marginal cost of hiring Q workers under optimal procurement with a minimum wage of $\underline{w}$ is increasing in Q and strictly exceeds $V(Q)$ for $Q > Q^p$. Consequently, the monopsony will optimally employ the efficient quantity $Q^p > Q^*$ of workers under a minimum wage of $\underline{w} = W(Q^p)$. Moreover, the monopsony will optimally procure these workers by setting a market-clearing wage of $W(Q^p)$ (see Theorem 2), thereby eliminating both involuntary unemployment and wage dispersion. Theorem 2 also establishes that the minimal cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity Q under a minimum wage of $\underline{w}$ is increasing in both Q and $\underline{w}$. This implies that, relative to the laissez-faire equilibrium, imposing a minimum wage of $\underline{w} = W(Q^p)$ increases workers' total pay. Since $W(Q) > V(Q)$ holds for all $Q > Q^p$, no minimum wage can induce the monopsony to hire more than the efficient quantity Q^p. Thus, setting a minimum wage of $\underline{w} = W(Q^p)$ maximizes total employment. Moreover, social surplus is maximized when the monopsony hires the efficient quantity of workers under a market-clearing wage, which is precisely what is achieved by setting $\underline{w} = W(Q^p)$. This establishes each statement of Proposition 2 concerning the introduction of a minimum wage of $\underline{w} = W(Q^p)$.

Next, consider introducing a minimum wage of $\underline{w} = W(Q_2(m))$. The marginal cost of hiring $Q \leq Q_2(m)$ workers is then $W(Q_2(m))$, while the marginal cost of hiring $Q > Q_2(m)$ workers is $\underline{C}'(Q)$ (see Theorem 2). By assumption, $Q^* \in (Q_1(m)), Q_2(m))$ holds for some $m \in \mathcal{M}$ and we therefore have $\underline{C}'(Q_2(m)) = \underline{C}'(Q^*) = V(Q^*)$. Consequently, for all $Q >$

[47] As shown in Section 3.2, if $Q < S(\underline{w})$ then the monopsony cannot reduce its procurement cost benefit from randomizing over wages that are at least as high as $\underline{w}$. Consequently the marginal cost of optimally procuring $Q < S(\underline{w})$ workers is $\underline{w}$.

$Q_2(m)$ we have $Q > Q^*$ and $\underline{C}'(Q) > \underline{C}'(Q_2(m)) = V(Q^*) > V(Q)$. This establishes that the monopsony will not hire more than $Q_2(m)$ workers. In the first paragraph of this proof we also established that $\underline{C}'(Q_2(m)) > W(Q_2(m))$, implying that $V(Q^*) > W(Q_2(m))$. This implies that the monopsony will hire strictly more than Q^* workers under a minimum wage of $\underline{w} = W(Q_2(m))$. Consequently, if $Q^p \geq Q_2(m)$ then the monopsony will hire precisely $Q_2(m)$ workers and involuntary unemployment is eliminated. If $Q^p < Q_2(m)$ then the monopsony will hire $V^{-1}(\underline{w}) \in (Q^*, Q^p)$ workers, rationing these workers at the minimum wage. However, involuntary unemployment will be lower relative to the laissez-faire equilibrium since in either case the total number of workers who participate is $Q_2(m)$ but given the minimum wage, $V^{-1}(\underline{w}) > Q^*$ workers are hired, implying that fewer are involuntarily unemployed. Repeating our previous argument for a minimum wage of $\underline{w} = W(Q^p)$ shows that introducing a minimum wage of $\underline{w} = W(Q_2(m))$ also increases workers' total pay relative to the laissez-faire equilibrium.

Next, we consider introducing a minimum wage $\underline{w} \in (w_1(Q^*), W(Q_2(m)))$. Then combining our arguments here with the results of Proposition 4 and Theorem 3 shows that employment increases in $\underline{w}$ on $\underline{w} < W(Q^p)$, is maximized at $\underline{w} = W(Q^p)$ and decreases in $\underline{w}$ on $\underline{w} > W(Q^p)$. As just argued, relative to the laissez-faire equilibrium, employment is higher under a minimum wage of $\underline{w} = W(Q_2(m))$. This implies that employment is higher under any minimum wage $\underline{w} \in (w_1(Q^*), W(Q_2(m)))$. To show that introducing a minimum wage of $\underline{w} \in (w_1(Q^*), W(Q_2(m)))$ increases workers' total pay relative to the laissez-faire equilibrium, we can again repeat our previous argument for the case where $\underline{w} = W(Q^p)$.

It remains to show that introducing a minimum wage of $\underline{w} \in (w_1(Q^*), W(Q_2(m)))$ also decreases involuntary unemployment relative to the laissez-faire equilibrium. The proof of Theorem 2 shows that under such a minimum wage, the optimal mechanism is either a two-wage mechanism or it involves rationing $S(\underline{w})$ workers at the minimum wage. In the latter case, involuntary unemployment decreases relative to the laissez-faire equilibrium since total employment increases and $S(\underline{w}) < Q_2(m)$ by assumption. Similarly, in the former case it suffices to show that the equilibrium mass of workers that participate in the mechanism decreases relative to the laissez-faire equilibrium. This is established in the proof of Case 1 of Proposition 4.

It now only remains to prove the final statement of the proposition. Any minimum wage $\underline{w}$ that eliminates involuntary unemployment is necessarily such that $\underline{w} \leq W(Q^p)$. Moreover, we know that employment increases in $\underline{w}$ on $\underline{w} < W(Q^p)$. Consequently, any minimum wage that eliminates involuntary unemployment necessarily increases total employment relative to the laissez-faire equilibrium, bringing it closer to the efficient level of Q^p. Such a minimum wage also eliminates the random, inefficient allocation that is associated with involuntary unemployment. Thus, any minimum wage that eliminates involuntary unemployment also increases social surplus relative to the laissez-faire equilibrium. $\square$

A.2 Proof of Proposition 3

Proof. We start by proving the first statement of the proposition. Suppose that there is involuntary unemployment and wage dispersion under a given minimum wage $\underline{w}$. Clearly, the proposition statement holds if $\underline{w}$ does not bind, so assume that a marginal increase in the minimum wage $\underline{w}$ results in a binding minimum wage. Case 1 of Proposition 4 then applies and there is wage dispersion and involuntary unemployment under a binding minimum wage of $\underline{w} + \varepsilon$, provided $\varepsilon > 0$ is sufficiently small. Since the lowest wage paid to workers is always equal to the minimum wage, this clearly increases under a marginal increase in $\underline{w}$ as required. That the efficiency wage decreases under a marginal increase in $\underline{w}$ is established in the course of proving Case 1 of Proposition 4. So it only remains to show that the average wage paid to workers also increases under a marginal increase in $\underline{w}$. Theorem 2 establishes that the minimal cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity Q under the minimum wage $\underline{w}$ is increasing in both $\underline{w}$ and Q and is convex in Q. Moreover, by Case 1 of Proposition 4, the equilibrium quantity $Q^*(\underline{w})$ of workers employed increases under a marginal increase in $\underline{w}$. Putting all of this together, we have

$$\frac{\underline{C}_R(Q^*(\underline{w}), \underline{w})}{Q^*(\underline{w})} \leq \frac{\underline{C}_R(Q^*(\underline{w}), \underline{w} + \varepsilon)}{Q^*(\underline{w})} \leq \frac{\underline{C}_R(Q^*(\underline{w} + \varepsilon), \underline{w} + \varepsilon)}{Q^*(\underline{w} + \varepsilon)}. \tag{9}$$

Here, the first inequality in (9) follows from the fact that $\underline{C}_R$ is increasing in $\underline{w}$. To establish the second inequality, notice that the convexity of $\underline{C}_R$ in Q implies that, for all $\underline{w}$, the function $\frac{\underline{C}_R(Q,\underline{w}) - \underline{C}_R(0,\underline{w})}{Q}$ is increasing in Q. Combining this with the fact that $\underline{C}_R(0, \underline{w}) = 0$ holds for all $\underline{w}$, and that $Q^*(\underline{w})$ is increasing in $\underline{w}$, then yields the second inequality in (9). Thus, (9) establishes that the average wage paid to workers is increasing in $\underline{w}$ as required.

We now prove the second statement of Proposition 3. Suppose that there is no involuntary unemployment under a given minimum wage $\underline{w}$ and that $\underline{w} \neq W(Q^p)$. Then the effects of a marginal increase in $\underline{w}$ are described in Case 2 of Proposition 4, and the equilibrium quantity of employed workers increases. There are two possible subcases. First, if there is no wage dispersion or involuntary unemployment following the marginal increase in $\underline{w}$, then all workers are paid the minimum wage before and after this increase. Second, if the marginal increase in the minimum wage induces wage dispersion and involuntary unemployment, then some employed workers are paid the higher minimum wage, while others are paid an even higher efficiency wage. $\square$

A.3 Proof of Theorem 2

Proof. This proof is divided into two parts. In the first part we prove that the minimum cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity Q under a minimum wage of $\underline{w}$ is given by (4). We then prove the stated properties of this cost function.

Part I: Proof that the minimal cost is $\underline{C}_R(Q, \underline{w})$ *as given in* (4)

The proof of Part I is largely contained within the body of the paper, so here we focus on elaborating on any omitted steps. The designer's problem is to determine the cost-minimizing mechanism for procuring a fixed quantity Q of labor, subject to the constraint that the menu of wages it offers does not include a wage below the minimum wage of $\underline{w}$. A worker's payoff when of type c and reporting to be of type $\hat{c}$ takes the form

$$x(\hat{c})(w(\hat{c}) - c).$$

Let $U(c) := x(c)(w(c) - c)$ denote the worker's payoff under truthful reporting. Individual rationality requires $U(c) \geq 0$ for all c. Incentive compatibility implies that x is non-increasing and that $U'(c) = -x(c)$ holds almost everywhere. For any $c, \hat{c} \in [\underline{c}, \overline{c}]$ we then have

$$U(c) = U(\hat{c}) + \int_c^{\hat{c}} x(y)dy.$$

Setting this equal to $x(c)(w(c) - c)$ and solving for the expected transfer $w(c)x(c)$ paid to type c yields

$$w(c)x(c) = U(\hat{c}) + x(c)c + \int_c^{\hat{c}} x(y)\, dy.$$

Observing that for $c < \hat{c}$, $U(c) \geq U(\hat{c})$ holds because $\int_c^{\hat{c}} x(y)\, dy \geq 0$, the individual rationality constraint is satisfied for all types if and only if $U(\overline{c}) \geq 0$. In an optimal mechanism satisfying incentive compatibility and individual rationality, we must have $U(\overline{c}) = 0$ because otherwise the designer leaves money on the table. Expressing $w(c)x(c)$ with $\hat{c} = \overline{c}$ and using $U(\overline{c}) = 0$, we obtain

$$w(c)x(c) = x(c)c + \int_c^{\overline{c}} x(y)dy.$$

The designer's procurement cost minimization problem, subject to the minimum wage constraint parameterized by $\underline{w}$, is thus given by

$$\min_x \int_{\underline{c}}^{\overline{c}} w(c)x(c)\,dG(c)$$

$$\text{s.t. } x \text{ is non-increasing}, \quad \int_{\underline{c}}^{\overline{c}} x(c)\,dG(c) = Q, \quad w(c) \geq \underline{w} \text{ for all } c \in [\underline{c}, \overline{c}].$$

We have a continuum of constraints—$w(c) \geq \underline{w}$ for all $c \in [\underline{c}, \overline{c}]$—associated with the minimum wage. We now show that it suffices to impose the constraint associated with the minimum wage on the lowest type $c = \underline{c}$. First, notice that individual rationality implies that no worker can be paid a wage w that is less than their opportunity cost. Consequently, for workers with $c > \underline{w}$, the constraint never binds. Next, using the fact that the constraint $w(c) \geq \underline{w}$ is equivalent to $h(c) := x(c)(\underline{w} - w(x)) \leq 0$, we show that $h(c)$ decreases in c on $[\underline{c}, \underline{w}]$. Specifically, letting $c_0, c_1 \in [\underline{c}, \underline{w}]$ with $c_0 < c_1$, we have

$$\begin{aligned} h(c_1) - h(c_0) &= \underline{w}(x(c_1) - x(c_0)) - (x(c_1)c_1 - x(c_0)c_0) + \int_{c_0}^{c_1} x(y)\,dy \\ &= (\underline{w} - c_1)(x(c_1) - x(c_0)) + \int_{c_0}^{c_1} x(y)\,dy - (c_1 - c_0)x(c_0) \leq 0, \end{aligned}$$

where the inequality is strict if x is not constant on $[c_0, c_1]$.[48] Consequently, it suffices to impose the constraint associated with the minimum wage on the lowest type $c = \underline{c}$.

Let $\lambda \geq 0$ denote the Lagrange multiplier associated with the minimum wage constraint for type $c = \underline{c}$ and consider the corresponding dual problem. Since strong duality holds, the primal problem is convex and solving the dual problem yields a solution that is also primal feasible, the solution to the dual problem also solves the primal problem (see, for example, Theorem 2.165 in Bonnans and Shapiro, 2000). So from this point forward it is without loss of generality to focus on the dual problem given in (2). In particular, using $w(c)x(c) = x(c)c + \int_c^{\overline{c}} x(y)\,dy$, the Lagrange dual function is given by

$$\begin{aligned} \mathcal{L}(x, \lambda) &= \int_{\underline{c}}^{\overline{c}} w(c)x(c)\,dG(c) + \lambda(\underline{w}x(\underline{c}) - w(\underline{c})x(\underline{c})) \\ &= \int_{\underline{c}}^{\overline{c}} \left(x(c)c + \int_c^{\overline{c}} x(y)\,dy \right) dF(v) + \lambda x(\underline{c})(\underline{w} - \underline{c}) - \lambda \int_{\underline{c}}^{\overline{c}} x(c)\,dc. \end{aligned}$$

[48] Since x is non-increasing, if x is not constant on $[c_0, c_1]$ we have $(c_1 - c_0)x(c_0) > \int_{c_0}^{c_1} x(y)\,dy$ and $(\underline{w} - c_1)(x(c_1) - x(c_0)) \leq 0$ with strict inequality if $c_1 < \underline{w}$.

Using

$$\int_{\underline{c}}^{\overline{c}} \int_{c}^{\overline{c}} g(c)x(y)\, dy\, dc = \int_{\underline{c}}^{\overline{c}} \int_{\underline{c}}^{y} g(c)x(y)\, dc\, dy = \int_{\underline{c}}^{\overline{c}} G(y)x(y)\, dy,$$

we have

$$\begin{aligned}\mathcal{L}(x,\lambda) &= \int_{\underline{c}}^{\overline{c}} \Gamma(c)x(c)\, dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}) - \lambda \int_{\underline{c}}^{\overline{c}} x(c)\, dc \\ &= \int_{\underline{c}}^{\overline{c}} \left(\Gamma(c) - \frac{\lambda}{g(c)}\right) x(c)\, dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}).\end{aligned}$$

Letting $H(x) = \mathbf{1}(x \geq 0)$ denote the Heaviside step function and using the probability measure $G_\lambda(c) = \frac{\lambda}{1+\lambda} H(\underline{c} - c) + \frac{1}{1+\lambda} G(c)$, we can rewrite the Lagrangian as

$$\mathcal{L}(x,\lambda) = (1+\lambda) \int_{\underline{c}}^{\overline{c}} \left[\left(\Gamma(c) - \frac{\lambda}{g(c)}\right) \mathbf{1}(c > \underline{c}) + (\underline{w} - \underline{c})\, \mathbf{1}\left(c = \underline{c}\right)\right] x(v)\, dG_\lambda(c).$$

We can therefore derive the optimal allocation rule x^* by ironing the function

$$\psi_\lambda(c) = \begin{cases} \underline{w} - \underline{c}, & c = \underline{c} \\ \Gamma(c) - \frac{\lambda}{g(c)}, & c \in (\underline{c}, \overline{c}] \end{cases}$$

with respect to the probability measure G_λ. That this implies that the minimal cost of procuring the quantity Q under a minimum wage $\underline{w}$ is given by (4), is already shown in Section 3.2.1, from (3) onward. The only remaining omitted step is to formally derive the corresponding concavification procedure.

Whenever a two-wage mechanism is optimal under a binding minimum wage $\underline{w}$, instead of ironing the function Ψ with respect to the probability measure G_λ, we can compute the optimal mechanism by performing an appropriate concavification procedure. We accomplish this by rewriting the Lagrangian in terms of quantiles of the type distribution (or, equivalently, as an integral with respect to the uniform probability measure). We make the change of variables $z = G(c)$ and let $y = x \circ G^{-1}$. Note that we then have $W(z) = G^{-1}(z)$ and $C(z) = G^{-1}(z)z$, which implies that $W'(z) = \frac{1}{g(G^{-1}(z))}$ and $C'(z) = G^{-1}(z) + \frac{z}{g(G^{-1}(z))}$. The Lagrangian

$$\mathcal{L}(x,\lambda) = \int_{\underline{c}}^{\overline{c}} \left(c + \frac{G(c)}{g(c)} - \frac{\lambda}{g(c)}\right) x(c)\, dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c})$$

therefore becomes

$$
\begin{aligned}
\mathcal{L}(y,\lambda) &= \int_0^1 \left(G^{-1}(z) + \frac{z}{g(G^{-1}(z))} - \frac{\lambda}{g(G^{-1}(z))} \right) y(z)\, dz + \lambda y(0)(\underline{w} - \underline{c}) \\
&= \int_0^1 \left(C'(z) - \lambda W'(z) \right) y(z)\, dz + \lambda y(0)(\underline{w} - \underline{c}).
\end{aligned}
$$

Integrating by parts then yields

$$
\begin{gathered}
\mathcal{L}(y,\lambda) = \int_0^1 \left(C'(z) - \lambda W'(z) \right) y(z)\, dz + \lambda y(0)(\underline{w} - \underline{c}) \\
= \left(C(1) - \lambda W(1) \right) y(1) - \left(C(0) - \lambda W(0) \right) y(0) + \int_0^1 \left(R(z) - \lambda P(z) \right) \left(-y'(z) \right) dz \\
+ \lambda y(0)(\underline{w} - \underline{c}).
\end{gathered}
$$

Since the optimal mechanism is a two-wage mechanism, we can set $y(0) = 1$. Moreover, since $V(1) < W(1)$ and $\underline{C}(1) = C(1)$, it is without loss of generality to restrict attention to mechanisms such that $y(1) = 0$. Combining these observations with $C(0) = 0$ and $W(0) = \underline{c}$, we have

$$
\mathcal{L}(y,\lambda) = \int_0^1 \left(C(z) - \lambda W(z) \right) \left(-y'(z) \right) dz + \lambda \underline{w}.
$$

The designer's full problem can then be written

$$
\begin{aligned}
&\max_{\lambda \geq 0} \min_{y(\cdot)} \left\{ \int_0^1 \left(C(z) - \lambda W(z) \right) \left(-y'(z) \right) dz + \lambda \underline{w} \right\} \\
&\quad \text{s.t.} \quad \int_0^1 y(z)\, dz = Q, \quad y \text{ non-increasing}.
\end{aligned}
$$

Solving the inner minimization problem then yields

$$
\mathrm{co}\left(C - \lambda W \right)(Q) + \lambda \underline{w},
$$

where $\mathrm{co}(C - \lambda W)$ denotes the convexification of the function $C(z) - \lambda W(z)$ on $z \in [0,1]$. Solving the outer maximization problem, the optimal value λ^* of the Lagrange multiplier is pinned down by the first-order condition

$$
-\frac{d}{d\lambda} \left(\mathrm{co}(C - \lambda W)(Q) \right) \Big|_{\lambda = \lambda^*} = \underline{w}
$$

as required.

Part II: Proof of the stated properties of $\underline{C}_R$

Given a minimum wage $\underline{w}$, if $\underline{w} \in (W(Q_1(m)), W(Q_2(m))$, then m is fixed. So for the remainder of this proof we omit the dependence of $Q_i(m)$ on m and simply write Q_i, for $i = 1, 2$. Before proving the stated properties of $\underline{C}_R$, we first need to complete our characterization of the optimal mechanism when $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$. In such cases, the monopsony solves

$$\min_{q_1 \in [0,Q), q_2 > Q} (1-\beta)C(q_1) + \beta C(q_2),$$

where $\beta = \frac{Q-q_1}{q_2-q_1}$, subject to the constraint $(1-\beta)W(q_1) + \beta W(q_2) \geq \underline{w}$. The corresponding Lagrangian is

$$\mathcal{L}(q_1, q_2, \lambda) = (1-\beta)C(q_1) + \beta C(q_2) - \lambda[(1-\beta)W(q_1) + \beta W(q_2) - \underline{w}],$$

where λ is the Lagrange multiplier associated with the minimum wage constraint. For $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ the constraint will bind (i.e. hold with equality at an optimum) and, consequently, $\lambda > 0$. Using $C_\lambda(Q) := W(Q)(Q - \lambda)$, the Lagrangian can equivalently be written as

$$\mathcal{L}(q_1, q_2, \lambda) = (1-\beta)C_\lambda(q_1) + \beta C_\lambda(q_2) + \lambda \underline{w}.$$

Using

$$\frac{\partial \beta}{\partial q_1} = -\frac{1-\beta}{q_2 - q_1} \quad \text{and} \quad \frac{\partial \beta}{\partial q_2} = -\frac{\beta}{q_2 - q_1},$$

the first-order conditions with respect to q_1 and q_2 given by

$$C'_\lambda(q_1) = \frac{C_\lambda(q_2) - C_\lambda(q_1)}{q_2 - q_1} = C'_\lambda(q_2), \tag{10}$$

while the first-order condition with respect to λ is

$$(1-\beta)W(q_1) + \beta W(q_2) = \underline{w}. \tag{11}$$

Introduce

$$H(q_2, q_1, \lambda) := \frac{C_\lambda(q_2) - C_\lambda(q_1)}{q_2 - q_1} > 0,$$

where the inequality holds because we have $q_2 > q_1$ by assumption and C is a strictly increasing function. Using subscripts to denote partial derivatives, we have

$$H_1(q_2, q_1, \lambda) = \frac{1}{q_2 - q_1} \left[C'_\lambda(q_2) - H(q_2, q_1, \lambda)\right], \; H_2(q_2, q_1, \lambda) = \frac{1}{q_2 - q_1} \left[H(q_2, q_1, \lambda) - C'_\lambda(q_1)\right],$$

$$H_3(q_2, q_1, \lambda) = \frac{W(q_1) - W(q_2)}{q_2 - q_1}.$$

Note that $H_3 < 0$ holds because $q_2 > q_1$ and W is an increasing function. Observe also that (10) is equivalent to

$$C'_\lambda(q_1) = H(q_2, q_1, \lambda) = C'_\lambda(q_2). \tag{12}$$

Denote by $q_1^*(\lambda)$ and $q_2^*(\lambda)$ the values of q_1 and q_2 that satisfy (12). Evaluated at these values, we have

$$H_1(q_2^*(\lambda), q_1^*(\lambda), \lambda) = 0 = H_2(q_2^*(\lambda), q_1^*(\lambda), \lambda).$$

This implies that the second partials of $\mathcal{L}(q_1, q_2, \lambda)$ with respect to q_1 and q_2, evaluated at $q_i = q_i^*$ are

$$\frac{\partial^2 \mathcal{L}(q_1^*, q_2^*, \lambda^*)}{\partial q_1^2} = (1-\beta) C''_\lambda(q_1^*), \quad \frac{\partial^2 \mathcal{L}(q_1^*, q_2^*, \lambda^*)}{\partial q_2^2} = \beta C''_\lambda(q_2^*) \quad \text{and} \quad \frac{\partial^2 \mathcal{L}(q_1^*, q_2^*, \lambda^*)}{\partial q_1 \partial q_2} = 0.$$

The corresponding Hessian matrix is thus

$$\begin{pmatrix} (1-\beta) C''_\lambda(q_1^*) & 0 \\ 0 & \beta C''_\lambda(q_2^*) \end{pmatrix}.$$

This is positive definite if and only if $(1-\beta)C''_\lambda(q_1^*) > 0$ and $\beta C''_\lambda(q_2^*) > 0$. Thus, for each $i \in \{1, 2\}$, at the optimum we have

$$C''_\lambda(q_i^*) > 0.$$

Totally differentiating $C'_\lambda(q_i^*) = H(q_2^*, q_1^*, \lambda)$ with respect to λ and using $H_1(q_2^*, q_1^*, \lambda) = 0 = H_2(q_2^*, q_1^*, \lambda)$ yields

$$\frac{dq_i^*}{d\lambda} = \frac{H_3(q_2^*, q_1^*, \lambda) + W'(q_i^*)}{C''_\lambda(q_i^*)}.$$

Because $C''_\lambda(q_i^*) > 0$, it follows that $\frac{dq_i^*}{d\lambda}$ has the same sign as

$$H_3(q_2^*, q_1^*, \lambda) + W'(q_i^*) = \frac{W(q_1^*) - W(q_2^*)}{q_2^* - q_1^*} + W'(q_i^*).$$

We next show that this expression is positive for $i = 1$ and negative for $i = 2$.

To that end, notice that for $q_1^* < q_2^*$ and $Q \in (q_1^*, q_2^*)$, $C_\lambda(Q)$ is not convex. That is, for all $Q \in (q_1^*, q_2^*)$ we have $\underline{C}_\lambda(Q) < C_\lambda(Q)$. Otherwise, there would be no need to convexify

$C_\lambda(Q)$. We now show that this implies that $W(Q)$ is not convex on $[q_1^*, q_2^*]$ by showing that convexity of W implies convexity of C_λ. In particular, for $a \in [0,1]$ and Q_A and Q_B satisfying $q_1^* \leq Q_A < Q_B \leq q_2^*$, define $Q^a := aQ_A + (1-a)Q_B$. Convexity of W on $[q_1^*, q_2^*]$ means that

$$W(Q^a) \leq aW(Q_A) + (1-a)W(Q_B).$$

Next, from the definition of C_λ, we have $C_\lambda(Q^a) = W(Q^a)(Q^a - \lambda)$. Convexity of W then implies that

$$\begin{aligned}
C_\lambda(Q^a) &\leq \left(aW(Q_A) + (1-a)W(Q_B)\right)\left(aQ_A + (1-a)Q_B - \lambda\right) \\
&= \left(aW(Q_A) + (1-a)W(Q_B)\right)\left(a(Q_A - \lambda) + (1-a)(Q_B - \lambda\right) \\
&= a\left(aW(Q_A) + (1-a)W(Q_B)\right)(Q_A - \lambda) + (1-a)\left(aW(Q_A) + (1-a)W(Q_B)\right)(Q_B - \lambda)) \\
&= aW(Q_A)(Q_A - \lambda) + (1-a)W(Q_B)(Q_B - \lambda) + a(1-a)(W(Q_B) - W(Q_A))(Q_A - Q_B) \\
&= aC_\lambda(Q_A) + (1-a)C_\lambda(Q_B) + a(1-a)(W(Q_B) - W(Q_A))(Q_Q - Q_B) \\
&\leq aC_\lambda(x_0) + (1-a)C_\lambda(x_1).
\end{aligned}$$

Here, the second inequality follows from $W(Q_B) - W(Q_A) > 0$ and $Q_A - Q_B < 0$ (which also implies that the inequality is strict if $a \in (0,1)$). Thus, we have that convexity of W implies convexity of C_λ. However, since we know that C_λ fails to be convex on $[q_1^*, q_2^*]$, we then have that $W(Q)$ is not convex on $[q_1^*, q_2^*]$. That is, for all $Q \in (q_1^*, q_2^*)$,

$$W(Q) > W(q_1^*) + (Q - q_1^*)\frac{W(q_2^*) - W(q_1^*)}{q_2^* - q_1^*}.$$

Finally, because $W(Q)$ intersects with the linear function $W(q_1^*) + (Q - q_1^*)\frac{W(q_2^*)-W(q_1^*)}{q_2^*-q_1^*}$ at $Q = q_2^*$ from above, it follows that the slope of W at that point is smaller than $\frac{W(q_2^*)-W(q_1^*)}{q_2^*-q_1^*}$. Consequently, we have $W'(q_2^*) < \frac{W(q_2^*)-W(q_1^*)}{q_2^*-q_1^*}$, which is equivalent to

$$\frac{W(q_1^*) - W(q_2^*)}{q_2^* - q_1^*} + W'(q_2^*) = H_3(q_2^*, q_1^*, \lambda) + W'(q_2^*) < 0.$$

This implies

$$\frac{dq_2^*(\lambda)}{d\lambda} < 0.$$

By the same token, $W(Q)$ intersects with the linear function $W(q_1^*) + (Q - q_1^*)\frac{W(q_2^*)-W(q_1^*)}{q_2^*-q_1^*}$ at $Q = q_1^*$ from below. This implies that $W(q_1^*) + (q_2^* - q_1^*)W'(q_1^*) > W(q_2^*)$, which is equivalent to

$$\frac{W(q_1^*) - W(q_2^*)}{q_2^* - q_1^*} + W'(q_1^*) = H_3(q_2^*, q_1^*, \lambda) + W'(q_1^*) > 0,$$

implying that

$$\frac{dq_1^*(\lambda)}{d\lambda} > 0.$$

Once we have established the comparative static properties of the solution value $\lambda^*(Q, \underline{w})$ with respect to Q and $\underline{w}$, the comparatives static properties of $q_i^*(Q, \underline{w})$ with respect to these parameters will follow from the definition of $q_i^*(Q, \underline{w})$ via $q_i^*(Q, \underline{w}) = q_i^*(\lambda^*(Q, \underline{w}))$ and the fact that $\frac{\partial q_1^*(\lambda)}{d\lambda} > 0 > \frac{\partial q_2^*(\lambda)}{d\lambda}$. Using (11) and totally differentiating $(1-\beta^*)W(q_1^*) + \beta^* W(q_2^*) = \underline{w}$ with respect to $\underline{w}$, where $\beta^* = \frac{Q - q_1^*}{q_2^* - q_1^*}$ and we have dropped dependence on λ^* for notational ease, yields

$$\left\{(1-\beta^*)\frac{dq_1^*}{d\lambda}\left(W'(q_1^*(\lambda)) + H_3\right) + \beta^* \frac{dq_2^*}{d\lambda}\left(W'(q_2^*(\lambda)) + H_3\right)\right\} \frac{\partial \lambda^*}{\partial \underline{w}} = 1.$$

Thus, $\frac{\partial \lambda^*}{\partial \underline{w}}$ is positive if the term in brackets is positive, which is the case if both summands are positive. To see that the second summand is positive, recall that $\frac{dq_2^*}{d\lambda} < 0$ and $W'(q_2^*(\lambda)) + \frac{W(q_1^*) - W(q_2^*)}{q_2^* - q_1^*} < 0$. To see that the first summand is positive, it suffices to recall that $\frac{dq_1^*}{d\lambda} > 0$ and that $W'(q_1^*) + \frac{W(q_1^*) - W(q_2^*)}{q_2^* - q_1^*} > 0$. Since $\frac{\partial q_i^*(Q, \underline{w})}{\partial \underline{w}} = \frac{dq_i^*(\lambda)}{d\lambda} \frac{\partial \lambda^*(Q, \underline{w})}{\partial \underline{w}}$, it follows that

$$\frac{\partial q_1^*(Q, \underline{w})}{\partial \underline{w}} > 0 > \frac{\partial q_2^*(Q, \underline{w})}{\partial \underline{w}}. \tag{13}$$

Similarly, totally differentiating $(1-\beta^*)W(q_1^*) + \beta^* W(q_2^*) = \underline{w}$ with respect to Q yields

$$\left\{(1-\beta^*)\frac{dq_1^*}{d\lambda}\left(W'(q_1^*(\lambda)) + H_3\right) + \beta^* \frac{dq_2^*}{d\lambda}\left(W'(q_2^*(\lambda)) + H_3\right)\right\} \frac{\partial \lambda^*}{\partial Q} = H_3.$$

Since the right-hand side is negative and the term in brackets on the left-hand side is, as just shown, positive, it follows that $\frac{\partial \lambda^*}{\partial Q} < 0$, implying

$$\frac{\partial q_1^*(Q, \underline{w})}{\partial Q} < 0 < \frac{\partial q_2^*(Q, \underline{w})}{\partial Q}. \tag{14}$$

Note that we also have

$$\frac{\partial \lambda^*}{\partial Q} = H_3 \frac{\partial \lambda^*}{\partial \underline{w}}.$$

We are now ready to prove the stated properties of the function $\underline{C}_R$.

Convexity. Let the minimum wage $\underline{w}$ be given. We start by showing that the function $\underline{C}_R(\cdot, \underline{w})$ is convex. As noted in Theorem 2, this implies that $\underline{C}_R(\cdot, \underline{w})$ is continuous in Q.

Take any two points $Q_A, Q_B \in [0, 1]$. Then we need to show that for any $a \in [0, 1]$ we have $\underline{C}_R(aQ_A + (1-a)Q_B, \underline{w}) \geq a\underline{C}_R(Q_A, \underline{w}) + (1-a)\underline{C}_R(Q_B, \underline{w})$. It suffices to show that there exists an incentive compatible and ex post individually rational procurement mechanism that procures the quantity $aQ_A + (1-a)Q_B$ at a cost of $a\underline{C}_R(Q_A, \underline{w}) + (1-a)\underline{C}_R(Q_B, \underline{w})$ without violating the minimum wage constraint. Let $\boldsymbol{x}_A$ ($\boldsymbol{x}_B$) denote the allocation rule and $\boldsymbol{t}_A$ ($\boldsymbol{t}_B$) denote the payment rule of the incentive compatible and individually rational procurement mechanism that procures the quantity Q_A (Q_B) at the minimal cost $\underline{C}_R(Q_A, \underline{w})$ ($\underline{C}_R(Q_B, \underline{w})$). Now consider the allocation rule $\boldsymbol{x}$ given by $\boldsymbol{x} := \alpha\boldsymbol{x}_A + (1-\alpha)\boldsymbol{x}_B$ and the wage schedule $\boldsymbol{w}$ that implements this allocation at the minimal cost. Since the weighted sum of two increasing function is also an increasing function, the allocation rule $\boldsymbol{x}$ can be implemented using an incentive compatible and ex post individually rational procurement mechanism. By construction, this mechanism procures the quantity $aQ_A + (1-a)Q_B$ at a cost of

$$\begin{aligned}\int_{\underline{c}}^{\overline{c}} \Gamma(c)x(c)\,dG(c) &= a\int_{\underline{c}}^{\overline{c}} \Gamma(c)x_A(c)\,dG(c) + (1-a)\int_{\underline{c}}^{\overline{c}} \Gamma(c)x_B(c)\,dG(c) \\ &= a\underline{C}_R(Q_A, \underline{w}) + (1-a)\underline{C}_R(Q_B, \underline{w})\end{aligned}$$

as required. This last expression also shows that $w(c)x(c) = aw_A(c)x_A(c) + (1-a)w_B(c)x_B(c)$ holds for all $c \in [\underline{c}, \overline{c}]$. It only remains to verify that the mechanism $\langle \boldsymbol{x}, \boldsymbol{w}\rangle$ does not violate the minimum wage constraint. Since the wage schedules $\boldsymbol{w}_A$ and $\boldsymbol{w}_B$ satisfy the minimum wage constraint, for all $c \in [\underline{c}, \overline{c}]$, we have

$$w(c)x(c) = aw_A(c)x_A(c) + (1-a)w_B(c)x_B(c) \geq a\underline{w}x_A(c) + (1-a)\underline{w}x_B(c) = \underline{w}x(c).$$

Thus, the wage schedule $\boldsymbol{w}$ satisfies the minimum wage constraint as required.

Monotonicity. Clearly, $\underline{C}_R(Q, \underline{w})$ is increasing in both Q and $\underline{w}$ on $Q \notin (S(\underline{w}), w_1^{-1}(\underline{w}))$. It remains to show that $\underline{C}_R(Q, \underline{w}) = \mathcal{D}^*(Q, \underline{w})$ is increasing in both Q and $\underline{w}$ on $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$. Since $\underline{C}_R$ is continuous in both Q and $\underline{w}$, this establishes that $\underline{C}_R$ is everywhere increasing in both Q and $\underline{w}$, as required. By construction, we have

$$\mathcal{D}^*(Q, \underline{w}) = (1-\beta^*)C_{\lambda^*}(q_1^*) + \beta^* C_{\lambda^*}(q_2^*) + \lambda^*\underline{w},$$

where $\lambda^* = \lambda^*(Q, \underline{w})$, $q_i^* = q_i^*(Q, \underline{w})$ and $\beta^* = \frac{Q-q_1^*}{q_2^*-q_1^*}$.[49] By the envelope theorem we have

$$\frac{\partial \mathcal{D}^*(Q, \underline{w})}{\partial \underline{w}} = \lambda^* > 0 \quad \text{and} \quad \frac{\partial \mathcal{D}^*(Q, \underline{w})}{\partial Q} = H(q_2^*, q_1^*, \lambda^*) > 0, \tag{15}$$

which establishes the required monotonicity properties.

Marginal cost properties. Recall that the marginal cost function $\underline{C}'_R(\cdot, \underline{w})$ is given by the left derivative of $\underline{C}_R(\cdot, \underline{w})$ with respect to Q. Since $\underline{C}_R(\cdot, \underline{w})$ is convex in Q it is almost everywhere differentiable in Q and admits left and right derivatives on its entire domain. Consequently, $\underline{C}'_R$ is a well-defined. Clearly, $\underline{C}'_R$ is continuous on $(Q, \omega) \in [0, 1] \times [W(0), W(1)]$ with $Q \neq S(\underline{w})$ and $Q \neq w_1^{-1}(\underline{w})$.[50] However, it remains to show that $\underline{C}'_R$ is continuous at $Q = w_1^{-1}(\underline{w})$. To that end, notice that $q_i^*(0) = Q_i$, and satisfying (11) then requires that $\underline{w} = w_1(Q)$. Consequently, $\lambda^*(Q, \underline{w}) \downarrow 0$ and $q_i^*(Q, \underline{w}) \to Q_i$ as $Q \uparrow w_1^{-1}(\underline{w})$. Since the parameters of the optimal two-wage mechanism are continuous at $Q = w_1^{-1}(\underline{w})$, it follows that $\underline{C}'_R$ is too.[51]

To complete this proof, it now only remains to show that, for $Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big)$, $\underline{C}'_R$ is bounded and

$$\frac{\partial \underline{C}'_R(Q, \underline{w})}{\partial Q} > 0 > \frac{\partial \underline{C}'_R(Q, \underline{w})}{\partial \underline{w}}.$$

Starting from (15) and taking the derivative with respect to Q once more yields

$$\begin{aligned}
\frac{\partial \underline{C}'_R(Q, \underline{w})}{\partial \underline{w}} &= \frac{\partial^2 \mathcal{D}^*(Q, \underline{w})}{\partial \underline{w} \partial Q} = \frac{\partial \lambda^*}{\partial Q} = H_3(q_2^*, q_1^*, \lambda^*) \frac{\partial \lambda^*}{\partial \underline{w}} < 0, \\
\frac{\partial \underline{C}'_R(Q, \underline{w})}{\partial Q} &= \frac{\partial^2 \mathcal{D}^*(Q, \underline{w})}{\partial Q^2} = H_3(q_2^*, q_1^*, \lambda^*) \frac{\partial \lambda^*}{\partial Q} > 0,
\end{aligned}$$

where the inequalities follows from $\frac{\partial \lambda^*}{\partial \underline{w}} > 0 > H_3(q_2^*, q_1^*, \lambda^*)$. That $\underline{C}'_R$ is bounded on $Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big)$ follows from the fact that $\underline{C}_R(\cdot, \underline{w})$ is convex in Q so $\underline{C}'_R(\cdot, \underline{w})$ increases in Q on $Q \in [0, 1]$. □

[49] Note that at $Q = w_1^{-1}(\underline{w})$ we have $\lambda^* = 0$ and $\mathcal{D}^*(Q, \underline{w}) = \underline{C}(Q)$ at $\underline{w} = w_1(Q)$. Likewise, at $Q = S(\underline{w})$ we have $\lambda^* = Q$, which implies that $C_{\lambda^*}(Q) = 0$ and $\mathcal{D}^*(Q, \underline{w}) = \underline{w}Q$.

[50] This follows from the continuity of the functions $\underline{w}$ and $\underline{C}'$, as well as the solution value $\mathcal{D}^*(Q, \underline{w})$ to the dual.

[51] Intuitively, the possible failure of continuity on $Q = S(\underline{w})$ is a result of the optimal mechanism changing from a two-wage mechanism to one involving a single wage.

A.4 Proof of Lemma 1

Proof. These comparative statics have already been proven in the course of proving Theorem 2 (see (13) and (14)). $\square$

A.5 Proof of Proposition 4

Proof. As established in the exposition that follows the proposition statement, it only remains to show that wage dispersion and involuntary unemployment are decreasing in $\underline{w}$ in the region covered by Case 1. Moreover, as established in footnote 30, it suffices to show that $q_2^*(Q^*(\underline{w}), \underline{w})$ decreases in this region. In particular, $Q^*(\underline{w})$ satisfies

$$V'(Q^*(\underline{w})) = H(q_2^*, q_1^*, \lambda^*),$$

where $H(q_2^*, q_1^*, \lambda^*)$ is the marginal cost of procurement derived in the proof of Theorem 2. Totally differentiating yields

$$\frac{dQ^*(\underline{w})}{d\underline{w}} = \frac{H_3}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}} \frac{\partial \lambda^*}{\partial \underline{w}} > 0,$$

where the inequality holds because $V' < 0$, $H_3 < 0$ and $\frac{d\lambda^*}{dQ} < 0 < \frac{d\lambda^*}{d\underline{w}}$. Using the definition of $q_2^*(Q, \underline{w}) = q_2^*(\lambda^*(Q, \underline{w}))$ and totally differentiation $q_2^*(\lambda^*(Q^*(\underline{w}, \underline{w}))$ with respect to $\underline{w}$ yields

$$\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} = \frac{\partial q_2^*}{\partial \lambda} \left[\frac{\partial \lambda^*}{\partial Q} \frac{\partial Q^*(\underline{w})}{\partial \underline{w}} + \frac{\partial \lambda^*}{\partial \underline{w}} \right] = \frac{\partial q_2^*}{\partial \lambda} \frac{\partial \lambda^*}{\partial \underline{w}} \left[H_3 \frac{\partial Q^*(\underline{w})}{\partial \underline{w}} + 1 \right].$$

Here, the second equality follows from $\frac{\partial \lambda^*}{\partial Q} = H_3 \frac{\partial \lambda^*}{\partial \underline{w}}$. Substituting

$$\frac{dQ^*(\underline{w})}{d\underline{w}} = \frac{H_3 \frac{\partial \lambda^*}{\partial \underline{w}}}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}}$$

into this last expression yields

$$\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} = \frac{\partial q_2^*}{\partial \lambda} \frac{\partial \lambda^*}{\partial \underline{w}} \left[\frac{(H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}}}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}} + 1 \right] = \frac{\partial q_2^*}{\partial \lambda} \frac{\partial \lambda^*}{\partial \underline{w}} \left[\frac{(H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}} + V' - H_3 \frac{\partial \lambda^*}{\partial Q}}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}} \right].$$

Since $\frac{\partial q_2^*}{\partial \lambda} < 0 < \frac{\partial \lambda^*}{\partial \underline{w}}$, $\frac{dq_2^*(\lambda^*(Q^*(\underline{w}),\underline{w}))}{d\underline{w}} < 0$ holds if the term in brackets is positive. To see that this is the case, we can again substitute $\frac{\partial \lambda^*}{\partial Q} = H_3 \frac{\partial \lambda^*}{\partial \underline{w}}$ to obtain

$$\frac{dq_2^*(\lambda^*(Q^*(\underline{w}),\underline{w}))}{d\underline{w}} = \frac{\partial q_2^*}{\partial \lambda}\frac{\partial \lambda^*}{\partial \underline{w}}\left[\frac{V'}{V' - (H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}}}\right].$$

Since $V' < 0$ and $V' - (H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}} < 0$, we have

$$\frac{dq_2^*(\lambda^*(Q^*(\underline{w}),\underline{w}))}{d\underline{w}} < 0$$

as desired. □

A.6 Proof of Proposition 5

Proof. Firm i's first-order condition is

$$V(y_i) = \frac{Q - y_i}{Q^2}\underline{C}(Q) + \frac{y_i}{Q}\underline{C}'(Q).$$

The left-hand side is decreasing in y_i. The partial derivative of the right-hand side with respect to y_i is $-\frac{1}{Q^2}(\underline{C}(Q) - Q\underline{C}'(Q))$, which is positive because $\underline{C}$ is convex. This implies that for any aggregate quantity Q there is a unique y_i that satisfies the first-order condition. This y_i must thus be the same for all i. Hence, any equilibrium is symmetric. Given this, we can write the first-order condition as

$$V\left(\frac{Q}{n}\right) = \frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q). \tag{16}$$

The left-hand side is decreasing in Q. The derivative of the right-hand side with respect to Q is

$$-\frac{n-1}{nQ^2}\left(\underline{C}(Q) - Q\underline{C}'(Q)\right) + \frac{1}{n}\underline{C}''(Q) \geq 0. \tag{17}$$

Here the inequality follows from the fact that $\underline{C}$ is convex, which in turn implies that $\underline{C}'' \geq 0$ and $Q\underline{C}'(Q) \geq \underline{C}(Q)$. Because at $Q = 0$, the left-hand side is larger than the right-hand side, there is a unique Q that satisfies (16). This proves that the equilibrium is unique and symmetric.

To see that Q_n^* is increasing in n, suppose to the contrary that it is not and we have

$Q^*_n \geq Q^*_{n+1}$ for some n. This implies $\frac{Q^*_n}{n} > \frac{Q^*_{n+1}}{n+1}$ and therefore

$$\begin{aligned} V\left(\frac{Q^*_{n+1}}{n+1}\right) > V\left(\frac{Q^*_n}{n}\right) &= \frac{n-1}{n}\frac{\underline{C}(Q^*_n)}{Q^*_n} + \frac{1}{n}\underline{C}'(Q^*_n) \\ &\geq \frac{n-1}{n}\frac{\underline{C}(Q^*_{n+1})}{Q^*_{n+1}} + \frac{1}{n}\underline{C}'(Q^*_{n+1}) \\ &\geq \frac{n}{n+1}\frac{\underline{C}(Q^*_{n+1})}{Q^*_{n+1}} + \frac{1}{n+1}\underline{C}'(Q^*_{n+1}). \end{aligned}$$

Here, the first inequality is due to (17) and the second follows from the fact that the derivative of $\frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q)$ with respect to n is

$$\frac{1}{n^2 Q}\left[\underline{C}(Q) - Q\underline{C}'(Q)\right] \leq 0,$$

where the inequality holds because $\underline{C}(Q)$ is convex. Since in equilibrium

$$V\left(\frac{Q^*_{n+1}}{n+1}\right) = \frac{n}{n+1}\frac{\underline{C}(Q^*_{n+1})}{Q^*_{n+1}} + \frac{1}{n+1}\underline{C}'(Q^*_{n+1}),$$

we have the desired contradiction.

That $Q^p_n < Q^*_n$ holds for n sufficiently small follows from the discussion after the proposition by choosing $n = 1$ since $h(Q,1) > W(Q)$ for all $Q \in (Q_1(m), Q_2(m))$. Moreover, $Q^p_n \leq Q^*_n$ requires $Q^*_n \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ since otherwise $h(Q,n) = W(Q) + \frac{Q}{n}W'(Q)$, which implies $Q^*_n < Q^p_n$. The arguments after the proposition imply that $h(Q,n) < W(Q)$ for some $Q \in (Q_1(m), Q_2(m))$ can only occur if n is sufficiently large.

Assume now that $\underline{C}(Q^e) = C(Q^e)$ and let $Q_\infty := \lim_{n\to\infty} Q^*_n$. Taking limits of both sides of (16) yields

$$V(0) = \frac{\underline{C}(Q_\infty)}{Q_\infty}. \tag{18}$$

The definition of Q^e then implies that $V(0) = \frac{\underline{C}(Q_\infty)}{Q_\infty} = W(Q^e) = \frac{\underline{C}(Q^e)}{Q^e}$. Using

$$\frac{d}{dQ}\left(\frac{\underline{C}(Q)}{Q}\right) = \frac{Q\underline{C}'(Q) - \underline{C}(Q)}{Q^2} \geq 0,$$

where the inequality holds because $\underline{C}$ is convex, we have that the solution to the equation $V(0) = \frac{\underline{C}(Q_\infty)}{Q_\infty}$ is unique. Since Q^e satisfies this equation we thus have $Q_\infty = Q^e$. Hence, if $Q^e \notin \cup_{m\in\mathcal{M}}(Q_1(m)), Q_2(m))$ then Q^e is also the aggregate quantity in the limit as claimed.

Assume now that $Q^e \in (Q_1(m_e)), Q_2(m_e))$ for some $m_e \in \mathcal{M}$. For $Q \in (Q_1(m_e)), Q_2(m_e))$, $\underline{C}(Q)$ increases linearly from $C(Q_1(m_e))$ to $C(Q_2(m_e))$ with a slope that is greater than $V(0)$.

The latter follows from our observation that $\underline{C}'(Q^e) > V(0)$. Because W is increasing we have

$$\frac{C(Q_1(m_e))}{Q_1(m_e)} = W(Q_1(m_e)) < W(Q^e) = V(0) < W(Q_2(m_e)) = \frac{C(Q_2(m_e))}{Q_2(m_e)}.$$

This implies there exists a unique number $\tilde{Q} \in (Q_1(m_e)), Q_2(m_e))$ such that $\frac{\underline{C}(\tilde{Q})}{\tilde{Q}} = V(0)$. If $Q^e \in (Q_1(m_e)), Q_2(m_e))$ this is then the aggregate quantity in the limit as claimed.

We are left to show that $\tilde{Q} > Q^e$ holds whenever $Q^e \in (Q_1(m_e), Q_2(m_e))$. To see that this holds, rearrange (18) to

$$Q_\infty V(0) = \underline{C}(Q_\infty)$$

and recall that $Q^e V(0) = C(Q^e)$. Since $C(Q^e) > \underline{C}(Q^e)$, $\tilde{Q} = Q_\infty > Q^e$ follows. □

A.7 Proof of Proposition 6

Proof. Note first that $\underline{C}'_R(Q, \underline{w})$ is continuous at $\underline{w} = w_1(Q)$ because discontinuities in $\underline{C}'_R(Q, \underline{w})$ only occur at $\underline{w} = W(Q)$. The equilibrium condition is thus

$$V\left(\frac{Q_n^*(\underline{w})}{n}\right) = h(Q_n^*(\underline{w}), n, \underline{w}) = \frac{n-1}{n} \frac{\underline{C}_R(Q_n^*(\underline{w}), \underline{w})}{Q_n^*(\underline{w})} + \frac{1}{n}\underline{C}'_R(Q_n^*(\underline{w}), \underline{w}).$$

Totally differentiating with respect to $\underline{w}$, dropping arguments and writing $\underline{C}'_R$ and $\underline{C}''_R$ in lieu of $\frac{\partial \underline{C}_R}{\partial Q}$ and $\frac{\partial^2 \underline{C}_R}{\partial Q^2}$ yields

$$\left[V' - (n-1)\left[\frac{Q_n^* \underline{C}'_R - \underline{C}_R}{(Q_n^*)^2}\right] - \underline{C}''_R\right] \frac{dQ_n^*}{d\underline{w}} = (n-1)\frac{\partial \underline{C}_R}{\partial \underline{w}} \frac{1}{Q_n^*} + \frac{\partial \underline{C}'_R}{\partial \underline{w}}.$$

Since the term in brackets on the left-hand side is negative, $\frac{dQ_n^*}{d\underline{w}}$ has the opposite sign of $(n-1)\frac{\partial \underline{C}_R}{\partial \underline{w}} \frac{1}{Q_n^*} + \frac{\partial \underline{C}'_R}{\partial \underline{w}}$. From the proof of Theorem 2, we know that $\frac{\partial \underline{C}_R}{\partial \underline{w}} = \lambda^* \geq 0$ and $\frac{\partial \underline{C}'_R}{\partial \underline{w}} = \frac{\partial \lambda^*}{\partial Q} \leq 0$, where λ^* is the solution value of the Lagrange multiplier associated with the minimum wage constraint. At $\underline{w} = w_1(Q)$, we have $\lambda^* = 0$ and $\frac{\partial \lambda^*}{\partial Q} < 0$. We therefore have $\frac{dQ_n^*}{d\underline{w}}|_{\underline{w}=w_1(Q_n^*)} > 0$ as required. □

A.8 Proof of Theorem 4

Proof. As noted, the minimum wage only binds if $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$. Fixing Q, define

$$h_\gamma(Q, n) := \lim_{\underline{w} \uparrow W(Q)} h(Q, n, \underline{w}).$$

Since $\underline{C}_R(Q, \underline{w})$ is continuous, it satisfies $\underline{C}_R(Q, W(Q)) = W(Q)Q$ and $\lim_{\underline{w}\uparrow W(Q)} \frac{\underline{C}_R(Q,\underline{w})}{Q} = W(Q)$. From the monopsony model, we know that $\lim_{\underline{w}\uparrow W(Q)} \underline{C}'_R(Q, \underline{w}) = \gamma(Q)$, which is continuous in Q. We thus obtain $h_\gamma(Q, n)$ as given in (8). Since $h_\gamma(Q, n)$ is continuous, V is continuously decreasing and $Q^*_n \in (Q_1(m), Q_2(m))$, there exist smallest and largest values of Q such that

$$V(Q/n) = h_\gamma(Q, n).$$

We denote these values of Q by $\hat{Q}_{L,n}$ and $\hat{Q}_{H,n}$, respectively. Since V is decreasing and $h_\gamma(Q, n) < C'(Q_2(m))$ holds for $Q \in (Q_1(m), Q_2(m))$, we have

$$Q^*_n < \hat{Q}_{L,n} \quad \text{and} \quad \hat{Q}_{H,n} \leq Q_2(m).$$

Moreover, since $h_\gamma(Q, n) > W(Q)$ holds unless $\underline{C}'_R(Q, \underline{w})$ is continuous at $\underline{w} = W(Q)$, we have

$$\hat{Q}_{H,n} \leq Q^p_n. \tag{19}$$

This last inequality is strict unless $h_\gamma(\hat{Q}_{H,n}, n) = W(\hat{Q}_{H,n})$. Since $h_\gamma(Q, n)$ converges to $W(Q)$ as $n \to \infty$, provided $Q^e \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, we have $\lim_{n\to\infty} \hat{Q}_{H,n} = \lim_{n\to\infty} Q^p_n = Q^e$.

It follows that for $\underline{w} \leq W(\hat{Q}_{L,n})$, the equilibrium given the minimum wage $\underline{w}$ involves wage dispersion and involuntary unemployment. Moreover, for $\underline{w} \in [W(\hat{Q}_{H,n}), W(Q_2(m)]$, there is no wage dispersion in equilibrium. Minimum wages $\underline{w} \in [W(\hat{Q}_{H,n}), W(Q^p_n)]$ correspond to the *pure Robinson-Stigler oligopsony* region, where increases in $\underline{w}$ increase equilibrium employment without inducing involuntary unemployment. □

B Supplementary material

B.1 Piecewise linear specification

For the purposes of illustration in Sections 1 to 4.1, we consider piecewise linear specifications of W in which, for $a > b > 0$ and $\underline{q} \in (0, 1)$, W is given by

$$W(Q) = \begin{cases} aQ, & Q \in \left[0, \underline{q}\right) \\ bQ + (a - b)\underline{q}, & Q \in \left[\underline{q}, 1\right], \end{cases} \tag{20}$$

which gives rise to $C(Q) = aQ^2$ for $Q \in \left[0, \underline{q}\right)$ and $C(Q) = bQ^2 + Q(a-b)\underline{q}$ for $Q \in \left[\underline{q}, 1\right]$. A specific numerical example that arises by setting $a = 4$, $b = 1/2$ and $\underline{q} = 1/4$ is the following piecewise linear input supply function and its corresponding cost function

$$W(Q) = \begin{cases} 4Q, & Q \in [0, 1/4) \\ Q/2 + 7/8, & Q \in [1/4, 1] \end{cases} \quad \text{and} \quad C(Q) = \begin{cases} 4Q^2, & Q \in [0, 1/4) \\ Q^2/2 + 7Q/8, & Q \in [1/4, 1] \end{cases}. \quad (21)$$

We use this as our leading example throughout Sections 1 to 4.1. Straightforward computations show that

$$Q_1 = \frac{4 + \sqrt{2}}{32} \approx 0.169 \quad \text{and} \quad Q_2 = \frac{1 + 2\sqrt{2}}{8} \approx 0.478.$$

Property of the cost given a minimum wage We now state a lemma that provides a property that holds in general for the low-wage function under a two-wage mechanism parameterized by q_1 and q_2, given a minimum wage $\underline{w}$. For the piecewise linear specification, the lemma implies that the slope of the low-wage function does not vary with $\underline{w}$.

Lemma B.1. *For any $Q \in (Q_1(m), Q_2(m))$ and $\underline{w} \in [w_1(Q), W(Q))$, the optimal values of q_1 and q_2 are such that*

$$\left(\frac{W(q_2) - W(q_1)}{q_2 - q_1} \right)^2 = W'(q_1)W'(q_2). \quad (22)$$

Proof. The optimal mechanism given $\underline{w}$ being a two-wage mechanism implies that q_1 and q_2 are such that

$$W(q_1) + (Q - q_1)\frac{W(q_2) - W(q_1)}{q_2 - q_1} = \underline{w}, \quad (23)$$

which is to say that q_1 and q_2 are such that the linear function

$$w_1(x, Q, \underline{w}) = W(q_1) + (x - q_1)\frac{W(q_2) - W(q_1)}{q_2 - q_1},$$

which is the low wage, is equal to $\underline{w}$ at $x = Q$. The cost of procurement as a function q_1 and q_2, denoted $K(q_1, q_2)$, is

$$\begin{aligned} K(q_1, q_2) &= q_1 w_1(Q, Q, \underline{w}) + (Q - q_1)W(q_2) \\ &= Q\underline{w} + (Q - q_1)(q_2 - Q)\frac{W(q_2) - W(q_1)}{q_2 - q_1}, \end{aligned} \quad (24)$$

where the second equality follows by using (23) and the fact that by construction $w_1(Q, Q, \underline{w}) = \underline{w}$.

To simplify notation in what follows, we use the short-hand notation $B = \frac{W(q_2)-W(q_1)}{q_2-q_1}$, $W_i' = W'(q_i)$ for $i = 1, 2$ and $\beta = \frac{Q-q_1}{q_2-q_1}$, bearing in mind that W_1' is *not* the derivative of $w_1(Q)$ nor that of $w_1(x, Q, \underline{w})$ nor directly related to these functions in any other way. The objective is thus to minimize $K(q_1, q_2)$ over q_1 and q_2 subject to the constraint (23). We arbitrarily choose q_2 as the control variable and let q_1 be an implicit function of q_2 given by (23). Totally differentiating (23) yields

$$\frac{dq_1}{dq_2} = -\frac{\beta}{1-\beta}\frac{W_2' - B}{W_1' - B}. \tag{25}$$

Partially differentiating B with respect to q_1 and q_2 gives

$$\frac{\partial B}{\partial q_1} = \frac{B - W_1'}{q_2 - q_1} \quad \text{and} \quad \frac{\partial B}{\partial q_2} = \frac{W_2' - B}{q_2 - q_1}.$$

This implies

$$(Q - q_1)\frac{\partial B}{\partial q_1} = \beta(B - W_1')$$

and

$$(q_2 - Q)\frac{\partial B}{\partial q_2} = (1-\beta)(W_2' - B) \quad \text{and} \quad (Q - q_1)\frac{\partial B}{\partial q_2} = \beta(W_2' - B),$$

which will prove useful below.

Letting $k(q_2) = K(q_1(q_2), q_2)$, we have

$$k'(q_2) = (Q - q_1)[\beta B + (1-\beta)W_2'] - \frac{dq_1}{dq_2}(q_2 - Q)[(1-\beta)B + \beta W_1'],$$

which at an optimum is 0. Substituting in $\frac{dq_1}{dq_2}$ from (25), using the fact that $\frac{Q-q_1}{q_2-Q} = \frac{\beta}{1-\beta}$ and somewhat tedious algebra reveals that $k'(q_2) = 0$ is equivalent to

$$B^2 = W_1'W_2',$$

which is what was to be shown. □

For the piecewise linear specification (see (20)), Lemma B.1 implies that for any $\underline{w} \in [w_1(Q), W(Q))$, $\frac{W(q_2)-W(q_1)}{q_2-q_1} = \frac{W(Q_2)-W(Q_1)}{Q_2-Q_1}$ holds because $W'(q_1)$ does not vary with $q_1 < \underline{q}$ and $W'(q_2)$ does not vary with q_2 for $q_2 > \underline{q}$. For example, for the specification in (21), we have $W'(q_1) = 4$ and $W'(q_2) = 1/2$ and hence $\frac{W(q_2)-W(q_1)}{q_2-q_1} = \sqrt{2}$. Observe also that $B^2 = W'(q_1)W'(q_2)$ with $B = \frac{W(q_2)-W(q_1)}{q_2-q_1}$ holds for $q_i = Q_i$ with $i = 1, 2$. To see this, rearrange the first-order conditions $C'(Q_i) = \frac{C(Q_2)-C(Q_1)}{Q_2-Q_1}$ to obtain, with $B = \frac{W(Q_2)-W(Q_1)}{Q_2-Q_1}$,

$$Q_1 W'(Q_1) = BQ_2 \quad \text{and} \quad Q_2 W'(Q_2) = BQ_1.$$

This implies $W'(Q_1)W'(Q_2) = B^2$.

For the piecewise linear specification, (22) implies q_1 and q_2 are such that $\frac{W(q_2)-W(q_1)}{q_2-q_1} = \frac{W(Q_2(m))-W(Q_1(m))}{Q_2(m)-Q_1(m)}$. For $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ and $x \in (S(\underline{w}), w_1^{-1}(\underline{w}))$, the low-wage function $w_1(x, Q, \underline{w}) = W(q_1)+(x-q_1)\frac{W(q_2)-W(q_1)}{q_2-q_1}$ is therefore a parallel shift of the low-wage function

$$w_1(x) = W(Q_1(m)) + (x - Q_1(m))\frac{W(Q_2(m)) - W(Q_1(m))}{Q_2(m) - Q_1(m)}$$

satisfying $w_1(q_1, Q, \underline{w}) = W(q_1)$, $w_1(Q, Q, \underline{w}) = \underline{w}$ and $w_1(q_2, Q, \underline{w}) = W(Q_2(m))$ whose derivative with respect to x is $\frac{W(Q_2(m))-W(Q_1(m))}{Q_2(m)-Q_1(m)}$.

The identify $W(q_1) + (Q - q_1)\frac{W(Q_2(m))-W(Q_1(m))}{Q_2(m)-Q_1(m)} = \underline{w}$ also makes it easy to see that, as stated in Lemma 1, q_1 increases in $\underline{w}$, for a fixed Q, since the left-hand side increase in q_1. (Intuitively, as the minimum wage increase, more units are procured at the minimum wage and fewer at the high wage.) For the same reason, as Q increases, keeping $\underline{w}$ fixed, q_1 decreases.

B.2 Quantity competition equilibrium

In Figures 15 and 16 the left-hand panels are plotted using $V(y_i) = 1.1 - 8y_i$ and the right-hand panels are plotted using $V(y_i) = 1.2 - 8y_i$. This implies that for the left-hand panels we have $Q^e = 0.45 \in (Q_1, Q_2) = (0.169, 0.478)$ and $\tilde{Q} = 0.4516$, while for the right-hand panels we have $Q^e = 0.65 > Q_2$.

B.3 Efficiency wages, migration and unemployment

Efficiency wage theory is customarily associated with the so-called Five-Dollar Day introduced by the Ford Motor Company in 1914.[52] A pervasive feature of that wage increase was that it caused workers to migrate to Detroit (see, for example, Sward, 1948, p.53). As we now show, when workers face a fixed cost of moving or participating in the labor market, this gives rise to a procurement cost function that is non-convex and consequently may make the use of an efficiency wage and involuntary unemployment optimal.

[52] Contrary to perceived wisdom, a wage of five dollars per day was not uniformly applied across all workers from the time of its introduction in 1914. See, for example, Sward (1948) who notes that according to the company's financial statement 30% of the overall workforce were paid less than that in 1916.

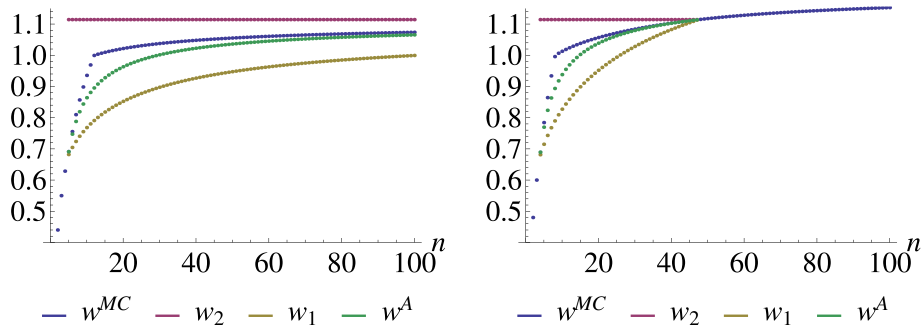

Figure 15: Equilibrium wages as a function on n, where w_1 denotes the lower equilibrium wage, w_2 denotes the higher equilibrium wage, $w^{MC} = W(Q_n^*)$ denotes the market-clearing wage and w^A the average wage $w^A = (w_1 + w_2)/2$. On the left, $W(Q^e) = 1.1 < 1.114 = w_2$ and on the right $W(Q^e) = 1.2 > w_2$.

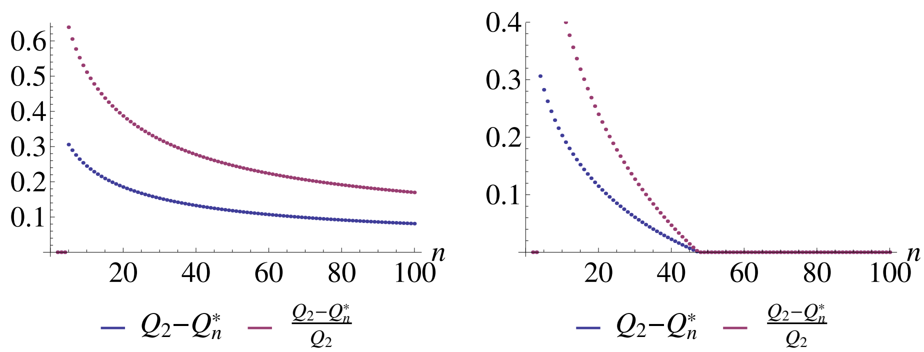

Figure 16: Involuntary unemployment and the unemployment rate as a function on n. On the left, there is involuntary unemployment of size $Q_2 - \tilde{Q} = 0.0269$ and an unemployment rate of 5.6% as $n \to \infty$.

Specifically, consider a model with a monopsony firm that operates in a market in which the inverse labor supply function is W_A. We assume that this function is increasing and differential. For ease of exposition, we also assume that it is convex. This implies that absent any migration, the cost $QW_A(Q)$ of procuring Q units of labor is convex in Q, which in turn implies that without migration the firm optimally sets a market-clearing wage. To model migration, we assume that there is another pool of workers whose opportunity costs of working after migrating are described by the inverse supply function W_B, which we also assume to be convex, differentiable and increasing. Each worker in this pool has the same fixed cost $k > 0$ of moving. For $i \in \{A, B\}$, let $S_i(w) = W_i^{-1}(w)$ and, for $w > W_B(0) + k$, let $S_{AB}(w) = S_A(w) + S_B(w - k)$ denote the supply function that the firm faces. Moreover, for $Q > S_A(W_B(0) + k) =: \check{Q}$, we let $W_{AB}(Q) = S_{AB}^{-1}(Q)$. Then the inverse labor supply function the firm faces is $W(Q) = W_A(Q)$ for $Q \leq \check{Q}$ and $W(Q) = W_{AB}(Q)$ for $Q > \check{Q}$, yielding the cost of procurement function $C(Q) = W(Q)Q$ that accounts for migration.[53] The key implication of this is that $C(Q)$ is not convex. As shown below, we have

$$\lim_{Q \uparrow \check{Q}} C'(Q) > \lim_{Q \downarrow \check{Q}} C'(Q), \tag{26}$$

and the marginal cost of procurement decreases at $\check{Q}$. Geographical migration is only one possible interpretation of problems involving fixed costs. One can also think of workers moving from one industry to another or as workers joining the labor force at some fixed cost.

This perspective resonates with the prevalent view that migration is a cause of unemployment in the region to which workers migrate. However, here involuntary unemployment occurs not because of frictions such as costly search or costly wage adjustment, but rather as a consequence of optimal pricing on the part of the firm. It also offers a novel interpretation of the episode at the Ford Motor Company in the mid 1910s. According to this interpretation, with high enough wages, workers were willing to bear the fixed cost of moving, making the cost of procurement non-convex in the short run and efficiency wages optimal: "the greatest cost cutting measure" according to the dictum often attributed to Henry Ford. As the demand for its cars and its demand for labor continued increasing, eventually it became optimal to set market-clearing wages again. More broadly, the model with fixed costs of migration or labor market participation and an optimal mechanism used by the firm offers a framework in which economic expansion may be a cause of involuntary unemployment.

To see that (26) holds, let $\breve{w} = W_B(0) + k$ (which is the same as $W_A(\check{Q})$). We then have

$$\lim_{Q \uparrow \check{Q}} C'(Q) = W_A(\check{Q}) + \check{Q}(S_A')^{-1}(\check{Q}) > W_{AB}(\check{Q}) + \check{Q}(S_{AB}')^{-1}(\check{Q}) = \lim_{Q \downarrow \check{Q}} C'(Q).$$

Here, the inequality holds because $W_A(\check{Q}) = W_{AB}(\check{Q}) = \breve{w}$ and, for $w \geq \breve{w}$, $S_{AB}(w) = S_A(w) + S_B(w - k)$. This implies that $S_{AB}'(w) = S_A'(w) + S_B'(w - k) > S_A'(w)$, which in turn implies that $(S_{AB}')^{-1}(\check{Q}) = \frac{1}{S_{AB}'(\breve{w})} < \frac{1}{S_A'(\hat{w})} = (S_A')^{-1}(\hat{Q})$. Consequently, the function C is not convex as required.

[53]For example, for $W_A(Q) = 4Q$, $W_B(Q) = \frac{4}{7}Q + \frac{1}{2}$ and $k = 1/2$, we obtain the specification in (21). To see this, note that $W_A(Q) = 4Q$ and $W_B(Q) = 4Q/7 + 1/2$ imply $S_A(w) = w/4$ and $S_B(w) = 7(w - 1/2)/4$ and hence using $k = 1/2$ for $w \geq \hat{w}$ we have $S_{AB}(w) = S_A(w) + S_B(w - k) = 2w - 7/4$. Inverting S_{AB} yields $W_{AB}(Q) = Q/2 + 7/8$, which is the second line in (21). It remains to verify that $\hat{Q} = 1/4$, which is the case since $S_A(W_B(0) + k) = (1/2 + 1/2)/4 = 1/4$.

B.4 Figure parameterizations

This appendix provides the parameterizations that generated each of the figures in the body of the paper. Figures 1 to 11 and Figure 13 are drawn for the piecewise linear specification from Appendix B.1 with $a = 4$, $b = 1/2$ and $\underline{q} = 1/4$. Under this parameterization, $Q_1 \approx 0.169$ and $Q_2 \approx 0.478$.

Figure 1 All panels use $V(Q) = 1.76 - Q$. Panel (a) assumes the monopsony is a price-taker, Panel (b) restricts the monopsony to using a uniform wage and in Panel (c) the monopsony uses the optimal procurement mechanism.

Figure 2 Panel (c) uses $V(Q) = 2.5 - 4Q$.

Figure 3 Panel (a) uses $V(Q) = 2.5 - 4Q$ and Panel (b) uses $V(Q) = 1.7 - Q$.

Figure 4 This figure is a schematic illustration.

Figure 5 Both panels use $V(Q) = 1.76 - Q$.

Figure 6 Panels (a) and (b) use $\underline{w} = 0.9$ and Panel (c) uses $\underline{w} = 0.9$ and $\underline{w} = 0.95$.

Figure 7 Both panels use $\underline{w} = 0.95$ and Panel (b) uses $\tilde{Q} = 0.28$.

Figure 8 Panel (a) uses $\underline{w} = 0.9$ and $\underline{w} = 0.95$ and a variety of linear V functions, each with gradient -2. Panel (b) uses $\underline{w} = 0.9$ and $\underline{w} = 0.95$ and $V(Q) = 2.895 - 8Q$. Panel (c) uses $\underline{W} = 1.05$ and $\underline{w} = 1.06$ and $V(Q) = 1.855 - 2Q$.

Figure 9 Panels (a) and (b) use $\underline{w} = 0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 1, 1.05, 1.1$ and Panel (c) uses $V(Q) = 2.5 - 4Q$.

Figure 10 Both panels use $V(Q) = 1.3 - Q/2$.

Figure 11 The left-hand panel uses $n = 3$, $n = 5$ and $n = 15$ and the right-uses $n = 15$ and $V(Q/n) = 1.2 - 14Q/n$

Figure 12 Both panels use $V(y_i) = 1 - y_i$ and $W(Q) = Q$, and the right-hand panel assumes $\underline{w} = 0.55$.

Figure 13 This figures uses $V(Q) = 1.0908 - Q/5$ and $n = 5$ with $\underline{w} = 0.9, 0.95, 1$.

Figure 14 Figure uses $V(Q_\ell) = v - Q_\ell$ with $v = 7/8$.