Optimal labor procurement under minimum wages and monopsony power*

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This version: August 1, 2023 First version: November 11, 2021

Abstract

We derive the optimal procurement mechanism for a monopsony under a minimum wage constraint. For a setting where a continuum of workers have private information about their opportunity cost of working, we show that under cost-minimizing procurement at most two wages are paid. Cost-minimizing procurement involves two wages and involuntary unemployment if and only if the procurement cost under optimal uniform wage-setting lies above its convexification at the optimal employment level. Setting the minimum wage equal to the highest wage offered under laissez-faire increases total employment and workers' pay, and decreases (and possibly eliminates) involuntary unemployment. Binding minimum wages can make a two-wage mechanism and involuntary unemployment optimal even if a uniform wage is optimal under laissez-faire. If a minimum wage does not induce involuntary unemployment or induces both involuntary unemployment and wage dispersion, then a marginal increase in the minimum wage generically increases employment and decreases involuntary unemployment.

Keywords: Market power, efficiency wage, price regulation, quantity competition, involuntary unemployment, wage dispersion
JEL-Classification: C72, D47, D82

*We thank Mohammad Akbarpour, Simon Anderson, Ivan Balbuzanov, David Byrne, David Card, Mick Coelli, Ryo Kambayashi, Zi Yang Kang, Paul Milgrom, Roland Rathelot, Christoph Schlom, Mitchell Watt, Frank Yang and seminar audiences at Berkeley, Yale, MIT/Harvard, Brown, the CEPR Virtual IO Seminar, the 9th Annual ASU Economic Theory Conference, the Australasian Local Economic Theory Seminar (ALETS), Korea University, Singapore Management University, the University of Melbourne, the University of Virginia, Universitat Autònoma de Barcelona, Peking University, UCLA, Hitotsubashi University, NYU Stern, Warwick, Stanford, the Melbourne IO workshop, the Frontiers of Economic Design (FED) seminar, the 2022 SAET conference at ANU, the 2022 Women in Economic Theory Conference (WiETCON) at the University of Chicago, the 2023 ASSA meetings in New Orleans, the 2023 WEAI conference in Melbourne, the 2023 IAB Conference on Imperfect Competition in the Labor Market in Nuremberg and the 2023 Hal White Antitrust Conference in D.C. for useful discussions and feedback. Zhichong Lu, Anand Bharadwaj and Hasini Senadheera provided excellent research assistance. Financial support through a Visiting Research Scholar grant from the Faculty of Business and Economics at the University of Melbourne is also gratefully acknowledged.

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1 Introduction

Minimum wage legislation has been around for over a century and continues to play a prominent role in policy debates.[1] However, the optimal labor procurement mechanism of a firm in the presence of a minimum wage is not yet known. In this paper, we fill this gap. Specifically, for a monopsony model with a continuum of workers of equal productivity who are privately informed about their opportunity costs of working and a commonly known labor supply function, we derive the optimal incentive compatible and individually rational procurement mechanism given a minimum wage constraint.

We show the following. First, with and without a binding minimum wage, at most two wages are paid in equilibrium. Second, under laissez-faire the firm may optimally induce involuntary unemployment by hiring some workers at a wage that is above the market-clearing wage. If this is the case, then a minimum wage equal to the highest wage paid in equilibrium reduces—and possibly eliminates—involuntary unemployment while increasing total employment. Third, a minimum wage can make a two-wage mechanism and involuntary unemployment optimal even if a market-clearing wage is always optimal under laissez-faire (which is the case if, for example, the firm's marginal revenue product of labor is constant). With a binding minimum wage, it is also possible that all the high-wage workers are paid more than their marginal revenue product. Notwithstanding the richness of these effects, the analysis offers clear predictions about the effects of marginally increasing a prevailing minimum wage based on whether or not there is wage dispersion and/or involuntary unemployment. A marginal increase in the minimum wage: increases employment and decreases unemployment if there is both involuntary unemployment and wage dispersion; increases employment if there is no involuntary unemployment (and no wage dispersion); and decreases employment and increases unemployment if there is involuntary unemployment and no wage dispersion. Thus, our analysis shows that a regulator observing involuntary unemployment can identify whether it is caused by market power or by a minimum wage on the basis of whether or not there is also wage dispersion (which indicates that the firm is engaging in a form of wage discrimination).

Solving the monopsony's labor procurement problem in the presence of a minimum wage constraint requires overcoming several challenges. First, under a minimum wage constraint the nature of implementation matters. Two natural ways to implement a given two-wage mechanism are *ex post implementation* and *ex ante implementation*. Under ex post implementation the monopsony simultaneously offers all workers both a low wage and a high

[1] While precursors to minimum wage legislation date back to the Hammurabi Code (c. 1755–1750 BC), New Zealand became the first country to implement a minimum wage in 1894, followed by the Australian state of Victoria in 1896, and the United Kingdom in 1909 (Starr, 1981).

wage. It randomly rations employment for the set of workers that select the high wage, so that in equilibrium low-cost workers select the low wage and high cost workers select the high wage. Under ex ante implementation the monopsony makes each worker a random take-it-or-leave-it offer of either a low wage or a high wage. Workers with a low cost always accept the offer whereas high cost workers only accept the high wage offer. As we show, the lowest wage paid to workers is always higher under ex post implementation than under ex ante implementation and more generally, among all possible implementations, the minimum wage constraint is least tight under ex post implementation. Thus, focusing on ex post implementation is without loss of generality. Second, as minimum wages constrain the transfer rule—whereas mechanism design problems are typically represented so that the object of choice is the allocation rule—we need to determine how they indirectly constrain the allocation rule. While in principle there is a continuum of constraints, the monotonicity of the allocation rule that follows from incentive compatibility implies that it suffices to impose the minimum wage constraint on the lowest cost type. The designer's constrained optimization problem then becomes one in which the integrand of the Lagrangian needs to be ironed, implying that both with and without a binding minimum wage constraint the optimal mechanism consists of at most two wages. What permits us to do the comparative statics summarized above—despite the countervailing effects that arise—is the fact that the optimal mechanism can always be described by at most three parameters.

In extensions, formally analyzed in an online appendix, we allow for quantity competition among firms and for horizontal differentiation of workers and jobs. Our model of quantity competition is related to the literature on Cournot competition (Cournot, 1838) and heeds David Card's call to move "beyond the 'no strategic interactions' case" (Card, 2022b). It assumes that the aggregate quantity is optimally procured at the minimum cost rather than at a uniform wage. We show that total employment and involuntary unemployment can both increase as the number of firms increases and that there is no intrinsic relationship between the intensity of competition and the level of involuntary unemployment. The main insights from the monopsony model with regard to the minimum wage effects carry over to the model with quantity competition. In particular, an appropriately chosen minimum wage still eliminates involuntary unemployment. With horizontally differentiated workers and jobs, optimal procurement may involve deliberate and inefficient mismatches between workers and jobs, in addition to involuntary unemployment.[2] In this setting an appropriately chosen minimum wage not only increases employment and eliminates involuntary unemployment but also eliminates any of these inefficient mismatches.

This paper relates to the literature on monopoly and monopsony pricing, including Mussa

[2]This analysis builds on Balestrieri et al. (2021) and Loertscher and Muir (2023).

and Rosen (1978), Bulow and Roberts (1989) and Loertscher and Muir (2022), by studying a setting without aggregate uncertainty and without imposing the regularity assumption of Myerson (1981).[3] Among these papers, it is closest to Mussa and Rosen (1978) and Loertscher and Muir (2022) in that it employs mechanism design techniques to solve the firm's problem. The procurement setting studied here builds on Lee and Saez (2012). Absent a binding minimum wage constraint, it is the monopsony version of the baseline monopoly pricing problem studied in Loertscher and Muir (2022) since the aforementioned paper does not allow for price regulation.[4] Since it deals with minimum wages and labor procurement, this paper also relates to the labor literature in which there is growing recognition that employers exert market power; see, for example, Card (2022a,b). In particular, our minimum wage analysis generalizes that of Robinson (1933), which restricts the firm to setting a uniform wage.[5]

While we do not derive optimally regulated prices, our methodology may prove useful for analyses of optimal regulation in the tradition of Ramsey (1927) and Boiteux (1956) without restricting the firm's contracting space; see also Wilson (1993). In light of the recent upsurge of interest in regulation, particularly in the context of "big tech" (see, for example, Australian Competition and Consumer Commission, 2019; Crémer et al., 2019; Furman et al., 2019; Stigler Center, 2019), there is potentially a wide range of applications for this methodology.

The idea that involuntary unemployment is beneficial for businesses and detrimental for workers is a popular one that dates at least as far as Friedrich Engels' and Karl Marx' notion of a *reserve army of labor*.[6] Our paper provides an optimal pricing theory of involuntary unemployment. By construction, the firm cannot do better by using more elaborate contracts, which is a problem that has plagued earlier theories of efficiency wages and involuntary unemployment.[7]

[3] There has been a recent upsurge of interest in mechanism design without regularity; see, for example, Condorelli (2012), Dworczak, Kominers, and Akbarpour (2021), Kleiner et al. (2021) and Akbarpour, Dworczak, and Kominers (2022). However, none of these papers consider pricing constraints.

[4] Like Mussa and Rosen (1978), Loertscher and Muir (2022) allow for vertically differentiated products. They analyze resale that arises due to randomization and study optimal selling mechanisms with resale.

[5] See also Stigler (1946). As shown by Bhaskar, Manning, and To (2002), this basic logic also extends to imperfectly competitive markets. There is empirical evidence consistent with these effects, the classic reference being Card and Krueger (1994). More recently, Wiltshire (2021) provides an analysis of the labor market effects of Walmart supercenters and the effects of minimum wages in the presence of monopsony power, as well as a comprehensive overview of this strand of literature.

[6] See Engels (1845) and Marx (1867).

[7] The collection of essays in Akerlof and Yellen (1986) provides an overview of the early literature that formalized the idea that firms deliberately offer wages exceeding the market-clearing level so that the resulting excess supply of labor (and corresponding level of involuntary unemployment) can be used to discipline their workforce. Krueger and Summers (1988) provide empirical evidence on industry wage structure. As Yellen (1984, p. 202) put it: "All these models suffer from a similar theoretical difficulty—that employment

The remainder of this paper is structured as follows. Section 2 introduces the baseline procurement setup. In Section 3, we derive the optimal procurement mechanism in the presence of a minimum wage. Section 4 derives the corresponding comparative statics and provides policy guidance. Section 5 discusses modelling assumptions, robustness and extensions. The paper concludes with Section 6. An online appendix extends the model to quantity competition and horizontal differentiation.

2 Setup

We consider the procurement problem of a monopsony firm whose *marginal revenue product of labor function* V is such that $V(Q)$ is the firm's willingness to pay for $Q \in [0,1]$ units of labor input. We assume that V is continuous and (weakly) decreasing (and so is differentiable almost everywhere).[8] For the special case where V is constant over $[0,1]$, we denote its constant value by v. For any intervals where V happens to be constant and the firm's demand for workers is not uniquely pinned down, we break ties by assuming that it employs the maximum number of workers so that its demand function given the wage $w \in [V(1), V(0)]$ is $D(w) = \max\{Q \in [0,1] : V(Q) \geq w\}$. This implies that if V is strictly decreasing, then $D(w) = V^{-1}(w)$. If $V(Q) = v$ for all $Q \in [0,1]$, then $D(w) = 1$ for $w \leq v$ and $D(w) = 0$ otherwise.

Let W denote the *inverse supply function* faced by the monopsony. We assume that W is continuous and strictly increasing, so it faces an upward sloping *labor supply schedule* $S := W^{-1}$. For expositional convenience we also assume W is continuously differentiable. The *cost function*

$$C(Q) := W(Q)Q$$

is then a strictly increasing and continuously differentiable function that specifies the cost of procuring $Q \in [0,1]$ units at the market-clearing wage $W(Q)$. We assume $V(0) > W(0)$ and $V(1) < W(1)$ so that under optimal procurement there are strictly positive masses of both employed and unemployed workers.

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contracts more ingenious than the simple wage schemes considered, can reduce or eliminate involuntary unemployment."

[8]If the firm uses Q units of input to generate a downstream profit of $\Pi(Q)$, where Π is concave and differentiable with $\Pi'(0) > 0$, then the firm's willingness to pay for the Q-th unit of input is given by $V(Q) = \Pi'(Q)$. For example, the firm could be a monopoly in the downstream market with access to a production technology that transforms one unit of labor into one unit of output. As we allow for the possibility that V is constant on some subintervals of $[0,1]$, this encompasses the possibility that there may be some regions where marginal revenue is constant due to ironing (see, for example, Loertscher and Muir, 2022). Alternatively, the firm could be a downstream price-taker using a production technology with a constant or diminishing marginal product of labor.

Following Lee and Saez (2012), we microfound the inverse supply schedule W by assuming that the monopsony faces a continuum of equally productive workers of mass 1, each of whom supplies a single unit of labor inelastically. Each worker has a *private* opportunity cost $c \in [\underline{c}, \overline{c}] := [W(0), W(1)]$ of supplying labor whose cumulative distribution function we denote by G with density $g > 0$ on $[\underline{c}, \overline{c}]$. Note that $G = S$ and $G^{-1}(Q) = W(Q)$ represents the opportunity cost of working for the worker with the Q-th lowest cost. We assume workers are risk-neutral with quasi-linear utility. The interim expected payoff of a worker with cost c that is hired by the firm at wage $w \in \mathbb{R}_{\geq 0}$ with probability $x \in [0, 1]$ is therefore $x(w - c)$.

Mechanisms A direct *procurement mechanism* $\langle x, w \rangle$ consists of an *allocation rule* $x : [\underline{c}, \overline{c}] \to [0, 1]$ that maps worker reports to their probability of employment and a *wage schedule* $w : [\underline{c}, \overline{c}] \to \mathbb{R}_{\geq 0}$ that maps worker reports to a wage that is paid conditional on employment. For all $c, \hat{c} \in [\underline{c}, \overline{c}]$, *incentive compatibility* then requires that

$$x(c)(w(c) - c) \geq x(\hat{c})(w(\hat{c}) - c). \tag{IC}$$

Assuming that each worker receives the same payoff from not working and normalizing this payoff to 0, *individual rationality* requires that, for all $c \in [\underline{c}, \overline{c}]$,

$$w(c) - c \geq 0. \tag{IR}$$

By the revelation principle, the focus on direct mechanisms is without loss of generality. Furthermore, since there is no aggregate uncertainty, it is also without loss of generality to restrict attention to direct mechanisms that determine the employment probability and wage of a given worker independently of the reports of the other workers. Notice that we are restricting attention to direct mechanisms such that $w(c)$ is the deterministic wage that a worker of type c is paid *conditional* on being employed by the firm. We refer to this as *ex post implementation*. As we will see, beyond resonating with real world practices, this representation is also without loss of generality and allows us to conveniently incorporate a minimum wage constraint $\underline{w}$ in the mechanism design problem by requiring that, for all $c \in [\underline{c}, \overline{c}]$, $w(c) \geq \underline{w}$.

We say that a mechanism $\langle x, w \rangle$ such that Q workers are hired in equilibrium involves an *efficiency wage* w with $w > W(Q)$ if the set $\{c \in [\underline{c}, \overline{c}] : w(c) > W(Q)\}$ has positive measure. Such a mechanism necessarily induces *involuntary unemployment* since there is a positive mass of workers willing to supply labor at an efficiency wage that are nevertheless unemployed.[9]

[9]Defined this way, involuntary unemployment is observable, given the appropriate data; see, for example,

Our setting makes two departures from an otherwise standard monopsony pricing problem. First, we do not restrict the monopsony's procurement mechanism above and beyond requiring it to respect the workers' incentive compatibility and individual rationality constraints. In particular, and in contrast to the existing literature, we do not require the firm to set a market-clearing or uniform wage. Second, we do not assume that C is convex.

Cost-minimizing procurement The *convexification* of the cost function C, denoted $\underline{C}$, plays an important role in the firm's cost-minimizing procurement problem. The convexification of C is the largest convex function that is everywhere less than C. It is characterized by a countable set $\mathcal{M}$ and a set of disjoint open intervals $\{(Q_1(m), Q_2(m))\}_{m\in\mathcal{M}}$ such that if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then

$$\underline{C}(Q) = C(Q_1(m)) + \frac{(Q - Q_1(m))(C(Q_2(m)) - C(Q_1(m)))}{Q_2(m) - Q_1(m)}$$

and, otherwise, $\underline{C}(Q) = C(Q)$. Equivalently, for all Q such that $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, introducing $\alpha(Q) := \frac{Q-Q_1(m)}{Q_2(m)-Q_1(m)}$ allows us to write $\underline{C}(Q)$ as the convex combination

$$\underline{C}(Q) = (1 - \alpha(Q))C(Q_1(m)) + \alpha(Q)C(Q_2(m)).$$

If $|\mathcal{M}| > 1$, then without loss of generality we can index the intervals $(Q_1(m), Q_2(m))$ in increasing order so that, for all $m > 1$, we have $Q_2(m-1) < Q_1(m)$. Note that since C is strictly increasing and differentiable, so too is $\underline{C}$. Moreover, since W is strictly increasing, we have $Q_1(1) > 0$.[10] For ease of exposition, we also assume that $Q_2(|\mathcal{M}|) < 1$. For each $m \in \mathcal{M}$, $Q_1(m)$ and $Q_2(m)$ then satisfy the first-order conditions

$$C'(Q_1(m)) = C'(Q_2(m)) = \frac{C(Q_2(m)) - C(Q_1(m))}{Q_2(m) - Q_1(m)}. \tag{1}$$

The importance of $\underline{C}(Q)$ is that it is the smallest cost of procuring the quantity Q under any mechanism satisfying (IC) and (IR). We now explicitly shows that $\underline{C}(Q)$ can be achieved subject to these constraints because the argument is instructive and it allows us to introduce the general notion of a *two-wage mechanism*. That the firm cannot do better follows immediately from Myerson (1981) and the convexification procedure considered here is of course equivalent to Myersonian ironing.

Breza, Kaur, and Shamdasani (2021).

[10]To see this, assume to the contrary that $Q_1(1) = 0$. Using $C'(0) = W(0)$ and $C(Q_2(1)) = Q_2(1)W(Q_2(1))$, the first equality in the first-order condition (1) becomes $W(0) = W(Q_2(1))$, which contradicts the assumption that W is strictly increasing.

First, notice that if $\underline{C}(Q) = C(Q)$, then the result immediately follows since the cost of hiring Q workers at the market-clearing wage is precisely $\underline{C}(Q)$. So assume $\underline{C}(Q) < C(Q)$. For any quantities q_1 and q_2 satisfying $q_1 < Q < q_2$, we can then choose wages w_1 and w_2 such that in equilibrium q_1 workers are hired at the wage w_1, while $q_2 - q_1$ workers are randomly rationed into $Q - q_1$ openings at the efficiency wage w_2. The individual rationality constraints of the marginal workers that participate in the mechanism imply that $w_2 = W(q_2)$. Moreover, incentive compatibility implies that workers with cost $W(q_1)$ must be indifferent between working at the low wage w_1 for sure and being hired at the wage w_2 with probability $\beta(Q, q_1, q_2) := \frac{Q-q_1}{q_2-q_1}$.[11] That is, $w_1 - W(q_1) = \beta(Q, q_1, q_2)(W(q_2) - W(q_1))$ must hold, which is equivalent to

$$w_1 = \beta(Q, q_1, q_2)W(q_2) + (1 - \beta(Q, q_1, q_2))W(q_1).$$

The cost of procuring Q using such a two-wage mechanism is then

$$q_1 w_1 + (Q - q_1)w_2 = \beta(Q, q_1, q_2)C(q_2) + (1 - \beta(Q, q_1, q_2))C(q_1).$$

Thus, this *two-wage mechanism* parameterized by the quantities (Q, q_1, q_2) can lower the procurement cost relative to setting a market-clearing wage if and only if $\underline{C}(Q) < C(Q)$. Moreover, if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then choosing $q_1 = Q_1(m)$ and $q_2 = Q_2(m)$ is optimal, and we have $\alpha(Q) = \beta(Q, Q_1(m), Q_2(m))$.

Summarizing, if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then the minimal cost of procurement is achieved by a two-wage mechanism that induces excess labor supply at the efficiency wage $w_2 = W(Q_2(m))$ and involuntary unemployment of size $Q_2(m) - Q$. The optimal level of employment under the laissez-faire equilibrium, denoted Q^ℓ, then satisfies $V(Q^\ell) = \underline{C}'(Q^\ell)$. If this intersection is not unique we take Q^ℓ to be the largest Q satisfying $V(Q) = \underline{C}'(Q)$. We also let $w_1(Q)$ denote the low wage paid under the optimal mechanism for procuring the quantity Q under the laissez-faire equilibrium. We have established that if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then

$$w_1(Q) = (1 - \alpha(Q))W(Q_1(m)) + \alpha(Q)W(Q_2(m))$$

and, otherwise, $w_1(Q) = W(Q)$.[12] Since w_1 is an increasing and continuous function its

[11]Note that rationing is random and independent of workers' types. The setup satisfies a single-crossing condition which ensures that all workers whose opportunity cost of supplying labor is less than $W(Q_1(m))$ prefer working with certainty at the low wage w_1 and all workers whose opportunity cost is greater than $W(Q_1(m))$ prefer entering the lottery to working at the low wage w_1.

[12]Note that $w_1(0) = W(0)$ and $w_1(1) = W(1)$.

inverse w_1^{-1} is well-defined.

Figure 1 provides an illustration in which the supply side is characterized by the piecewise linear specification

$$W(Q) = \begin{cases} 4Q & Q \in [0, 1/4)\,, \\ Q/2 + 7/8 & Q \in [1/4, 1]\,. \end{cases} \tag{2}$$

This implies $C(Q) = 4Q^2$ for $Q \in [0, 1/4)$ and $C(Q) = Q^2/2 + 7Q/8$ otherwise and yields a single ironing interval with $Q_1 = (4 + \sqrt{2})/32 \approx 0.169$ $Q_2 = (1 + 2\sqrt{2})/8 \approx 0.478$.[13] Note that this is a special case of a more general piecewise linear specification of W that is provided in Appendix OC.

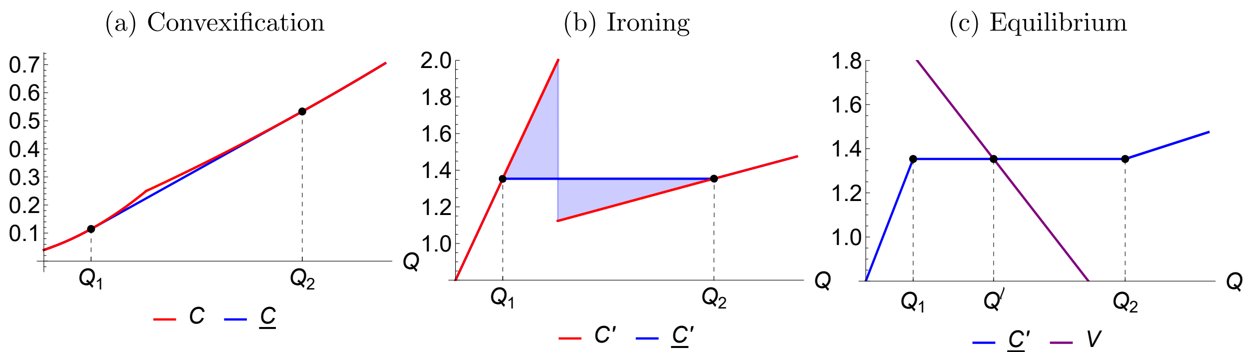

Figure 1: Panel (a) illustrates the convexification $\underline{C}$ of C for the piecewise linear specification of W given in (2). Panel (b) illustrates the corresponding marginal cost function C' and ironed marginal cost function $\underline{C}'$. Panel (c) illustrates an example with $V(Q) = 2.5 - Q$ and $Q^\ell \in (Q_1, Q_2)$.

Ex post and ex ante implementation As we have just seen, when $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, the procurement cost of $\underline{C}(Q)$ can be achieved by posting two wages and allowing workers to self-select into low- and high-wage openings, before rationing the excess supply of high-wage workers. As previously noted, since the firm rations workers after they have self-selected into their desired opening, we refer to this as ex post implementation. There may be multiple Nash equilibria when hiring at each wage occurs simultaneously. However, there is also a dynamic implementation that induces a unique dominant strategy equilibrium: the monopsony first hires $Q_1(m)$ workers at the low wage and before then rationing to fill the remaining $Q - Q_1(m)$ vacancies at the high wage.

The ex post rationing that occurs at the high wage can be achieved in numerous ways. For example, low-wage workers can be thought of as permanent employees and high-wage

[13] Although this specification violates our assumption that W is continuously differentiable, this is of no consequence because the only point at which W fails to be differentiable is within the ironing range (Q_1, Q_2).

workers can be thought of as casual staff, where $\alpha(Q)$ is the probability of being hired on a given day, or the fraction of time a casual worker is employed. Alternatively, workers may be randomly selected at the high wage if hiring occurs on a first-come-first-serve basis and the order of arrival is independent of workers' costs; if hiring occurs based on observable worker characteristics that are not correlated with their opportunity costs of working (and are in that sense irrelevant); or if hiring occurs literally via a lottery as was the case in the so-called "shape-up" that was commonly used for hiring dock workers.[14]

The monopsony can also implement any incentive compatible allocation rule using a decentralized dominant strategy implementation in which it makes a randomized take-it-or-leave-it offer to each worker. We refer to this as *ex ante implementation* because it is such that the monopsony randomizes before workers choose between two deterministic options. Suppose that $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Then under the ex ante implementation of the optimal allocation rule $\alpha(Q)$ workers are offered a wage of $W(Q_2(m))$ and $1 - \alpha(Q)$ workers are offered a wage of $W(Q_1(m))$. The dominant strategy of workers with costs below $W(Q_1(m))$ is to accept both offers and as a result these workers are paid an average wage of $(1 - \alpha(Q))W(Q_1(m)) + \alpha(Q)W(Q_2(m)) = w_1(Q)$. Workers with costs between $W(Q_1(m))$ and $W(Q_2(m))$ only accept high wage offers and $\alpha(Q)$ is therefore the proportion of these workers hired in equilibrium. All other workers reject both offers. Consequently, this implementation replicates the outcome under ex post implementation, where the monopsony instead offers each worker a menu of two wages.

3 Optimal procurement under a minimum wage

In this section we derive the optimal procurement mechanism in the presence of a minimum wage constraint. The exposition, which provides an overview of the analysis, skips many technical details. The full details of each of the proofs can be found in the appendix.

3.1 Preliminaries

Our task in this section is to characterize the minimum cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity $Q \in [0, 1]$ when the monopsony faces a minimum wage of $\underline{w} \in [W(0), \min\{V(0), W(1)\})$, as well as the associated optimal mechanism. If we momentarily restrict the monopsony to *uniform pricing*, the cost $C_U(Q, \underline{w})$ of procuring the quantity Q under a minimum wage $\underline{w}$

[14]This mechanism came to the general public's awareness through the movie "On the Waterfront" and the series of newspaper articles it was based on (see Johnson and Schulberg, 2005).

is simply given by $C_U(Q, \underline{w}) = \underline{w}Q$ if $Q \le S(\underline{w})$ and $C_U(Q, \underline{w}) = C(Q)$ otherwise.[15] Interestingly, the cost of procuring the quantity Q under *optimal* procurement with a minimum wage of $\underline{w}$ is *not* simply given by evaluating the convexification of $C_U(\cdot, \underline{w})$ at the quantity Q. When the monopsony faces a minimum wage of $\underline{w}$ and is restricted to procuring workers at a uniform wage, it cannot separate any workers with costs $c \le \underline{w}$. This property is preserved under convexification. However, if we allow the monopsony to offer workers a menu of wages, then it can still screen over workers with costs below $\underline{w}$ by coupling each wage with an appropriately chosen employment lottery. So the difficulty in solving the problem at hand is to determine how the designer *optimally* screens workers with $c \le \underline{w}$ when the minimum wage constraint makes it relatively more costly for the firm to separate these types.

Note that when we have a minimum wage constraint, the choice of implementation matters. In particular, the minimum wage constraint is *tighter* under ex ante implementation than it is under ex post implementation.[16] The monopsony minimizes the lowest wage it pays in equilibrium by paying every type of worker a deterministic wage conditional on securing employment. However, this is precisely what occurs under ex post implementation. Thus, it is without loss of generality to restrict attention to the class of direct mechanisms introduced in Section 2. Interestingly, if the monopsony were restricted to ex ante implementation, then the minimal cost of procuring Q units under a minimum wage of $\underline{w}$ *would* be given by evaluating the convexification of $C_U(\cdot, \underline{w})$ at the quantity Q.[17] This is so because—just like under uniform wage-setting—the monopsony cannot screen over workers with $c \le \underline{w}$ when it is restricted to ex ante implementation with a minimum wage of $\underline{w}$. In contrast, ex post implementation allows the monopsony to do strictly better.[18]

[15] Let $C'_U(Q, \underline{w}) := \lim_{\epsilon \uparrow 0} \frac{C_U(Q+\epsilon, \underline{w}) - C_U(Q, \underline{w})}{\epsilon}$ denote the corresponding marginal cost schedule and notice that $C'(Q) > W(Q)$ holds for all $Q \in (0, 1]$. Consequently, provided $\underline{w} \in (W(0), W(1))$, the function $C'_U(\cdot, \underline{w})$ is discontinuous at $Q = S(\underline{w})$.

[16] To illustrate this, reconsider the general two-wage mechanisms introduced in Section 2. Let $w_1(Q, q_1, q_2) = (1 - \beta(Q, q_1, q_2))W(q_2) + \beta(Q, q_1, q_2)W(q_1)$ denote the low wage. Under ex post implementation, imposing a minimum wage of $\underline{w}$ leads to the constraint $w_1(Q, q_1, q_2) \ge \underline{w}$. Under ex ante implementation (where the monopsony offers $\beta(Q, q_1, q_2)$ workers a wage of $W(q_2)$ and $1 - \beta(Q, q_1, q_2)$ workers a wage of $W(q_1)$), the minimum wage constraint becomes $W(q_1) \ge \underline{w}$. Since $w_1(Q, q_1, q_2) > W(q_1)$, the minimum wage constraint is tighter under ex ante implementation.

[17] Let $\underline{C_U}(Q, \underline{w})$ denote the convexification of $C_U(\cdot, \underline{w})$ evaluated at the quantity Q and $\underline{C'_U}(Q, \underline{w}) := \lim_{\epsilon \uparrow 0} \frac{\underline{C_U}(Q+\epsilon, \underline{w}) - \underline{C_U}(Q, \underline{w})}{\epsilon}$ denote the corresponding marginal cost schedule. Panel (a) in Figure 2 provides an example of such a cost schedule.

[18] Panel (a) in Figure 2 highlights the difference between optimal procurement under a minimum wage when the monopsony is restricted to ex ante implementation and optimal procurement under a minimum wage when the monopsony employs ex post implementation.

3.2 The mechanism design problem

Formally, given a minimum wage $\underline{w} \in [W(0), W(1))$, the firm's problem is

$$\underline{C}_R(Q, \underline{w}) = \min_{x,w} \int_{\underline{c}}^{\overline{c}} w(c)x(c)\, dG(c), \tag{3}$$
$$\text{s.t.} \quad \text{(IC)}, \quad \text{(IR)}, \quad w(c) \geq \underline{w}\ \forall c \in [\underline{c}, \overline{c}], \quad \int_{\underline{c}}^{\overline{c}} x(c)\, dG(c) = Q.$$

Introducing the virtual cost function $\Gamma(c) = c + \frac{G(c)}{g(c)}$ and combining standard mechanism design arguments with the fact that it suffices to impose the minimum wage constraint on the lowest type $c = \underline{c}$ (see proof of Theorem 1), we can rewrite this problem as

$$\underline{C}_R(Q, \underline{w}) = \min_{x} \int_{\underline{c}}^{\overline{c}} \Gamma(c)x(c)\, dG(c), \tag{4}$$
$$\text{s.t.} \quad x \text{ is non-increasing}, \quad x(\underline{c})\underline{c} + \int_{\underline{c}}^{\overline{c}} x(c)\, dc \geq \underline{w}x(\underline{c}), \quad \int_{\underline{c}}^{\overline{c}} x(c)\, dG(c) = Q.$$

Letting $\lambda \geq 0$ denote the Lagrange multiplier associated with the minimum wage constraint, the proof of Theorem 1 also shows that from this point forward it is without loss of generality to focus on the dual problem:

$$\underline{C}_R(Q, \underline{w}) = \max_{\lambda \geq 0} \min_{x} \int_{\underline{c}}^{\overline{c}} \left(\Gamma(c) - \frac{\lambda}{g(c)} \right) x(c)\, dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}), \tag{5}$$
$$\text{s.t.} \quad x \text{ is non-increasing}, \quad \int_{\underline{c}}^{\overline{c}} x(c)\, dG(c) = Q.$$

Introducing the probability measure $G_\lambda(c) := \frac{\lambda}{1+\lambda}\mathbf{1}(\underline{c} = c) + \frac{1}{1+\lambda}G(c)$, we can rewrite the Lagrangian as

$$(1 + \lambda) \int_{\underline{c}}^{\overline{c}} \left[\left(\Gamma(c) - \frac{\lambda}{g(c)} \right) \mathbf{1}(c > \underline{c}) + (\underline{w} - \underline{c})\, \mathbf{1}\,(c = \underline{c}) \right] x(c)\, dG_\lambda(c).$$

Solving the dual problem then requires that we iron the function

$$\psi_\lambda(c) := \left(\Gamma(c) - \frac{\lambda}{g(c)} \right) \mathbf{1}(c > \underline{c}) + (\underline{w} - \underline{c})\, \mathbf{1}\,(c = \underline{c})$$

with respect to the probability measure G_λ. We denote this ironed function by $\underline{\psi}_\lambda$. Imposing the quantity constraint and pointwise minimizing the ironed objective function then gives us an optimal allocation rule of $x^*(c) = \mathbf{1}(c \le G^{-1}(Q))$. This finally yields

$$\underline{C}_R(Q, \underline{w}) = \max_{\lambda \ge 0} \left\{ (1+\lambda) \int_{\underline{c}}^{\overline{c}} \underline{\psi}_\lambda(c) \mathbf{1}(c \le G^{-1}(Q))\, dG_\lambda(c) \right\}. \tag{6}$$

3.3 Optimal procurement mechanisms

Having solved the firm's cost-minimization problem, we are now in a position to characterize the corresponding optimal mechanisms. This in turn will allow us to rewrite (6) in a more illuminating fashion.

Since the solution to the firm's cost-minimization problem can be represented in terms of an ironing procedure, this immediately implies that the optimal selling mechanisms always involve setting at most two prices. Moreover, examining (6) we see that if $\lambda > 0$ and the minimum wage constraint binds, it is possible that $\underline{\psi}_\lambda(\underline{c}) > \lim_{c \downarrow \underline{c}} \underline{\psi}_\lambda(c)$. Consequently, $\underline{\psi}_\lambda$ may exhibit an ironing interval at the origin (i.e. an ironing interval of the form (Q_1, Q_2), where $Q_1 = 0$ and $Q_2 > 0$). This then leaves us with only three possibilities for the optimal mechanism: setting a market-clearing wage (corresponding to quantities outside an ironing interval), rationing workers at the minimum wage (corresponding to quantities within an ironing interval at the origin) or using a two-wage mechanism as introduced in Section 2 (corresponding to quantities within ironing intervals away from the origin).

Given a minimum wage $\underline{w}$, we next characterize the nature of the optimal selling mechanisms as a function of Q. If $Q \ge w_1^{-1}(\underline{w})$, then the minimum wage constraint does not bind and we have $\lambda = 0$. So we focus on quantities $Q < w_1^{-1}(\underline{w})$ such that the minimum wage constraint binds and $\lambda > 0$. To that end, notice that $S(\underline{w}) \le w_1^{-1}(\underline{w})$ always holds, while $S(\underline{w}) < w_1^{-1}(\underline{w})$ holds if and only if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$. So we have two subcases to consider: $Q \le S(\underline{w})$ and $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$, where the second subcase is non-empty if and only if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$. First, suppose that $Q \le S(\underline{w})$. In this case we must have $\underline{C}_R(Q, \underline{w}) = \underline{w}Q$ because this cost cannot be reduced by randomizing over wages that are at least as high as $\underline{w}$. Consequently, the optimal selling mechanism involves rationing workers at the minimum wage if $Q < S(\underline{w})$ and employing Q workers at the market-clearing wage if $Q = S(\underline{w})$. Second, suppose that $Q > S(\underline{w})$. In this case it is not feasible to hire Q workers by rationing at the minimum wage. Consequently, if $S(\underline{w}) < w_1^{-1}(\underline{w})$, then the optimal mechanism is a two-wage mechanism for any $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$. Since these mechanisms do not randomize over the lowest-cost workers,[19] they can be computed using a convexification procedure that simpli-

[19]This is the same as the feature commonly known as "no randomization at the top" that arises in optimal selling mechanisms. Here, it means no randomization at the bottom (of the type space).

fies (6). Specifically, let $\underline{\Psi}$ denote the convexification the function $\Psi(\cdot,\lambda) := C(\cdot) - \lambda W(\cdot)$ with respect to its first argument. Then rewriting (6) with respect to the uniform measure and integrating by parts yields $\underline{C}_R(Q,\underline{w}) = \underline{\Psi}(Q,\lambda^*) + \lambda^* \underline{w}$, where λ^* is pinned down by the first-order condition $-\left.\frac{d\underline{\Psi}(Q,\lambda)}{d\lambda}\right|_{\lambda=\lambda^*} = \underline{w}$. Intuitively, we end up with an objective function of the form $C(\cdot) - \lambda W(\cdot)$ because, fixing an arbitrary mechanism, one can compute the procurement cost by taking an appropriate convex combination of the function C, and taking the corresponding convex combination of the function W yields the lowest wage paid under that mechanism.[20]

Putting all of this together, we have

$$\underline{C}_R(Q,\underline{w}) = \begin{cases} \underline{w}Q, & Q \in [0, S(\underline{w})] \\ \mathcal{D}^*(Q,\underline{w}), & Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w})\right) \\ \underline{C}(Q), & Q \geq w_1^{-1}(\underline{w}), \end{cases} \tag{7}$$

where $\mathcal{D}^*(Q,\underline{w})$ is the value of the dual problem

$$\mathcal{D}^*(Q,\underline{w}) := \max_{\lambda\geq 0} \min_{q_1\in[0,Q],\, q_2\geq Q} \{(1-\beta(Q,q_1,q_2))\Psi(q_1,\lambda) + \beta(Q,q_1,q_2)\Psi(q_2,\lambda) + \lambda\underline{w}\} \tag{8}$$

with $\beta(Q,q_1,q_2) = \frac{Q-q_1}{q_2-q_1}$.

The following theorem formally summarizes the mechanism design analysis and establishes a number of useful properties of $\underline{C}_R$, as well as the corresponding marginal cost function $\underline{C}'_R(Q,\underline{w}) := \lim_{\epsilon\uparrow 0} \frac{\underline{C}_R(Q+\epsilon,\underline{w}) - \underline{C}_R(Q,\underline{w})}{\epsilon}$, which is the *left derivative* of $\underline{C}_R$ with respect to Q.

Theorem 1. *The minimal cost $\underline{C}_R(Q,\underline{w})$ of procuring the quantity $Q \in [0,1]$ under the minimum wage $\underline{w} \in [W(0), W(1)]$ is given by (7) and can always be achieved by a procurement mechanism involving no more than two wages. This function is convex (and hence continuous) in Q and increasing in both Q and $\underline{w}$. The marginal cost function $\underline{C}'_R$ is well-defined and continuous on $(Q,\underline{w}) \in [0,1] \times [W(0), W(1)]$ with $Q \neq S(\underline{w})$. Moreover, for $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w})\right)$, $\underline{C}'_R$ is bounded and $\frac{\partial \underline{C}'_R(Q,\underline{w})}{\partial Q} > 0 > \frac{\partial \underline{C}'_R(Q,\underline{w})}{\partial \underline{w}}$. Although $\underline{C}_R(Q,\underline{w}) = \underline{C}_U(Q,\underline{w})$ does not hold in general, cost-minimizing procurement involves two wages and involuntary unemployment if and only if $\underline{C}_U(Q,\underline{w}) < C_U(Q,\underline{w})$.*

[20] Adopting the approach of much of the prior literature and applying a Carathéodory-like theorem directly to (4) would not allow us to rule out the possibility that mechanisms involving up to three wages may be optimal (see, for example, Kang (2023) and references therein). Our dual approach shows that two-wage mechanisms remain optimal even with a minimum wage constraint because the objective function and the constraint are of the same functional form. The second-best mechanisms studied by Myerson and Satterthwaite (1983) similarly exhibit this feature as well because there both social surplus and revenue have the same functional form.

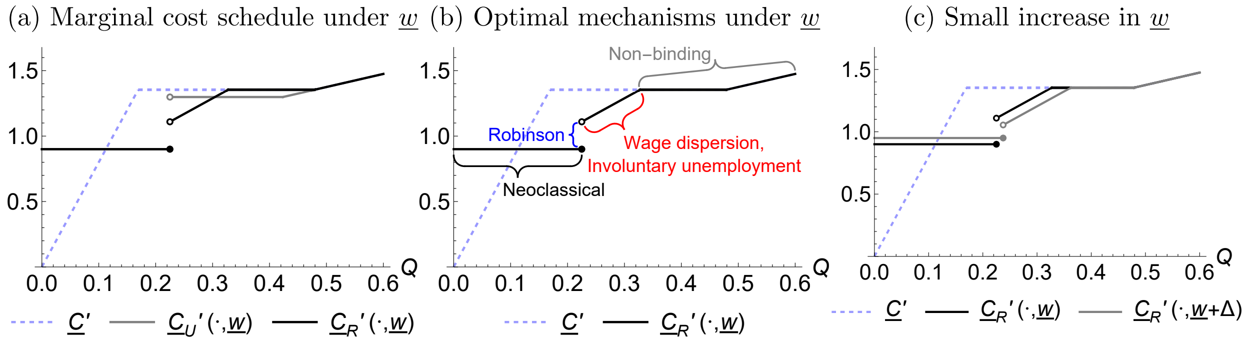

Figure 2: This figure uses the specification of W given in (2) with $\underline{w} = 0.5$ and $\Delta = 0.05$. Panel (a) illustrates $\underline{C}'_R(\cdot,\underline{w})$, $\underline{C}'_U(\cdot,\underline{w})$ (the corresponding marginal cost schedule when the monopsony is restricted to ex ante implementation; see Footnote 17) and $\underline{C}'$ for $\underline{w} = 0.9$. Panel (b) relates different segments of $\underline{C}'_R(\cdot,\underline{w})$ to the optimal mechanisms associated with it. Panel (c) illustrates how $\underline{C}'_R(\cdot,\underline{w})$ shifts in response to a small increase in $\underline{w}$ to $\underline{w} = 0.95$.

Panel (a) in Figure 2 provides a representative illustration of the marginal cost function $\underline{C}'_R(\cdot,\underline{w})$ for a given minimum wage $\underline{w}$. Panel (b) relates the different segments of $\underline{C}'_R(\cdot,\underline{w})$ to the corresponding optimal mechanisms. The first horizontal segment corresponds to what we call *neoclassical mechanisms*, which procure $Q < S(\underline{w})$ at the uniform wage $\underline{w}$ and ration any excess supply. The discontinuity at $Q = S(\underline{w})$ corresponds to a *Robinson mechanism*, whereby the quantity $S(\underline{w})$ is procured at the uniform wage $\underline{w}$ without any rationing. The increasing segment of $\underline{C}'_R(\cdot,\underline{w})$ on $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ is achieved using a two-wage mechanism that is constrained by the minimum wage, implying that the low wage is equal to $\underline{w}$. For quantities beyond this segment, the minimum wage is not binding and $\underline{C}'_R(Q,\underline{w}) = \underline{C}'(Q)$. Panel (c) illustrates the implications of a marginal increase in $\underline{w}$ on $\underline{C}'_R(\cdot,\underline{w})$. For $Q \geq w_1^{-1}(\underline{w})$ the minimum wage constraint does not bind and $\underline{C}'_R(\cdot,\underline{w})$ simply coincides with $\underline{C}'$. A marginal increase in $\underline{w}$ decreases (in a set inclusion sense) the set of Q values such that this case applies. On the interval $Q \in [0, S(\underline{w})]$, where the optimal mechanism involves rationing workers at the minimum wage, $\underline{C}'_R(\cdot,\underline{w})$ is constant and equal to $\underline{w}$. $\underline{C}'_R(\cdot,\underline{w})$ may be discontinuous at the point $Q = S(\underline{w})$, where the optimal procurement mechanism involves posting a market-clearing wage of $\underline{w}$.[21] As Theorem 1 shows and Panel (c) in Figure 2 illustrates: An increase in $\underline{w}$ expands the interval $[0, S(\underline{w})]$—shifting any discontinuity in $\underline{C}'_R(\cdot,\underline{w})$ at $Q = S(\underline{w})$ to the right—and increasing the value of $\underline{C}'_R(\cdot,\underline{w})$ on $[0, S(\underline{w})]$. Over the interval $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ the optimal mechanism is a two-wage

[21] When the monopsony is restricted to uniform wage-setting under a minimum wage of $\underline{w}$, the corresponding marginal cost schedule $\underline{C}'_U(\cdot,\underline{w})$ is discontinuous at $Q = S(\underline{w})$ for all $\underline{w} \in (W(0), W(1))$ (see page 11). Allowing the monopsony to engage in optimal procurement smooths the marginal cost schedule and, consequently, $\underline{C}'_R(\cdot,\underline{w})$ may be continuous at $Q = S(\underline{w})$ for some values of $\underline{w} \in (W(0), W(1))$.

mechanism but the "ironed" marginal cost function $\underline{C}'_R(\cdot,\underline{w})$ is strictly increasing.[22] A marginal increase in $\underline{w}$ decreases $\underline{C}'_R(\cdot,\underline{w})$ over this region.

Computing comparative statics in mechanism design problems involving constraints—where uniform pricing is not necessarily optimal—is challenging. Nevertheless, the mechanism design machinery developed in this paper can be used to derive comparative statics pertaining to the parameters of the optimal mechanism over the interval $Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big)$. In particular, we have the following lemma which is illustrated in Figure 3 for the specification of W given in (2).

Lemma 1. *Suppose that $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and that $Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big)$. For $i \in \{1,2\}$, let $q_i^*(Q,\underline{w})$ denote the solution value of q_i in (8). Then $q_1^*(Q,\underline{w})$ increases in $\underline{w}$ and decreases in Q and $q_2^*(Q,\underline{w})$ decreases in $\underline{w}$ and increases in Q.*

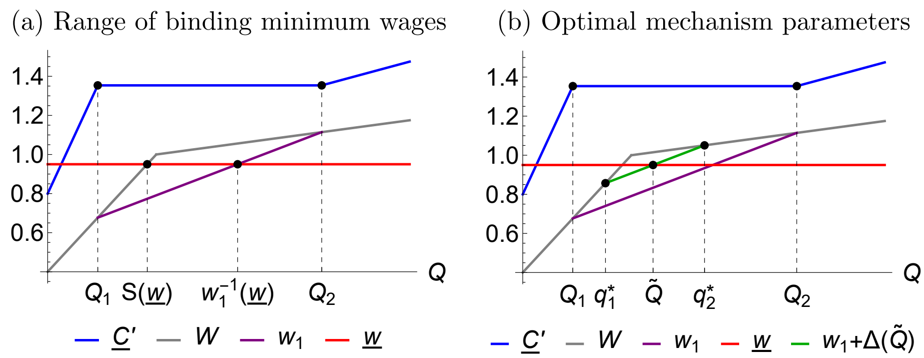

Figure 3: This figure uses the specification of W given in (2) with $\underline{w} = 0.95$. As Panel (a) illustrates, for $Q \in (Q_1, Q_2)$ the lowest wage offered under the laissez-faire equilibrium is characterized by the function w_1, which is a linear combination of $W(Q_1)$ and $W(Q_2)$. A minimum wage $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ is then binding for any $Q < w_1^{-1}(\underline{w})$. For the piecewise linear specification of W and a given quantity $\tilde{Q} \in (S(\underline{w}), w_1^{-1}(\underline{w}))$, the parameters of the optimal two-wage mechanism can then be computed by taking a parallel shift $w_1 + \Delta(\tilde{Q})$ of the w_1 function, where $\Delta(\tilde{Q})$ is a constant. This property is illustrated in Panel (b) and formally established in Appendix OC.

That $q_1^*(Q,\underline{w})$ increases in $\underline{w}$ formalizes the intuition that, as $\underline{w}$ increases, the monopsony will procure more units at the minimum wage since it is a price-taker on these units.

[22] In more standard problems—such as those considered by Mussa and Rosen (1978) and Myerson (1981)—ironed functions are constant over an ironing interval. Here, the slope of the function $\underline{C}_R(\cdot,\underline{w})$ varies with Q over the interval $Q \in \big(S(\underline{w}), w_1^{-1}(\underline{w})\big)$ because the Lagrange multiplier associated with the minimum wage constraint (i.e. the shadow price of that constraint) decreases as Q increases.

3.4 Optimal employment

We let $Q^*(\underline{w})$ denote the optimal level of employment under the minimum wage $\underline{w}$. In line with our convention of letting $D(w)$ denote the largest profit-maximizing quantity demanded by a firm that takes the wage w as given, we break any ties by assuming that $Q^*(\underline{w})$ is the largest profit-maximizing quantity for a firm that uses an optimal procurement mechanism given a minimum wage $\underline{w}$. We then have the following corollary to Theorem 1:

Corollary 1. *If there is a $Q^*(\underline{w})$ satisfying $V(Q^*(\underline{w})) = \underline{C}'_R(Q^*(\underline{w}), \underline{w})$, then $Q^*(\underline{w})$ characterizes the optimal level of employment under a given minimum wage $\underline{w}$. If there is no $Q^*(\underline{w})$ such that $V(Q^*(\underline{w})) = \underline{C}'_R(Q^*(\underline{w}), \underline{w})$, then the optimal level of employment given $\underline{w}$ is $Q^*(\underline{w}) = S(\underline{w})$.*

The "gap" case—that is, the case when the inverse demand function $V(Q)$ lies between $\underline{w}$ and $C'(Q)$ at $Q = S(\underline{w})$—is what gives rise to the employment-increasing effects of minimum wages in the analysis of Robinson, which assumes uniform pricing. This case continues to be a possibility when the firm is not restricted in its choice of mechanisms, above and beyond those imposed by the minimum wage. If this case applies, then marginal increases in the minimum wage still have an employment-increasing effect. However, minimum wage increases can also have employment-increasing effects outside the "gap" case while at the same time affecting involuntary unemployment, as we shall see next.

4 Comparative statics and their policy implications

In this section, we investigate the comparative statics properties of optimal procurement under a minimum wage constraint, as well as their policy implications. We begin with two illustrative examples. Section 4.2 then derives the comparative statics associated with marginal changes in minimum wages, while Section 4.3 provides a global characterization of minimum wage effects. Section 4.4 gives guidance for introducing a minimum wage and Section 4.5 discusses the implications of minimum wages for worker welfare, pay and redistributive effects.

Recall that in Section 2 we introduced the quantity Q^ℓ, which is the level of employment under the laissez-faire equilibrium. In the analysis that follows, the perfectly competitive quantity Q^p will also play an important role. This quantity, which satisfies $V(Q^p) = W(Q^p)$ and $Q^p > Q^\ell$, is the efficient employment level that would emerge under price-taking behaviour.[23]

[23]Since V is decreasing and W is strictly increasing and these functions satisfy $V(0) > W(0)$ and $V(1) < W(1)$, Q^p exists and is unique. In the proof of Proposition 1 we show that $Q^p > Q^\ell$ always holds.

4.1 Overview and illustrative examples

We now present two examples that illustrate the comparative statics associated with minimum wages. The supply side for both examples is the piecewise linear specification for W given in (2), and so both examples exhibit a single ironing range (Q_1, Q_2).

Example 1 The first example assumes $V(Q) = 1.76 - Q$. As illustrated in Panel (a) of Figure 4, this implies that there is involuntary unemployment and wage dispersion under the laissez-faire equilibrium. Panels (b) and (c) of Figure 4 then show how employment, involuntary unemployment and wages evolve as a function of the minimum wage $\underline{w}$. For $\underline{w} \leq w_1(Q^\ell)$, the minimum wage is not binding and has no effect on equilibrium outcomes. Otherwise, and as illustrated in Figure 5, the effects of minimum wages on equilibrium outcomes can be divided into three regions. The first region involves wage dispersion and involuntary unemployment. The second region—which we refer to as the *Robinson region* since this is where the firm optimally uses a Robinson mechanism (see page 15)—does not involve wage dispersion or involuntary unemployment. The third region—which we refer to as the *neoclassical region* since this is where the firm optimally uses a neoclassical mechanism (see page 15)—involves involuntary unemployment and no wage dispersion.

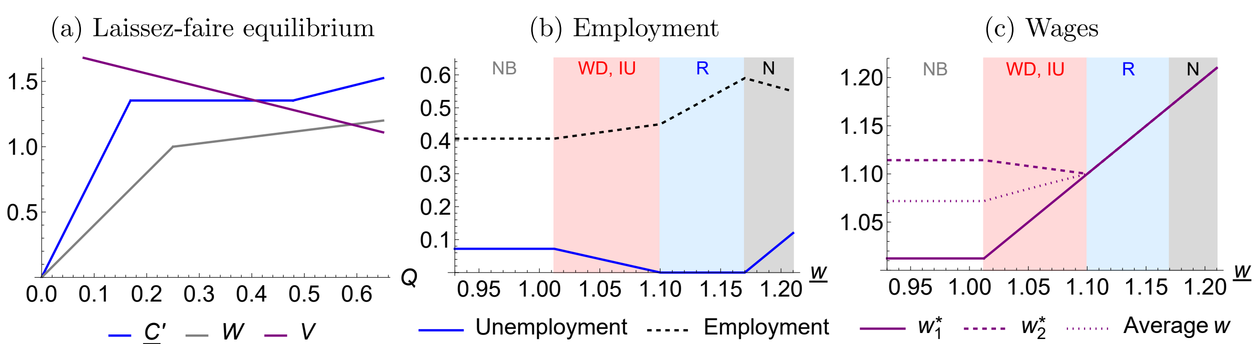

Figure 4: Panel (a) illustrates the laissez-faire equilibrium for Example 1. Panel (b) and Panel (c) then show how equilibrium employment, involuntary unemployment and the wage schedule (including the low wage w_1^*, the high wage w_2^* and the average wage) evolve as a function of $\underline{w}$. Each of the regions identified in Figure 5 are shaded and labeled.

In this example, the monopsony firm optimally selects a two-wage mechanism involving involuntary unemployment and wage dispersion when it faces a sufficiently low binding minimum wage. This region is highlighted in red in figures 4 and 5 and is such that $\underline{w} \in (w_1(Q^\ell), W(\hat{Q}))$.[24] Intuitively, although the minimum wage reduces the firm's market

[24]The quantity $\hat{Q}$ is defined as the smallest quantity $Q \leq Q^p$ such that for all $\underline{w} \in [W(Q), W(Q^p)]$, the monopsony optimally hires $S(\underline{w})$ workers at the minimum wage $\underline{w}$.

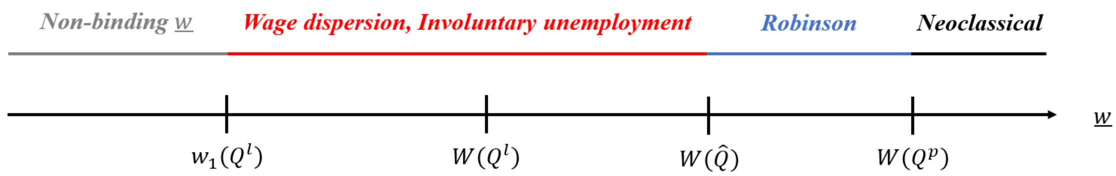

Figure 5: An illustration of the regions that characterize the comparative statics associated with marginal changes in the minimum wage $\underline{w}$ for Example 1. The quantity $\hat{Q}$ is defined in Footnote 24.

power (if it were to hire $Q \leq S(\underline{w})$ workers, it would be a price-taker on these units), the monopsony still benefits from engaging in wage discrimination if the minimum wage is sufficiently low. As is shown in Panel (b) of Figure 4, within this region an increase in the minimum wage $\underline{w}$ has the pro-competitive effect of increasing employment. This is an immediate implication of Theorem 1, which establishes that the firm's marginal cost envelope $\underline{C}'_R(Q, \underline{w})$ is decreasing in $\underline{w}$ in this region. Panels (b) and (c) of Figure 4 also show that both involuntary unemployment and wage dispersion are decreasing in $\underline{w}$, while the average wage increases in $\underline{w}$. This latter effect is an implication of the convexity of $\underline{C}_R(Q, \underline{w})$ in Q and of the fact that $\underline{C}_R(Q, \underline{w})$ increases in $\underline{w}$ (see Theorem 1). While the low wage is equal to $\underline{w}$ and so trivially increases in $\underline{w}$ in this region, the fact that in equilibrium both the high wage and involuntary unemployment decrease in $\underline{w}$ is not obvious due to the countervailing effects involved (as is stated in Lemma 1, $q_2^*(Q, \underline{w})$ decreases in $\underline{w}$ and increases in Q).

For intermediate values of the minimum wage $\underline{w}$, the firm optimally sets a uniform wage of $\underline{w}$ and procures $S(\underline{w})$ workers. This region—the Robinson region—is characterized by the absence of both involuntary unemployment and wage dispersion. It is shown in blue in figures 4 and 5 and corresponds to $\underline{w} \in [W(\hat{Q}), W(Q^p))$. This region exhibits precisely the pro-competitive comparative statics identified by Robinson (1933): increasing the minimum wage $\underline{w}$ increases employment without causing involuntary unemployment or wage dispersion. Intuitively, within this region the monopsony still exerts some market power but is constrained to the point that it no longer benefits from using a two-wage mechanism.

For sufficiently large values of the minimum wage, the monopsony acts as a price-taker and workers are rationed at the minimum wage. This region—the neoclassical region—is characterized by the presence of involuntary unemployment without any accompanying wage dispersion. It is shown in black in Figure 4 and Figure 5 and corresponds to $\underline{w} > W(Q^p)$. The presence of involuntary unemployment without wage dispersion indicates that the monopsony is acting as a price-taker and that the involuntary unemployment is *caused* by the minimum wage itself and not by market power. Increasing the minimum wage within the neoclassical region decreases employment and increases involuntary unemployment. These effects correspond to those found when a binding minimum wage is introduced to the neoclassical model of a perfectly competitive labor market.

Example 2 The supply side in Example 2 is the same as in Example 1 but, in contrast, the marginal revenue product of labor is now constant. Specifically, we assume $V(Q) = 1.15$ for all $Q \in [0, 1]$. As illustrated in Panel (a) of Figure 6, this means that under the laissez-faire equilibrium the firm sets a market-clearing wage and procures a quantity Q^ℓ that is smaller than the lower bound Q_1 of the ironing range (Q_1, Q_2). Figures 6 and 7 show that the comparative statics are characterized by similar regions to those identified in Example 1. However, these regions exhibit a different structure since the firm now sets a market-clearing wage under the laissez-faire equilibrium.

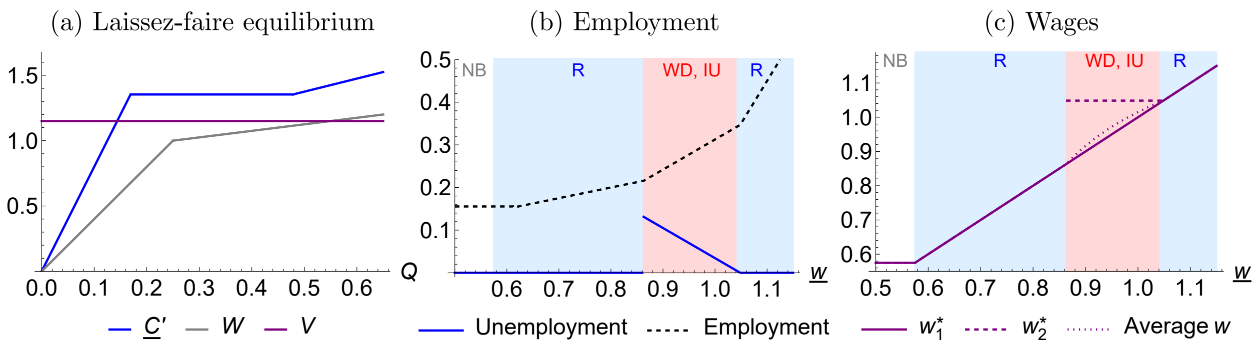

Figure 6: Panel (a) illustrates the laissez-faire equilibrium for Example 2. Panels (b) and (c) then show how equilibrium employment, involuntary unemployment and the wage schedule (including the low wage w_1^*, the high wage w_2^* and the average wage) evolve as a function of $\underline{w}$. The regions identified in Figure 7 are shaded and labeled.

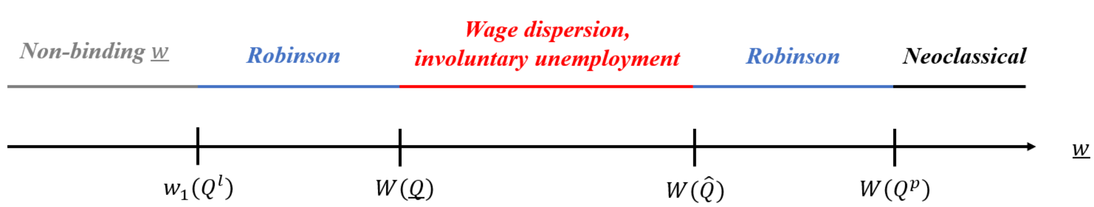

Figure 7: An illustration of the regions that characterize the comparative statics associated with marginal changes in the minimum wage $\underline{w}$ for Example 2. The quantities $\hat{Q}$ and $\underline{Q}$ are respectively defined in footnotes 24 and 25.

For a sufficiently low binding minimum wage, i.e. for $\underline{w} \in (w_1(Q^\ell, W(\underline{Q}))$, we are in the Robinson region and the firm optimally procures $S(\underline{w})$ workers at the market-clearing wage.[25] As previously noted, in the Robinson region increasing the minimum wage $\underline{w}$ has the

[25]The quantity $\underline{Q}$ is defined as the smallest quantity $Q \leq \hat{Q}$ such that for all $\underline{w} \in [W(Q), W(\hat{Q})]$, the

pro-competitive effect of increasing equilibrium employment. Eventually, this induces the firm to employ a quantity of workers within the ironing range. Consequently, the equilibrium transitions from the Robinson region to a region involving wage dispersion and involuntary unemployment. As Figure 6 shows, this transition from the Robinson region into a region with involuntary unemployment and wage dispersion is associated with a discontinuous increase in involuntary unemployment and a discontinuous change in the wage schedule. This implies that this transition also involves a discontinuous decrease in social surplus. Upon entering the ironing range, this example then exhibits identical comparative statics to those of Example 1 (with the exception that the high wage does not vary with $\underline{w}$ when the firm utilizes a two-wage mechanism; as we will see in Section 4.2, this is a general property when V is constant).

If V is constant, then absent a binding minimum wage constraint, a two-wage mechanism —inducing involuntary unemployment—is never strictly optimal. This is, of course, the analogue to the observation by Bulow and Roberts (1989) that in a monopoly context, rationing is never strictly optimal when the firm has constant marginal costs. However, as this example illustrates, even when V is constant a two-wage mechanism can become uniquely optimal under a binding minimum wage constraint.

Our analysis in Section 4.2 formally shows how employment, involuntary unemployment and the wage schedule adjust in response to marginal changes in the minimum wage within the three types of regions (regions with involuntary unemployment and wage dispersion, the Robinson region and the neoclassical region) identified in examples 1 and 2. As we have seen, both market power and minimum wages can cause involuntary unemployment. Notwithstanding these complications, Theorem 2 in Section 4.2 shows that a regulator who only observes wage dispersion and involuntary unemployment can always identify which region they are in and predict how employment, involuntary unemployment and the wage schedule will adjust in response to a small change in the minimum wage. In particular, a regulator can always distinguish involuntary unemployment caused by the minimum wage from involuntary unemployment caused by market power by identifying whether or not the involuntary unemployment is accompanied by wage dispersion and an efficiency wage.

Our analysis in Section 4.3 provides a global characterization of the effects of minimum wages by showing how transitions between the three types of regions identified here occur as a function of the minimum wage $\underline{w}$ in general (see Theorem 3 in Section 4.3). In particular, beyond the examples considered here there may be more than one relevant ironing range between the laissez-faire level of employment and the efficient level of employment.

monopsony optimally hires Q workers using a two-wage mechanism.

4.2 Marginal minimum wage effects

We now analyze how marginal increases in the minimum wage affect equilibrium employment, involuntary unemployment and wage dispersion. As we have just seen, such increases have rich and non-monotone effects on equilibrium employment, involuntary unemployment and wages. This raises the following question: could a regulator predict the effects of making a small change to a prevailing minimum wage purely on the basis of observable outcomes and without detailed knowledge of the functions V and W? Theorem 2 answers this question affirmatively and only requires that the regulator can separately observe whether or not there is involuntary unemployment and wage dispersion under the prevailing minimum wage.

Theorem 2.

1. *If there is involuntary unemployment and wage dispersion under a given minimum wage, then $\underline{w} < W(Q^p)$. A marginal increase in the minimum wage then increases employment and decreases involuntary unemployment and wage dispersion. Moreover, the low wage paid by the firm increases and the high wage decreases.*

2. *If there is no involuntary unemployment under a given minimum wage, then $\underline{w} \leq W(Q^p)$. Provided $\underline{w} \neq W(Q^p)$, a marginal increase in the minimum wage then increases employment.*

3. *If there is involuntary unemployment and no wage dispersion under a given minimum wage, then $\underline{w} > W(Q^p)$. A marginal increase in the minimum wage then decreases employment and increases involuntary unemployment without affecting wage dispersion.*

Theorem 2 generalizes the within region comparative statics identified in examples 1 and 2 in the previous section. In Figure 8 we provide an illustration of the comparative statics concerning equilibrium employment. The first case covered by Theorem 2 corresponds to regions where the firm uses a two-wage mechanism. As the comparative statics proven in Theorem 1 and Corollary 1 imply and Panel (a) of Figure 8 shows, in this region a marginal increase in the minimum wage increases equilibrium employment. The second case covered by Theorem 2 corresponds to the Robinson region, where the monopsony sets a market-clearing wage equal to the minimum wage. If we are in the interior of the Robinson region then a marginal increase in the minimum wage $\underline{w}$ shifts the discontinuity in $\underline{C}_R(\cdot, \underline{w})$ at $Q = S(\underline{w})$ to the right, resulting in an increase in equilibrium employment.[26] In the interior

[26] In the setting considered by Robinson (1933)—which restricts attention to uniform wage-setting—a marginal increase in the minimum wage $\underline{w} \in (W(0), W(1))$ increases employment precisely when the marginal value function V intersects the corresponding marginal cost schedule $C'_U(\cdot, \underline{w})$ in the interior of the discontinuity at $Q = S(\underline{w})$ (see Footnote 15).

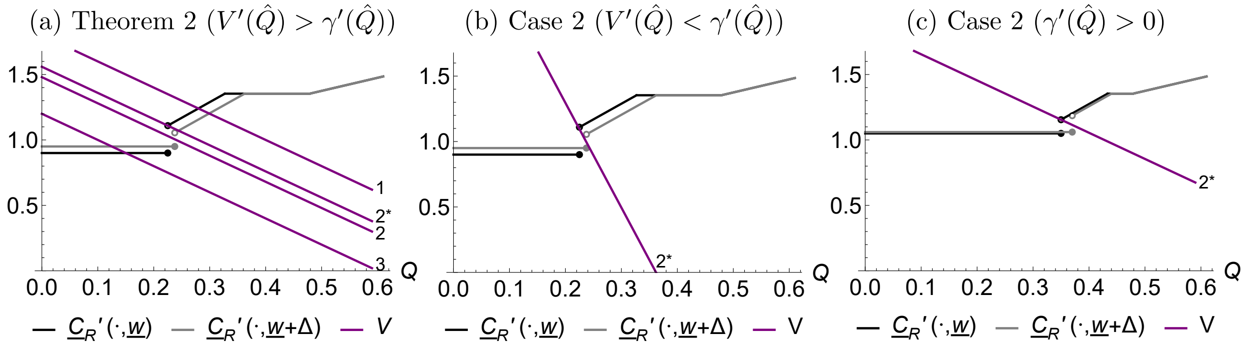

Figure 8: This figure uses the specification of W given in (2). Panel (a) illustrates Theorem 2 using a range of V functions labelled according to the cases they represent. Panel (a) includes a knife-edge instance of Case 2 (corresponding to the V function labelled 2^*) where a marginal increase in the minimum wage results in a transition from the Robinson region to a region with involuntary unemployment and wage dispersion. Panel (b) and Panel (c) illustrate two instances of this knife-edge case where a marginal increase in the minimum wage does not result in a transition out of the Robinson region. The function γ will be introduced in the following section.

of the Robinson region, involuntary unemployment and wage dispersion are also unaffected. However, in general it is possible for a marginal increase in the minimum wage to result in a transition from the Robinson region to a region with involuntary unemployment and wage dispersion (as was the case in Example 2) or to the neoclassical region. However, this latter case is precluded from Theorem 2 by the requirement that $\underline{w} \neq W(Q^p)$. The third case in Theorem 2 corresponds to the neoclassical region, where the monopsony rations workers at the minimum wage. Here, the comparative statics proven in Theorem 1 again imply that a marginal increase in the minimum wage decreases employment and increases involuntary unemployment without having any effect on wage dispersion.

As this discussion illustrates, the main work involved in proving Theorem 2 relates to proving the comparative statics concerning involuntary unemployment and wage dispersion covered by the first case. Theorem 1 and Corollary 1 imply that $Q^*(\underline{w})$ increases in $\underline{w}$ whenever there is wage dispersion. At the same time, we know from Lemma 1 that $q_2^*(Q, \underline{w})$ increases in Q and decreases in $\underline{w}$. The challenge is to show that despite the countervailing effects involved, $q_2^*(Q^*(\underline{w}), \underline{w})$ is decreasing in $\underline{w}$. To that end, we have the following lemma.

Lemma 2. *Given $\underline{w}$ such that $Q^*(\underline{w}) \in (S(\underline{w}), w_1^{-1}(\underline{w}))$, we have*

$$\frac{dq_1^*(Q^*(\underline{w}), \underline{w})}{d\underline{w}} \geq 0 \geq \frac{dq_2^*(Q^*(\underline{w}), \underline{w})}{d\underline{w}},$$

where the inequalities are strict if and only if $V'(Q^*(\underline{w})) < 0$.

Lemma 2 implies that whenever $Q^*(\underline{w}) \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ holds, $q_2^*(Q^*(\underline{w}), \underline{w})$ is decreasing in $\underline{w}$. Combining this with the fact that $Q^*(\underline{w})$ also increases in $\underline{w}$ in this case, we immediately have that involuntary unemployment $q_2^*(Q^*(\underline{w}), \underline{w}) - Q^*(\underline{w})$ decreases in $\underline{w}$. We also have that wage dispersion is decreasing in $\underline{w}$ since the low wage is simply $\underline{w}$ (which trivially increases in $\underline{w}$), while the high wage $W(q_2^*(Q^*(\underline{w}), \underline{w}))$ decreases in $\underline{w}$. Interestingly, the final statement of Lemma 2 shows that the high wage $W(q_2^*(Q^*(\underline{w}), \underline{w}))$ does not vary with the minimum wage if and only if V is constant at $Q^*(\underline{w})$.

4.3 Global minimum wage effects

We now provide a global characterization of the effects of minimum wages by showing how transitions between the three different types of regions (regions with wage dispersion and involuntary unemployment, the Robinson region and the neoclassical region) arise as a function of the minimum wage. The key to this characterization is tracing out how the boundary of the discontinuity in $\underline{C}'_R(\cdot, \underline{w})$ at $Q = S(\underline{w})$ varies with $\underline{w}$.

To unpack this, we begin by delineating the case when there is no region with wage dispersion. In particular, if there is no overlap between $[Q^\ell, Q^p]$ and any ironing interval (that is, if $[Q^\ell, Q^p] \bigcap \bigcup_{m \in \mathcal{M}} (Q_1(m), Q_2(m)) = \emptyset$) then there can never be equilibrium wage dispersion. The minimum wage effects are then precisely those identified by standard monopsony pricing models in the tradition of Robinson (1933), in which the monopsony always sets a uniform wage. For $\underline{w} < W(Q^\ell)$, the minimum wage does not bind and the monopsony hires Q^ℓ workers at the wage $W(Q^\ell)$. For $\underline{w} \in [W(Q^\ell), W(Q^p)]$, the monopsony hires $S(\underline{w})$ workers at the minimum wage, and employment increases in $\underline{w}$. In either case, there is no involuntary unemployment. Finally, for $\underline{w} > W(Q^p)$, we are in the neoclassical region and the monopsony hires $D(\underline{w})$ workers at the minimum wage. Here, involuntary unemployment $S(\underline{w}) - D(\underline{w})$ increases in $\underline{w}$ and employment $D(\underline{w})$ decreases in $\underline{w}$.

Now suppose that there exists $m \in \mathcal{M}$ such that $[Q^\ell, Q^p] \bigcap (Q_1(m), Q_2(m)) \neq \emptyset$. Given a binding minimum wage $\underline{w}$, there is equilibrium wage dispersion if and only if $w_1^{-1}(\underline{w}) > S(\underline{w})$ and the function V intersects with $\underline{C}'_R(\cdot, \underline{w})$ on $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ (i.e. the region to the right of the discontinuity in $\underline{C}'_R(\cdot, \underline{w})$ where the minimum wage is binding and $\underline{C}'_R(\cdot, \underline{w})$ is strictly increasing). To trace out where the function $\underline{C}'_R(\cdot, \underline{w})$ starts to strictly increase in Q under a binding minimum wage, we define the function

$$\gamma(Q) := \lim_{\underline{w} \uparrow W(Q)} \underline{C}'_R(Q, \underline{w}),$$

which gives the marginal cost of procuring $Q \in [0, 1]$ as $\underline{w}$ approaches the market-clearing wage $W(Q)$ from below.[27] Note that γ is well-defined and continuous.[28] If Q is such that $\underline{C}(Q) = C(Q)$ then γ satisfies $\gamma(Q) = C'(Q)$. If Q is such that $\underline{C}(Q) < C(Q)$ then γ traces out the right limit of the discontinuity in $\underline{C}'_R(Q, \cdot)$ at $\underline{w} = W(Q)$ that arises as a result of the transition in the optimal procurement mechanism from a two-wage mechanism to a mechanism involving a single wage of $\underline{w}$.

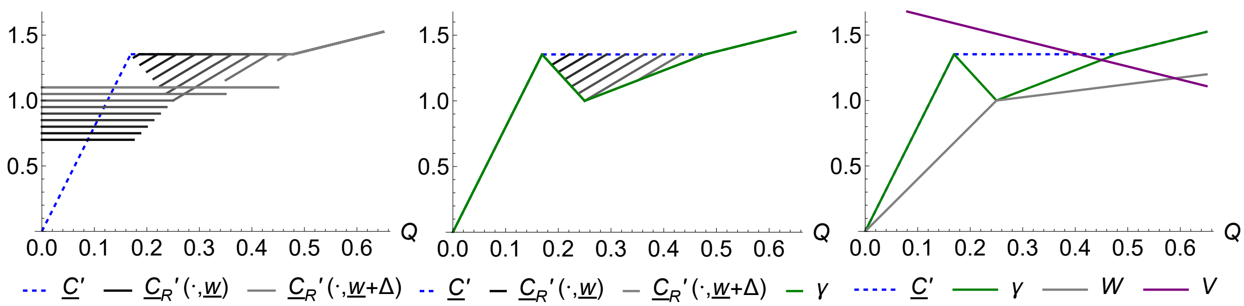

Figure 9: An illustration of how the contour of the strictly increasing region of $\underline{C}'_R(\cdot, \underline{w})$ (where this function is illustrated for a variety of $\underline{w}$ values) under a binding minimum wage defines the function γ for Example 1 from Section 4.1.

Figure 9 illustrates the region where $\underline{C}'_R(\cdot, \underline{w})$ is strictly increasing under a binding minimum wage and traces out the function γ for an example of the piecewise linear specification of W from Appendix OC with a single "kink" at $Q = \underline{q}$ and ironing interval (Q_1, Q_2). As Panel (c) in Figure 9 shows, aside from the point $\underline{q}$ such that $\gamma(\underline{q}) = W(\underline{q})$, $\gamma(Q) > W(Q)$ holds for all $Q \in (Q_1, Q_2)$.[29] This implies that $\underline{C}'_R(\cdot, \underline{w})$ is continuous at $Q = S(\underline{w})$ only if $\underline{w} = W(\underline{q})$ and is discontinuous at $Q = S(\underline{w})$ for all $\underline{w} \in (W(0), W(1)) \setminus \{W(\underline{q})\}$. For $Q^\ell \in (Q_1, Q_2)$ and V linear and strictly decreasing, the piecewise linear specification of W from Appendix OC then generically exhibits the structure depicted in Figure 5. Specifically, letting $\hat{Q}$ denote the unique point of intersection between V and γ, we have equilibrium wage dispersion and involuntary unemployment for all $\underline{w} \in (w_1(Q^\ell), W(\hat{Q}))$. Moreover, for all $\underline{w} \in [W(\hat{Q}), W(Q^p))$, we are in the Robinson region where the monopsony optimally hires $S(\underline{w})$ workers at the minimum wage $\underline{w}$. The non-generic case occurs if $Q^p = \underline{q}$, which implies that $\hat{Q} = Q^p$ and consequently, there is no Robinson region. However, aside from this knife-edge, the Robinson region always exists for the piecewise linear specification.

More generally, one needs to compare the functions V and γ to determine whether there is

[27] If $\underline{w}$ approaches $W(Q)$ from above, the marginal cost is simply $W(Q)$ (i.e. $\lim_{\underline{w} \downarrow W(Q)} \underline{C}'_R(Q, \underline{w}) = \underline{C}'_R(Q, W(Q)) = W(Q)$).

[28] Consider the function $\underline{C}'^{+}_R(Q, \underline{w}) := \lim_{\epsilon \downarrow 0} \frac{\underline{C}_R(Q+\epsilon, \underline{w}) - \underline{C}_R(Q, \underline{w})}{\epsilon}$, which is the *right derivative* of $\underline{C}_R$ with respect to Q. Then $\underline{C}'^{+}_R$ is the continuous extension of $\underline{C}'_R$ on the closed set $\{(Q, \underline{w}) \in [0,1] \times [W(0), W(1)] : \underline{w} \leq W(Q)\}$. The function γ is well-defined and continuous because $\gamma(Q) = \underline{C}'^{+}_R(Q, W(Q))$ holds for all $Q \in [0,1]$ and, consequently, γ is simply the composition of two continuous functions.

[29] These general properties are established in Appendix OC.

equilibrium wage dispersion and involuntary unemployment under a given minimum wage.[30] Consider the sets $\mathcal{W} := (w_1(Q^\ell), W(Q^p))$ and

$$\overline{\mathcal{W}} := \bigcup_{m \in \mathcal{M}} (W(Q_1(m)), W(Q_2(m))) \cap \mathcal{W}.$$

The optimal mechanism involves wage dispersion if and only if $\underline{w} \in \overline{\mathcal{W}}$ and $V(Q) > \gamma(Q)$, where Q is such that $\underline{w} = W(Q)$. Equivalently, the optimal mechanism involves wage dispersion if and only if $\underline{w} \in \overline{\mathcal{W}}$ and $V(S(\underline{w})) > \gamma(S(\underline{w}))$. Let $\tilde{V}(\underline{w}) := V(S(\underline{w}))$ and $\tilde{\gamma}(\underline{w}) := \gamma(S(\underline{w}))$ and define the sets

$$\mathcal{T} := \left\{ \underline{w} \in \overline{\mathcal{W}} : \tilde{V}(\underline{w}) > \tilde{\gamma}(\underline{w}) \right\} \quad \text{and} \quad \mathcal{S} := \mathcal{W} \setminus \mathcal{T}.$$

A two-wage mechanism is then used under a binding minimum wage if and only if $\underline{w} \in \mathcal{T}$ and a market-clearing wage is set under a binding minimum wage if and only if $\underline{w} \in \mathcal{S}$. Since the functions $\tilde{V}$ and $\tilde{\gamma}$ are continuous the sets $\mathcal{T}$ and $\mathcal{S}$ can be written as a union of disjoint intervals. Transitions from uniform wage (two-wage) mechanisms to two-wage (uniform wage) mechanisms occur as $\underline{w}$ transitions from the set $\mathcal{S}$ ($\mathcal{T}$) into the set $\mathcal{T}$ ($\mathcal{S}$). The following theorem summarizes this analysis and formally generalizes the transitions illustrated in figures 5 and 7.

Theorem 3. *For all $\underline{w} \in [W(0), w_1(Q^\ell)]$ the minimum wage constraint is not binding. All minimum wages $\underline{w} \in (W(Q^p), W(1)]$ constitute the neoclassical region. The Robinson region is given by the set $\mathcal{S}$, and $\mathcal{T}$ is the set of binding minimum wages where the optimal procurement mechanism involves wage dispersion and involuntary unemployment.*

For cases where $V(Q) = v$ holds for all $Q \in [0,1]$, Figure 10 depicts the analogue to Panel (c) in Figure 9. As the figure shows, in these cases there are two points of intersection between γ with V, which correspond to the quantities $\underline{Q}$ and $\hat{Q}$ in Figure 7. A marginal increase in $\underline{w}$ at $\underline{w} = W(\underline{Q})$ induces a transition from a mechanism with a market-clearing wage to a two-wage mechanism where employment is randomly rationed for the high-wage workers, resulting in a discontinuous decrease in social surplus and total worker surplus.[31]

[30] Transitions into and out of the Robinson region occur at points $\hat{Q}$ where the functions V and γ intersect, and depend on the sign and relative slopes of these functions at their points of intersection. In Figure 8, Panel (a) illustrates a transition out of the Robinson region, while Panels (b) and (c) illustrate transitions into the Robinson region.

[31] When Q workers are hired, social surplus is $\int_0^Q V(x)\,dx - \int_0^Q w(x)\,dx$ when the firm sets a uniform wage and $\int_0^{q_1^*(Q,\underline{w})} W(x)\,dx + \beta(Q, q_1^*(Q,\underline{w}), q_2^*(Q,\underline{w})) \int_{q_1^*(Q,\underline{w})}^{q_2^*(Q,\underline{w})} W(x)\,dx$ otherwise. Likewise, when Q workers are employed, worker surplus is given by subtracting the workers' aggregate cost of working from $\underline{C}_R(Q,\underline{w})$.

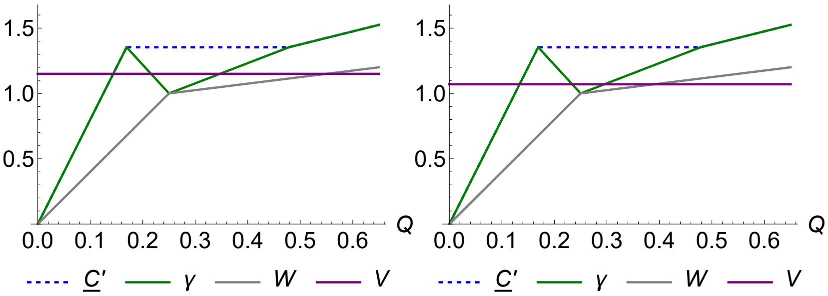

Figure 10: Panel (a) illustrates the γ contour for Example 2 from Section 4.1, which is such that $Q^p > Q_2$. Panel (b) uses $V(Q) = 1.07$ for all $Q \in [0, 1]$, which is such that $Q^p < Q_2$.

4.4 Guidance for introducing a minimum wage

In Section 4.2 we were concerned with the effects associated with a marginal change in a prevailing minimum wage. But, of course, the effects of introducing a minimum wage may be of equal or even greater interest. We now now briefly discuss these, assuming that there is involuntary unemployment and wage dispersion under the laissez-faire equilibrium.

Proposition 1. *Suppose that $Q^\ell \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Then introducing a minimum wage of $\underline{w} = W(Q^p)$ increases equilibrium employment to $Q^p > Q^\ell$, increases workers' total pay and eliminates both involuntary unemployment and wage dispersion. It also maximizes total employment and social surplus. Relative to the laissez-faire case, any minimum wage $\underline{w} \in (w_1(Q^\ell), W(Q_2(m))]$ increases total employment and workers' pay and decreases involuntary unemployment. Furthermore, a minimum wage of $\underline{w} = W(Q_2(m))$ eliminates involuntary unemployment if and only if $Q_2(m) \leq Q^p$. Any minimum wage that eliminates involuntary unemployment increases social surplus relative to laissez-faire.*

Proposition 1 shows that, relative to the laissez-faire case, setting $\underline{w} = W(Q^p)$ increases total employment and the total wage bill paid to workers, and eliminates involuntary unemployment. Such a minimum wage also maximizes both total employment and social surplus. Of course, in practice, it may be difficult for a regulator to observe or estimate $W(Q^p)$. However, as Proposition 1 also shows, even setting $\underline{w} = W(Q_2(m))$ (i.e. setting a minimum wage equal to the highest wage observed under the laissez-faire equilibrium) is guaranteed to increase employment and decrease involuntary unemployment, possibly to the point of eliminating it. If $\underline{w} = W(Q_2(m))$ eliminates involuntary unemployment, it also increases social surplus relative to the laissez-faire equilibrium because it brings total employment closer to the efficient level and eliminates the random, inefficient allocation associated with involuntary unemployment.

The intuition underlying Proposition 1 is simple. Given $\underline{w} = W(Q^p)$, the monopsony

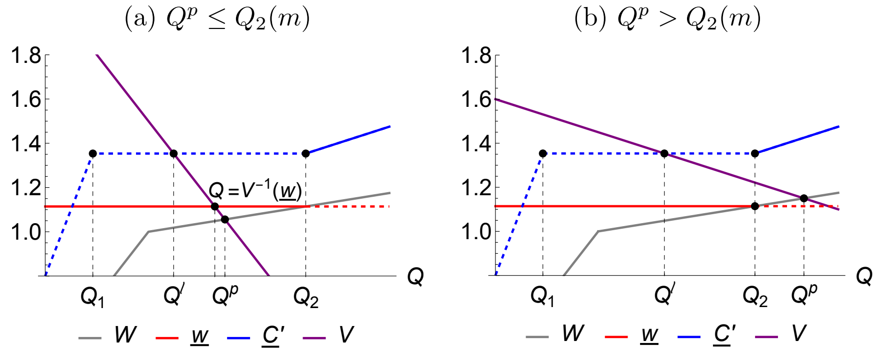

Figure 11: The effects of introducing a minimum wage of $\underline{w} = W(Q_2(m))$ for the piecewise linear specification of W from (2). In each panel the solid sections of the $\underline{w}$ (red) and $\underline{C}'$ (blue) curves indicate the marginal cost schedule associated with optimal procurement. In Panel (a), $Q^p \leq Q_2$ and $\underline{w}$ induces involuntary unemployment. In Panel (b), $Q^p > Q_2$ and $\underline{w}$ eliminates involuntary unemployment.

will optimally hire at least Q^p workers because the marginal benefit $V(Q)$ of hiring $Q < Q^p$ workers is weakly greater than the marginal cost $\underline{w} = V(Q^p)$.[32] It will not hire any additional workers because—as shown in Theorem 1—the marginal cost of hiring $Q > Q^p$ workers under a minimum wage of $\underline{w} = W(Q^p)$ strictly exceeds $V(Q)$. This also implies that total employment is maximized under a minimum wage of $\underline{w} = W(Q^p)$ and that setting $\underline{w} > W(Q^p)$ will cause involuntary unemployment and result in inefficiently low employment. Consequently, the minimum wage $\underline{w} = W(Q^p)$ maximizes social surplus. Similarly, under a minimum wage of $\underline{w} = W(Q_2(m))$ the monopsony is a price-taker on all Q units when $Q \leq Q_2(m)$, and it will never hire more than $Q_2(m)$ workers if it hires fewer than $Q_2(m)$ under laissez-faire. Since the monopsony now faces a strictly lower marginal cost of hiring any $Q \in [Q^\ell, Q_2(m))$ workers, it will always hire more than Q^ℓ workers. Moreover, if $Q^p \geq Q_2(m)$, then the monopsony will hire precisely $Q_2(m)$ workers and involuntary unemployment is eliminated (see Panel (a) of Figure 11). If $Q^p < Q_2(m)$, then the monopsony will hire $V^{-1}(\underline{w}) \in (Q^\ell, Q^p)$ workers (see Panel (b) of Figure 11). In this case $\underline{w} > W(Q^p)$ and the minimum wage *causes* involuntary unemployment in the sense that setting a lower minimum wage of $\underline{w} = W(Q^p)$ would have eliminated it. Even so, involuntary unemployment decreases relative to the laissez-faire equilibrium since the total number of workers who participate is always $Q_2(m)$, while $D(\underline{w}) > Q^\ell$ workers are hired.

[32]This follows immediately from the fact that V is decreasing together with our assumption that the monopsony hires the largest profit-maximizing quantity of workers.

4.5 Redistribution, worker welfare and pay

As figures 4 and 6 show for the examples in Section 4.1, whenever there is involuntary unemployment and wage dispersion, both the average wage (the total wage payments divided by the level of employment) and the low wage paid to workers increase in the minimum wage. Moreover, the high wage is decreasing in the minimum wage if and only if V is decreasing at $Q^*(\underline{w})$, which is an implication of Lemma 2. This is observation is formally generalized in the following proposition and points to a potential conflict of interest among workers when V is decreasing: high-wage workers are worse off under a marginal increase in the minimum wage, while those earning the minimum wage are better off.

Proposition 2. *If there is involuntary unemployment and wage dispersion under a given minimum wage $\underline{w}$, then a marginal increase in $\underline{w}$ increases the average wage and the low wage paid to workers. Moreover, it decreases the high wage if and only if V is decreasing at $Q^*(\underline{w})$. If there is no involuntary unemployment under a given minimum wage $\underline{w}$, then a marginal increase in $\underline{w}$ increases the wage of all employed workers.*

Proposition 2 considers cases in which a marginal increase in the minimum wage increases equilibrium employment. In contrast, when there is involuntary unemployment without wage dispersion at a given minimum wage, a marginal increase in the minimum wage increases the wage paid to all employed workers but decreases total employment. Interestingly, minimum wages can also have the effect that high-wage workers are paid more than their marginal revenue product. That is, the firm pays these workers more than its willingness to pay. To see this in the simplest and most transparent way, consider a specification in which the firm's marginal revenue product is constant and given by $V(Q) = v$. As implied by Lemma 2, given some minimum wage $\underline{w}$, whenever it is optimal to procure $Q^*(\underline{w})$ using a two-wage mechanism, the equilibrium values of q_1^* and q_2^* respectively correspond to $Q_1(m)$ and $Q_2(m)$ for some $m \in \mathcal{M}$. Assume further that v and $\underline{w}$ are such that $Q^*(\underline{w})$ is optimally procured with a two-wage mechanism in which the high wage is $w_2 = W(Q_2(m))$. If in addition $v < W(Q_2(m))$ holds, as is the case in the right-hand panel in Figure 10, then all the workers who are hired at the high wage are paid more than v.[33]

[33] To see that this possibility naturally arises, consider the general piecewise linear specification from Appendix OC, which exhibits a "kink" at the quantity $Q = \underline{q}$. This specification has the property that at $\underline{w} = W(\underline{q})$, the marginal cost of procurement $\underline{C}'_R(.\underline{w})$ is continuous in Q. This is illustrated in Panel (c) of Figure 9 and in Figure 10, as $\gamma(\underline{q}) = W(\underline{q})$ holds. Thus, for $\underline{w} = W(\underline{q})$, any $v \in (W(\underline{q}), W(Q_2))$ satisfies the aforementioned properties.

5 Discussion and extensions

In this section we briefly discuss our modelling assumptions and interpretation and sketch two extensions that are provided in the online appendix.

5.1 Discussion

We first discuss the robustness of two-wage mechanisms, part-time work, and the relationship between market power and non-regularity. To simplify the exposition, we focus on cases without binding minimum wages.

Robustness Like Lee and Saez (2012), we consider risk-neutral agents with quasilinear utility and focus on the extensive margin in labor supply, which Lee and Saez (2012) argue is the empirically relevant margin. Assuming risk neutrality and quasilinear utility ensures that two-wage mechanisms are optimal absent wage regulation. This makes the more involved problem of deriving the optimal mechanism under a given minimum wage tractable.

However, the two-wage mechanism that is optimal whenever $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ is robust to the introduction of risk-averse workers in the following sense. Suppose all workers have the same initial wealth level—which without loss of generality can be normalized to zero—and the same, strictly concave utility function u. A worker with opportunity cost $W(Q)$ working at wage $w \geq W(Q)$ then has a utility of $u(w - W(Q))$, while an unemployed worker has a utility of $u(0)$. The participation constraint for the marginal worker then still requires that $w_2 = W(Q_2(m))$. However, the wage $\hat{w}_1$ that makes workers with opportunity cost $W(Q_1(m))$ indifferent now satisfies $u(\hat{w}_1 - W(Q_1(m))) = \alpha(Q)u(W(Q_2(m)) - W(Q_1(m))) + (1 - \alpha(Q))u(0)$. Since u is strictly concave, we have $w_1 > \hat{w}_1$.[34] Unsurprisingly, the insurance benefit associated with certain employment works in favor of the firm's scheme, reducing its procurement cost relative to the case with risk-neutral workers. However, with risk-averse agents the *optimal* procurement mechanism may involve offering more than two wages.

The superiority of using a two-wage mechanism over a market-clearing wage whenever $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ is also robust to small errors in setting these wages.[35] Moreover, while randomly rationing workers with costs between $W(Q_1(m))$ and $W(Q_2(m))$ is optimal for the monopsony, the superiority of a two-wage mechanism does not hinge on the assumption that rationing is uniform. As mentioned, workers' incentive compatibility

[34] Moreover, the single-crossing condition (see Footnote 11) is satisfied.

[35] By continuity, if the two-wage mechanism is constructed with sufficiently accurate estimates of $Q_1(m)$ and $Q_2(m)$, then this yields an approximately optimal mechanism with a lower procurement cost than hiring workers at the market-clearing wage $W(Q)$.

constraints require that workers with lower costs are hired with weakly higher probability, so the only alternative rationing schemes are such that the allocation is more efficient than under uniform random rationing. If one parameterizes rationing schemes as convex combinations of the uniform random and the efficient allocation, the scheme with wage dispersion and involuntary unemployment remains optimal provided the probability that the efficient allocation obtains is less than one.

Part-time work As noted in Section 2, the rationing involved in implementing a two-wage mechanism can be interpreted in terms of part-time work, so that high-wage workers are *underemployed*, while low-wage workers are fully employed. In practice, it is not uncommon for part-time workers to be paid a higher hourly wage than their full-time counterparts. For example, a 2009 Bureau of Labor Statistics report documents that in various occupations in the U.S., in particular in health services, part-time workers earn higher hourly wages than full-time employees.[36] Another case in point is the restaurant industry in France, where full-time waiters are paid hourly wages of 12 or 13 euros and part-time waiters (called extras) are paid 16 euros per hour.[37]

Market power and non-regularity There is increasing recognition that employers exert wage-setting power in labor markets (Card, 2022a,b). Our baseline model considers a monopsony firm which provides a tractable model of labor market power, and in Appendix OA we show that the analysis in Section 3 can be extended to an oligopsony model involving n firms.

Non-convex cost functions naturally arise when workers face a fixed cost of moving, changing occupation or participating in the labor market (see Appendix OD). Similarly, if two labor markets that differ with respect to the lowest opportunity cost of working are integrated, then the integrated labor market always exhibits a non-convex cost function, even when as standalone markets each market exhibits a convex cost function.[38] While we are not aware of any previous empirical studies that have investigated the curvature properties of inverse labor supply schedules W and their corresponding cost functions C, a monopsony that faces a non-convex cost function is analogous to a monopoly that faces a non-concave revenue function. Each of these problems correspond to mechanism design problems that fail the *regularity* assumption of Myerson (1981). When the assumption of concave revenue (or, equivalently, monotone marginal revenue) has been tested empirically, it is frequently

[36]See *A Comparison of Hourly Wage Rates for Full- and Part-Time Workers by Occupation, 2007*.

[37]See *Grille des salaires : Extracadabra sort son étude 2022* and *Extracadabra*, respectively.

[38]This is analogous to the observation made in Loertscher and Muir (2022) that integrating output markets can render revenue non-concave.

rejected.[39] We anticipate similar failures of regularity to arise in labor markets, as it would be surprising if an assumption that is frequently rejected in output markets were to hold systematically in input markets.

5.2 Extensions

The minimum wage analysis and the results can be generalized beyond the monopsony model outlined in Section 2. In the online appendix, we provide extensions to quantity competition and to a monopsony model with horizontally differentiated jobs.

Specifically, in Appendix OA, we consider a model involving quantity competition among n symmetric firms, each with a strictly decreasing marginal revenue product of labor V. Each firm i simultaneously chooses a quantity y_i of labor to procure. A Walrasian auctioneer then uses an optimal procurement mechanism to procure the aggregate quantity $Q = \sum_{i=1}^{n} y_i$ at the minimum cost $\underline{C}(Q)$. Each firm i then pays a total cost of $\frac{y_i}{Q}\underline{C}(Q)$. Besides the fact that we do not restrict the auctioneer to procure workers at a uniform wage, this model is *identical* to the standard model of Cournot competition, where each firm i pays $\frac{y_i}{Q}C(Q) = y_i W(Q)$. We show that absent any minimum wage regulation, there is a unique and symmetric equilibrium and that the aggregate quantity of labor procured in equilibrium increases in n (see Proposition OA.1). However, there is not a monotone relationship between the number of firms n and the degree of involuntary unemployment.[40] Theorem OA.1 then generalizes the results of Theorem 2 to this model of quantity competition. It shows, among other things, that if there is involuntary unemployment under the laissez-faire equilibrium, then an appropriately chosen minimum wage can eliminate involuntary unemployment and increase total employment and workers' pay.

In Appendix OB we consider horizontally differentiated workers by studying a variant of the Hotelling model. Specifically, we assume that the monopsony employs workers at locations 0 and 1, with a strictly decreasing marginal revenue product of labor V for each location. There is also a continuum of workers whose private locations are uniformly distributed on the interval $[0, 1]$. Workers face linear transportation costs so that the payoff of worker at location $z \in [0, 1]$ who works at location 0 for the wage w_0 is $w_0 - z$, while the payoff from working at location 1 for the wage w_1 is $w_1 - (1 - z)$.[41] The optimal procurement mechanism is characterized in Proposition OB.1. In addition to generating involuntary un-

[39] See, for example, Celis, Lewis, Mobius, and Nazerzadeh (2014), Appendix D in Larsen and Zhang (2018) and Section 5 in Larsen (2021), as well as the related discussion in Loertscher and Muir (2022).

[40] As the number of firms n and aggregate quantity Q increase may result in this quantity entering an ironing range. There may also be multiple ironing ranges.

[41] Note that regardless of a worker's location $z \in [0, 1]$, the expected transportation cost for a worker that either works at location 0 or 1 with equal probability is 1/2.

employment, the monopsony may also induce worker-job mismatches. Randomly matching any workers at $z \in [1/4, 3/4]$ to a job at location 0 or 1 with equal probability ensures the participation constraints of these workers ("generalists") binds. Relative to mechanisms that do not induce worker-job mismatches, this allows the monopsony to employ workers with $z < 1/4$ at 0 and workers with $z > 3/4$ at 1 ("specialists") at lower wages. The minimum wage effects in this model are similar to those in the model with homogeneous workers and jobs (see Proposition OB.2). However, in the Hotelling model outlined here, minimum wages can also reduce or eliminate worker-jobs mismatches, which creates additional scope for minimum wages to benefit both social and worker surplus.

6 Conclusions

We conclude with a short discussion of avenues for future research. First, beyond minimum wages, our model provides scope for analyzing the effects of prohibiting wage discrimination.[42] An open question is whether total employment, worker surplus and social surplus are larger with or without wage discrimination. Second, one could extend the baseline model to allow for vertically differentiated tasks, which gives rise to a model of multi-tasking based on price theory. The effects of task-specific minimum wages are not known to date. Third, one could analyze the effects of introducing unemployment insurance in models in which there is involuntary unemployment under the laissez-faire equilibrium. Finally, the analysis of this paper also naturally raises the question of what form optimal price regulation more generally takes when a monopoly or monopsony may engage in price discrimination.

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[42] In concurrent policy debates, there is pressure for greater transparency concerning the wages paid by employers. Policies imposing wage transparency may have similar effects to prohibiting wage discrimination. For empirical evidence concerning the effects of requiring wage transparency and a comprehensive list of recent references, see Cullen and Pakzad-Hurson (2023).

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A Omitted proofs

A.1 Proof of Theorem 1

Proof. This proof is divided into two parts. In the first part we prove that the minimum cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity Q under a minimum wage of $\underline{w}$ is given by (7). We then prove the stated properties of this cost function.

Part I: Proof that the minimal cost is $\underline{C}_R(Q, \underline{w})$ *as given in* (7)

This proof is largely contained in Section 3.2, so here we focus on elaborating on any omitted steps. First, we derive (5). Let $U(c) := x(c)(w(c) - c)$ denote the equilibrium payoff of a worker of type c under an arbitrary incentive compatible and individually rational direct mechanism. By the envelope theorem the incentive compatibility constraints are equivalent to requiring that x is non-increasing and that $U'(c) = -x(c)$ holds almost everywhere. For any $c, \hat{c} \in [\underline{c}, \overline{c}]$, this implies that $U(c) = U(\hat{c}) + \int_c^{\hat{c}} x(y)dy$. Applying the definition of $U(c)$, the expected transfer $w(c)x(c)$ paid to type c is then characterized by $w(c)x(c) = U(\hat{c}) + x(c)c + \int_c^{\hat{c}} x(y)\,dy$. For $c < \hat{c}$, we have $\int_c^{\hat{c}} x(y)\,dy \geq 0$ and, consequently, $U(c) \geq U(\hat{c})$. Thus, the individual rationality constraint is satisfied for each type if and only if $U(\overline{c}) \geq 0$. Moreover, under any *cost-minimizing* mechanism satisfying incentive compatibility and individual rationality, we must have $U(\overline{c}) = 0$. We can therefore write

$$w(c)x(c) = x(c)c + \int_c^{\overline{c}} x(y)dy. \tag{9}$$

Next, we show that it suffices to impose the constraint associated with the minimum wage on the lowest type $c = \underline{c}$. Notice that individual rationality implies that no worker can be paid a wage w that is less than their opportunity cost. Consequently, for workers with $c > \underline{w}$, the constraint never binds. Next, using the fact that the constraint $w(c) \geq \underline{w}$ is equivalent to $h(c) := x(c)(\underline{w} - w(c)) \leq 0$, we show that $h(c)$ decreases in c on $[\underline{c}, \underline{w}]$. Specifically, letting $c_0, c_1 \in [\underline{c}, \underline{w}]$ with $c_0 < c_1$, we have

$$h(c_1) - h(c_0) = (\underline{w} - c_1)(x(c_1) - x(c_0)) + \int_{c_0}^{c_1} x(y)\,dy - (c_1 - c_0)x(c_0) \leq 0,$$

where the inequality is strict if x is not constant on $[c_0, c_1]$.[43] Consequently, it suffices to impose the constraint associated with the minimum wage on the lowest type $c = \underline{c}$.

Let $\lambda \geq 0$ denote the Lagrange multiplier associated with the minimum wage constraint for type $c = \underline{c}$ and consider the corresponding dual problem. Since strong duality holds, the primal problem is convex and solving the dual problem yields a solution that is also primal feasible, the solution to the dual problem also solves the primal problem (see, for example, Theorem 2.165 in Bonnans and Shapiro, 2000 which extends the analogous results from linear programming from finite-dimensional vector spaces to Banach spaces). So from this point forward it is without loss of generality to focus on the dual problem given in (5). In particular, using (9), the Lagrange dual function corresponding to (3) is given by

$$\mathcal{L}(x, \lambda) = \int_{\underline{c}}^{\overline{c}} \left(x(c)c + \int_c^{\overline{c}} x(y)\,dy \right) dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}) - \lambda \int_{\underline{c}}^{\overline{c}} x(c)\,dc.$$

[43]Since x is non-increasing, if x is not constant on $[c_0, c_1]$ we have $(c_1 - c_0)x(c_0) > \int_{c_0}^{c_1} x(y)\,dy$ and $(\underline{w} - c_1)(x(c_1) - x(c_0)) \leq 0$ with strict inequality if $c_1 < \underline{w}$.

Using $\int_{\underline{c}}^{\overline{c}} \int_{c}^{\overline{c}} g(c)x(y)\,dy\,dc = \int_{\underline{c}}^{\overline{c}} \int_{\underline{c}}^{y} g(c)x(y)\,dc\,dy = \int_{\underline{c}}^{\overline{c}} G(y)x(y)\,dy$, we have

$$\mathcal{L}(x,\lambda) = \int_{\underline{c}}^{\overline{c}} \left(\Gamma(c) - \frac{\lambda}{g(c)}\right) x(c)\,dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}),$$

which completes our derivation of (5).

All remaining steps in the derivation of (7) are provided in Section 3.2, from (5) onward. The only remaining omitted step is to formally derive the corresponding convexification procedure. Whenever a two-wage mechanism is optimal under a binding minimum wage $\underline{w}$, instead of ironing the function Ψ with respect to the probability measure G_λ, we can compute the optimal mechanism by performing an appropriate convexification procedure. We accomplish this by rewriting the Lagrangian in terms of quantiles of the type distribution (or, equivalently, as an integral with respect to the uniform probability measure). We make the change of variables $z = G(c)$ and let $y = x \circ G^{-1}$. Note that we then have $W(z) = G^{-1}(z)$ and $C(z) = G^{-1}(z)z$, which implies that $W'(z) = \frac{1}{g(G^{-1}(z))}$ and $C'(z) = G^{-1}(z) + \frac{z}{g(G^{-1}(z))}$. The Lagrangian $\mathcal{L}(x,\lambda) = \int_{\underline{c}}^{\overline{c}} \left(c + \frac{G(c)}{g(c)} - \frac{\lambda}{g(c)}\right) x(c)\,dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c})$ therefore becomes

$$\begin{aligned}\mathcal{L}(y,\lambda) &= \int_0^1 \left(G^{-1}(z) + \frac{z}{g(G^{-1}(z))} - \frac{\lambda}{g(G^{-1}(z))}\right) y(z)\,dz + \lambda y(0)(\underline{w} - \underline{c}) \\ &= \int_0^1 \left(C'(z) - \lambda W'(z)\right) y(z)\,dz + \lambda y(0)(\underline{w} - \underline{c}).\end{aligned}$$

Integrating by parts then yields $\mathcal{L}(y,\lambda) = \left(C(1) - \lambda W(1)\right) y(1) - \left(C(0) - \lambda W(0)\right) y(0) + \int_0^1 \left(R(z) - \lambda P(z)\right)\left(-y'(z)\right) dz + \lambda y(0)(\underline{w} - \underline{c})$. Since the optimal mechanism is a two-wage mechanism, we can set $y(0) = 1$. Moreover, since $V(1) < W(1)$ and $\underline{C}(1) = C(1)$, it is without loss of generality to restrict attention to mechanisms such that $y(1) = 0$. Combining these observations with $C(0) = 0$ and $W(0) = \underline{c}$, yields $\mathcal{L}(y,\lambda) = \int_0^1 \left(C(z) - \lambda W(z)\right)\left(-y'(z)\right) dz + \lambda \underline{w}$. The designer's full problem becomes

$$\begin{aligned}\max_{\lambda \geq 0} \min_{y(\cdot)} &\left\{ \int_0^1 \left(C(z) - \lambda W(z)\right)\left(-y'(z)\right) dz + \lambda \underline{w} \right\} \\ \text{s.t.} &\int_0^1 y(z)\,dz = Q, \quad y \text{ non-increasing.}\end{aligned}$$

Solving the inner minimization problem then yields $\mathrm{co}\left(C - \lambda W\right)(Q) + \lambda \underline{w}$, where $\mathrm{co}(C - \lambda W)$ denotes the convexification of the function $C(z) - \lambda W(z)$ on $z \in [0,1]$. Solving the outer maximization problem, the optimal value λ^* of the Lagrange multiplier is pinned down by the first-order condition $-\frac{d}{d\lambda}\left(\mathrm{co}(C - \lambda W)(Q)\right)\big|_{\lambda=\lambda^*} = \underline{w}$ as required.

Part II: Proof of the stated properties of $\underline{C}_R$

Given a minimum wage $\underline{w}$, if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$, then m is fixed. For the remainder of this proof we omit the dependence of $Q_i(m)$ on m and simply write Q_i.

Preliminaries. Before proving the stated properties of $\underline{C}_R$, we first need to provide a more detailed characterization of the optimal mechanism when $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$. In such cases, the monopsony solves $\min_{q_1 \in [0,Q), q_2 > Q}(1-\beta)C(q_1) + \beta C(q_2)$, where $\beta = \frac{Q-q_1}{q_2-q_1}$, subject to the constraint $(1-\beta)W(q_1) + \beta W(q_2) \geq \underline{w}$. The corresponding Lagrangian is $\mathcal{L}(q_1, q_2, \lambda) = (1-\beta)C(q_1) + \beta C(q_2) - \lambda[(1-\beta)W(q_1) + \beta W(q_2) - \underline{w}]$, where λ is the Lagrange multiplier associated with the minimum wage constraint. For $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ the constraint will bind (i.e. hold with equality at an optimum) and, consequently, $\lambda > 0$. Using $C_\lambda(Q) := W(Q)(Q - \lambda)$, the Lagrangian can equivalently be written as $\mathcal{L}(q_1, q_2, \lambda) = (1-\beta)C_\lambda(q_1) + \beta C_\lambda(q_2) + \lambda \underline{w}$. Using $\frac{\partial \beta}{\partial q_1} = -\frac{1-\beta}{q_2-q_1}$ and $\frac{\partial \beta}{\partial q_2} = -\frac{\beta}{q_2-q_1}$, the first-order conditions with respect to q_1 and q_2 are

$$C'_\lambda(q_1) = \frac{C_\lambda(q_2) - C_\lambda(q_1)}{q_2 - q_1} = C'_\lambda(q_2), \tag{10}$$

while the first-order condition with respect to λ is

$$(1-\beta)W(q_1) + \beta W(q_2) = \underline{w}. \tag{11}$$

We next introduce $H(q_2, q_1, \lambda) := \frac{C_\lambda(q_2) - C_\lambda(q_1)}{q_2 - q_1} > 0$, where the inequality holds because we have $q_2 > q_1$ by assumption and C is a strictly increasing function. Using subscripts to denote partial derivatives, we have $H_1(q_2, q_1, \lambda) = \frac{1}{q_2-q_1}[C'_\lambda(q_2) - H(q_2, q_1, \lambda)]$, $H_2(q_2, q_1, \lambda) = \frac{1}{q_2-q_1}[H(q_2, q_1, \lambda) - C'_\lambda(q_1)]$ and $H_3(q_2, q_1, \lambda) = \frac{W(q_1)-W(q_2)}{q_2-q_1}$. Note that $H_3 < 0$ holds because $q_2 > q_1$ and W is an increasing function. Observe also that (10) is equivalent to $C'_\lambda(q_1) = H(q_2, q_1, \lambda) = C'_\lambda(q_2)$. Let $\tilde{q}_1(\lambda)$ and $\tilde{q}_2(\lambda)$ denote the values of q_1 and q_2 that satisfy this first-order condition. By construction we have $H_1(\tilde{q}_2(\lambda), \tilde{q}_1(\lambda), \lambda) = H_2(\tilde{q}_2(\lambda), q_1(\lambda), \lambda) = 0$ and the corresponding Hessian matrix is

$$\begin{pmatrix} \frac{\partial^2 \mathcal{L}(\tilde{q}_1, \tilde{q}_2, \lambda^*)}{\partial q_1^2} & \frac{\partial^2 \mathcal{L}(\tilde{q}_1, \tilde{q}_2, \lambda^*)}{\partial q_1 \partial q_2} \\ \frac{\partial^2 \mathcal{L}(\tilde{q}_1, \tilde{q}_2, \lambda^*)}{\partial q_2 \partial q_1} & \frac{\partial^2 \mathcal{L}(\tilde{q}_1, \tilde{q}_2, \lambda^*)}{\partial q_2^2} \end{pmatrix} = \begin{pmatrix} (1-\beta)C''_\lambda(\tilde{q}_1) & 0 \\ 0 & \beta C''_\lambda(\tilde{q}_2) \end{pmatrix}.$$

This is positive definite if and only if $(1-\beta)C''_\lambda(\tilde{q}_1) > 0$ and $\beta C''_\lambda(\tilde{q}_2) > 0$. Thus, for $i \in \{1, 2\}$, we have $C''_\lambda(\tilde{q}_i) > 0$. Totally differentiating $C'_\lambda(\tilde{q}_i) = H(\tilde{q}_2, \tilde{q}_1, \lambda)$ with respect to λ and using $H_1(\tilde{q}_2, \tilde{q}_1, \lambda) = 0 = H_2(\tilde{q}_2, \tilde{q}_1, \lambda)$ yields $\frac{d\tilde{q}_i}{d\lambda} = \frac{H_3(\tilde{q}_2, \tilde{q}_1, \lambda) + W'(\tilde{q}_i)}{C''_\lambda(\tilde{q}_i)}$. Since $C''_\lambda(\tilde{q}_i) > 0$, $\frac{d\tilde{q}_i}{d\lambda}$ has the same sign as $H_3(\tilde{q}_2, \tilde{q}_1, \lambda) + W'(\tilde{q}_i) = \frac{W(\tilde{q}_1) - W(\tilde{q}_2)}{\tilde{q}_2 - \tilde{q}_1} + W'(\tilde{q}_i)$. We now show that this expression is positive for $i = 1$ and negative for $i = 2$.

To that end, notice that for $\tilde{q}_1 < \tilde{q}_2$ and $Q \in (\tilde{q}_1, \tilde{q}_2)$, $C_\lambda(Q)$ is not convex. That is, for all $Q \in (\tilde{q}_1, \tilde{q}_2)$ we have $\underline{C}_\lambda(Q) < C_\lambda(Q)$. Otherwise, there would be no need to convexify $C_\lambda(Q)$. We now show that this implies that $W(Q)$ is not convex on $[\tilde{q}_1, \tilde{q}_2]$ by showing that convexity of W implies convexity of C_λ. In particular, for $a \in [0, 1]$ and Q_A and Q_B satisfying $\tilde{q}_1 \leq Q_A < Q_B \leq \tilde{q}_2$, define $Q^a := aQ_A + (1-a)Q_B$. Convexity of W on $[\tilde{q}_1, \tilde{q}_2]$ means that $W(Q^a) \leq aW(Q_A) + (1-a)W(Q_B)$. Using the definition of C_λ, we have $C_\lambda(Q^a) = W(Q^a)(Q^a - \lambda)$. Convexity of W then implies that

$$\begin{aligned} C_\lambda(Q^a) &\leq (aW(Q_A) + (1-a)W(Q_B))\,(aQ_A + (1-a)Q_B - \lambda) \\ &= aC_\lambda(Q_A) + (1-a)C_\lambda(Q_B) + a(1-a)(W(Q_B) - W(Q_A))(Q_A - Q_B) \\ &\leq aC_\lambda(Q_A) + (1-a)C_\lambda(Q_B). \end{aligned}$$

Here, the second inequality follows from $W(Q_B) - W(Q_A) > 0$ and $Q_A - Q_B < 0$ (which also implies that the inequality is strict if $a \in (0, 1)$). Thus, we have that convexity of W implies convexity of C_λ. However, since we know that C_λ fails to be convex on $[\tilde{q}_1, \tilde{q}_2]$, we then have that $W(Q)$ is not convex on $[\tilde{q}_1, \tilde{q}_2]$. That is, for all $Q \in (\tilde{q}_1, \tilde{q}_2)$, $W(Q) > W(\tilde{q}_1) + (Q - \tilde{q}_1)\frac{W(\tilde{q}_2) - W(\tilde{q}_1)}{\tilde{q}_2 - \tilde{q}_1}$. Finally, because $W(Q)$ intersects with the linear function $W(\tilde{q}_1) + (Q - \tilde{q}_1)\frac{W(\tilde{q}_2) - W(\tilde{q}_1)}{\tilde{q}_2 - \tilde{q}_1}$ at $Q = \tilde{q}_2$ from above, it follows that the slope of W at that point is smaller than $\frac{W(\tilde{q}_2) - W(\tilde{q}_1)}{\tilde{q}_2 - \tilde{q}_1}$. Consequently, we have $W'(\tilde{q}_2) < \frac{W(\tilde{q}_2) - W(\tilde{q}_1)}{\tilde{q}_2 - \tilde{q}_1}$, which is equivalent to $\frac{W(\tilde{q}_1) - W(\tilde{q}_2)}{\tilde{q}_2 - \tilde{q}_1} + W'(\tilde{q}_2) = H_3(\tilde{q}_2, \tilde{q}_1, \lambda) + W'(\tilde{q}_2) < 0$. This implies $\frac{d\tilde{q}_2(\lambda)}{d\lambda} < 0$. By the same token, $W(Q)$ intersects with the linear function $W(\tilde{q}_1) + (Q - \tilde{q}_1)\frac{W(\tilde{q}_2) - W(\tilde{q}_1)}{\tilde{q}_2 - \tilde{q}_1}$ at $Q = \tilde{q}_1$ from below. This implies that $W(\tilde{q}_1) + (\tilde{q}_2 - \tilde{q}_1)W'(\tilde{q}_1) > W(\tilde{q}_2)$, which is equivalent to $\frac{W(\tilde{q}_1) - W(\tilde{q}_2)}{\tilde{q}_2 - \tilde{q}_1} + W'(\tilde{q}_1) = H_3(\tilde{q}_2, \tilde{q}_1, \lambda) + W'(\tilde{q}_1) > 0$, implying that $\frac{d\tilde{q}_1(\lambda)}{d\lambda} > 0$.

Establishing the comparative statics properties of $\lambda^*(Q, \underline{w})$ with respect to Q and $\underline{w}$, will yield the comparatives statics properties of $q_i^*(Q, \underline{w})$ with respect to Q and $\underline{w}$ using $q_i^*(Q, \underline{w}) = \tilde{q}_i(\lambda^*(Q, \underline{w}))$ and $\frac{dq_1^*(\lambda)}{d\lambda} > 0 > \frac{dq_2^*(\lambda)}{d\lambda}$. Using (11) and totally differentiating $(1 - \beta^*)W(\tilde{q}_1) + \beta^* W(\tilde{q}_2) = \underline{w}$ with respect to $\underline{w}$, where $\beta^* = \frac{Q - \tilde{q}_1}{\tilde{q}_2 - \tilde{q}_1}$ and where we have dropped dependence on λ^* for notational brevity, yields

$$\left\{(1 - \beta^*)\frac{d\tilde{q}_1}{d\lambda}\left(W'(\tilde{q}_1(\lambda)) + H_3\right) + \beta^* \frac{d\tilde{q}_2}{d\lambda}\left(W'(\tilde{q}_2(\lambda)) + H_3\right)\right\}\frac{\partial \lambda^*}{\partial \underline{w}} = 1.$$

Thus, $\frac{\partial \lambda^*}{\partial \underline{w}} > 0$ if the term in brackets is positive, which is the case if both summands are positive. To see that the second summand is positive, recall that $\frac{d\tilde{q}_2}{d\lambda} < 0$ and $W'(\tilde{q}_2(\lambda)) + \frac{W(\tilde{q}_1) - W(\tilde{q}_2)}{\tilde{q}_2 - \tilde{q}_1} < 0$. To see that the first summand is positive, it suffices to recall that $\frac{d\tilde{q}_1}{d\lambda} > 0$

and that $W'(\tilde{q}_1) + \frac{W(\tilde{q}_1)-W(\tilde{q}_2)}{\tilde{q}_2-\tilde{q}_1} > 0$. Hence, $\frac{\partial\lambda^*}{\partial\underline{w}} > 0$ holds.

Since $\frac{\partial q_i^*(Q,\underline{w})}{\partial\underline{w}} = \frac{d\tilde{q}_i(\lambda)}{d\lambda}\frac{\partial\lambda^*(Q,\underline{w})}{\partial\underline{w}}$, it follows that

$$\frac{\partial q_1^*(Q,\underline{w})}{\partial\underline{w}} > 0 > \frac{\partial q_2^*(Q,\underline{w})}{\partial\underline{w}}. \tag{12}$$

Similarly, totally differentiating $(1-\beta^*)W(\tilde{q}_1) + \beta^* W(\tilde{q}_2) = \underline{w}$ with respect to Q yields

$$\left\{(1-\beta^*)\frac{d\tilde{q}_1}{d\lambda}\left(W'(\tilde{q}_1(\lambda^*)) + H_3\right) + \beta^*\frac{d\tilde{q}_2}{d\lambda}\left(W'(\tilde{q}_2(\lambda^*)) + H_3\right)\right\}\frac{\partial\lambda^*}{\partial Q} = H_3.$$

Since the right-hand side is negative and the term in brackets on the left-hand side is, as just shown, positive, it follows that $\frac{\partial\lambda^*}{\partial Q} < 0$, implying

$$\frac{\partial q_1^*(Q,\underline{w})}{\partial Q} < 0 < \frac{\partial q_2^*(Q,\underline{w})}{\partial Q}. \tag{13}$$

Note that we also have $\frac{\partial\lambda^*}{\partial Q} = H_3\frac{\partial\lambda^*}{\partial\underline{w}}$. We are now ready to prove the stated properties of the function $\underline{C}_R$.

Convexity. Let the minimum wage $\underline{w}$ be given. We start by showing that the function $\underline{C}_R(\cdot,\underline{w})$ is convex. As noted in Theorem 1, this implies that $\underline{C}_R(\cdot,\underline{w})$ is continuous in Q.

Take any two points $Q_A, Q_B \in [0,1]$. Then we need to show that for any $a \in [0,1]$ we have $\underline{C}_R(aQ_A + (1-a)Q_B,\underline{w}) \geq a\underline{C}_R(Q_A,\underline{w}) + (1-a)\underline{C}_R(Q_B,\underline{w})$. It suffices to show that there exists an incentive compatible and ex post individually rational procurement mechanism that procures the quantity $aQ_A + (1-a)Q_B$ at a cost of $a\underline{C}_R(Q_A,\underline{w}) + (1-a)\underline{C}_R(Q_B,\underline{w})$ without violating the minimum wage constraint. Let x_A (x_B) denote the allocation rule and t_A (t_B) denote the payment rule of the incentive compatible and individually rational procurement mechanism that procures the quantity Q_A (Q_B) at the minimal cost $\underline{C}_R(Q_A,\underline{w})$ ($\underline{C}_R(Q_B,\underline{w})$). Now consider the allocation rule x given by $x := \alpha x_A + (1-\alpha)x_B$ and the wage schedule w that implements this allocation at the minimal cost. Since the weighted sum of two increasing function is also an increasing function, the allocation rule x can be implemented using an incentive compatible and ex post individually rational procurement mechanism. By construction, this mechanism procures the quantity $aQ_A + (1-a)Q_B$ at a cost of

$$\begin{aligned}\int_{\underline{c}}^{\overline{c}} \Gamma(c)x(c)\,dG(c) &= a\int_{\underline{c}}^{\overline{c}} \Gamma(c)x_A(c)\,dG(c) + (1-a)\int_{\underline{c}}^{\overline{c}} \Gamma(c)x_B(c)\,dG(c)\\ &= a\underline{C}_R(Q_A,\underline{w}) + (1-a)\underline{C}_R(Q_B,\underline{w})\end{aligned}$$

as required. This last expression also shows that $w(c)x(c) = aw_A(c)x_A(c)+(1-a)w_B(c)x_B(c)$ holds for all $c \in [\underline{c}, \overline{c}]$. It only remains to verify that the mechanism $\langle x, w\rangle$ does not violate the minimum wage constraint. Since the wage schedules w_A and w_B satisfy the minimum wage constraint, for all $c \in [\underline{c}, \overline{c}]$, we have

$$w(c)x(c) = aw_A(c)x_A(c) + (1-a)w_B(c)x_B(c) \geq a\underline{w}x_A(c) + (1-a)\underline{w}x_B(c) = \underline{w}x(c).$$

Thus, the wage schedule w satisfies the minimum wage constraint as required.

Monotonicity. Clearly, $\underline{C}_R(Q, \underline{w})$ is increasing in both Q and $\underline{w}$ on $Q \notin (S(\underline{w}), w_1^{-1}(\underline{w}))$. It remains to show that $\underline{C}_R(Q, \underline{w}) = \mathcal{D}^*(Q, \underline{w})$ is increasing in both Q and $\underline{w}$ on $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$. Since $\underline{C}_R$ is continuous in both Q and $\underline{w}$, this establishes that $\underline{C}_R$ is everywhere increasing in both Q and $\underline{w}$, as required. By construction, we have $\mathcal{D}^*(Q, \underline{w}) = (1-\beta^*)C_{\lambda^*}(q_1^*) + \beta^* C_{\lambda^*}(q_2^*) + \lambda^*\underline{w}$, where $\lambda^* = \lambda^*(Q, \underline{w})$, $q_i^* = q_i^*(Q, \underline{w})$ and $\beta^* = \frac{Q-q_1^*}{q_2^*-q_1^*}$.[44] Moreover, by the envelope theorem we have

$$\frac{\partial \mathcal{D}^*(Q, \underline{w})}{\partial \underline{w}} = \lambda^* > 0 \quad \text{and} \quad \frac{\partial \mathcal{D}^*(Q, \underline{w})}{\partial Q} = H(q_2^*, q_1^*, \lambda^*) > 0, \tag{14}$$

which establishes the required monotonicity properties.

Marginal cost properties. The marginal cost function $\underline{C}'_R(\cdot, \underline{w})$ is given by the left derivative of $\underline{C}_R(\cdot, \underline{w})$ with respect to Q. Since $\underline{C}_R(\cdot, \underline{w})$ is convex in Q it is almost everywhere differentiable in Q and admits left and right derivatives on its entire domain. Consequently, $\underline{C}'_R$ is a well-defined. Clearly, $\underline{C}'_R$ is continuous on $(Q, \underline{w}) \in [0,1] \times [W(0), W(1)]$ with $Q \neq S(\underline{w})$ and $Q \neq w_1^{-1}(\underline{w})$.[45] However, it remains to show that $\underline{C}'_R$ is continuous at $Q = w_1^{-1}(\underline{w})$. To that end, notice that $q_i^*(0) = Q_i$, and satisfying (11) then requires that $\underline{w} = w_1(Q)$. Consequently, $\lambda^*(Q, \underline{w}) \downarrow 0$ and $q_i^*(Q, \underline{w}) \to Q_i$ as $Q \uparrow w_1^{-1}(\underline{w})$. Since the parameters of the optimal two-wage mechanism are continuous at $Q = w_1^{-1}(\underline{w})$, it follows that $\underline{C}'_R$ is also continuous.

We next show that for $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w})\right)$, $\underline{C}'_R$ is bounded and $\frac{\partial \underline{C}'_R(Q,\underline{w})}{\partial Q} > 0 > \frac{\partial \underline{C}'_R(Q,\underline{w})}{\partial \underline{w}}$. Starting from (14) and taking the derivative with respect to Q once more yields $\frac{\partial \underline{C}'_R(Q,\underline{w})}{\partial \underline{w}} = \frac{\partial^2 \mathcal{D}^*(Q,\underline{w})}{\partial \underline{w} \partial Q} = \frac{\partial \lambda^*}{\partial Q} = H_3(q_2^*, q_1^*, \lambda^*)\frac{\partial \lambda^*}{\partial \underline{w}} < 0$ and $\frac{\partial \underline{C}'_R(Q,\underline{w})}{\partial Q} = \frac{\partial^2 \mathcal{D}^*(Q,\underline{w})}{\partial Q^2} = H_3(q_2^*, q_1^*, \lambda^*)\frac{\partial \lambda^*}{\partial Q} > 0$, where the inequalities follows from $\frac{\partial \lambda^*}{\partial \underline{w}} > 0 > H_3(q_2^*, q_1^*, \lambda^*)$. That $\underline{C}'_R$ is bounded on $Q \in \left(S(\underline{w}), w_1^{-1}(\underline{w})\right)$ follows from the fact that $\underline{C}_R(\cdot, \underline{w})$ is convex in Q on $Q \in [0,1]$.

[44] Note that at $Q = w_1^{-1}(\underline{w})$ we have $\lambda^* = 0$ and $\mathcal{D}^*(Q, \underline{w}) = \underline{C}(Q)$ at $\underline{w} = w_1(Q)$. Likewise, at $Q = S(\underline{w})$ we have $\lambda^* = Q$, which implies that $C_{\lambda^*}(Q) = 0$ and $\mathcal{D}^*(Q, \underline{w}) = \underline{w}Q$.

[45] This follows from the continuity of the functions $\underline{w}$ and $\underline{C}'$, as well as the dual solution value $\mathcal{D}^*(Q, \underline{w})$.

Connection to ex ante implementation. Since the minimum wage constraint is tighter under ex ante implementation, we must have $\underline{C}_R(Q, \underline{w}) \geq \underline{C}_U(Q, \underline{w})$. The example illustrated in Panel (a) of Figure 2 shows that $\underline{C}_R(Q, \underline{w}) \neq \underline{C}_U(Q, \underline{w})$ does not hold in general. It remains to show that cost-minimizing procurement involves two wages and involuntary unemployment if and only if $\underline{C}_U(Q, \underline{w}) < C_U(Q, \underline{w})$. This is trivially true if $\underline{w} \notin (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ (as in this case $\underline{C}_R(Q, \underline{w}) = \underline{C}_U(Q, \underline{w})$). So suppose that $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$. In this case the result is then true because a two-wage mechanism is always used to procure the non-trivial quantities $Q \in (S(\underline{w}), Q_2(m))$ under both cost-minimizing procurement and cost-minimizing procurement that is restricted to ex ante implementation. □

A.2 Proof of Lemma 1

Proof. See (12) and (13) in proof of Theorem 1. □

A.3 Proof of Theorem 2

Proof. As discussed in the body of the paper, Theorem 2 follows immediately from Theorem 1, Lemma 1 and Lemma 2. □

A.4 Proof of Lemma 2

Proof. $Q^*(\underline{w})$ satisfies $V(Q^*(\underline{w})) = H(q_2^*, q_1^*, \lambda^*)$, where $H(q_2^*, q_1^*, \lambda^*)$ is the marginal cost of procurement derived in the proof of Theorem 1. Totally differentiating yields $\frac{dQ^*(\underline{w})}{d\underline{w}} = \frac{H_3}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}} \frac{\partial \lambda^*}{\partial \underline{w}} > 0$, where the inequality holds because $V' \leq 0$, $H_3 < 0$ and $\frac{d\lambda^*}{dQ} < 0 < \frac{d\lambda^*}{d\underline{w}}$.

The following inequalities, which have been established in the proof of Theorem 1 (see pages 39 and 40), will be used throughout: $\frac{\partial q_2^*(\lambda)}{\partial \lambda} < 0 < \frac{\partial q_1^*(\lambda)}{\partial \lambda}$ and $\frac{\partial \lambda^*}{\partial \underline{w}} > 0$.

Using $q_2^*(Q, \underline{w}) = q_2^*(\lambda^*(Q, \underline{w}))$ and totally differentiating q_2^* with respect to $\underline{w}$ yields $\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} = \frac{\partial q_2^*}{\partial \lambda} \left[\frac{\partial \lambda^*}{\partial Q} \frac{\partial Q^*(\underline{w})}{\partial \underline{w}} + \frac{\partial \lambda^*}{\partial \underline{w}} \right] = \frac{\partial q_2^*}{\partial \lambda} \frac{\partial \lambda^*}{\partial \underline{w}} \left[H_3 \frac{\partial Q^*(\underline{w})}{\partial \underline{w}} + 1 \right]$. Here, the second equality follows from $\frac{\partial \lambda^*}{\partial Q} = H_3 \frac{\partial \lambda^*}{\partial \underline{w}}$. Substituting $\frac{dQ^*(\underline{w})}{d\underline{w}} = \frac{H_3 \frac{\partial \lambda^*}{\partial \underline{w}}}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}}$ into this last expression yields $\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} = \frac{\partial q_2^*}{\partial \lambda} \frac{\partial \lambda^*}{\partial \underline{w}} \left[\frac{(H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}}}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}} + 1 \right] = \frac{\partial q_2^*}{\partial \lambda} \frac{\partial \lambda^*}{\partial \underline{w}} \left[\frac{(H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}} + V' - H_3 \frac{\partial \lambda^*}{\partial Q}}{V' - H_3 \frac{\partial \lambda^*}{\partial Q}} \right]$. Since $\frac{\partial q_2^*}{\partial \lambda} < 0 < \frac{\partial \lambda^*}{\partial \underline{w}}$, $\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} < 0$ holds if the term in brackets is positive and $\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} = 0$ holds if the term in brackets is 0. To see that the term in brackets is non-negative, we can again substitute $\frac{\partial \lambda^*}{\partial Q} = H_3 \frac{\partial \lambda^*}{\partial \underline{w}}$ to obtain $\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} = \frac{\partial q_2^*}{\partial \lambda} \frac{\partial \lambda^*}{\partial \underline{w}} \left[\frac{V'}{V' - (H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}}} \right]$. Since $V' \leq 0$ and $V' - (H_3)^2 \frac{\partial \lambda^*}{\partial \underline{w}} < 0$, we have $\frac{dq_2^*(\lambda^*(Q^*(\underline{w}), \underline{w}))}{d\underline{w}} \leq 0$ as required. This last inequality is strict if and only if $V' < 0$ at $Q^*(\underline{w})$.

Similarly, using $q_1^*(Q, \underline{w}) = q_1^*(\lambda^*(Q, \underline{w}))$ and totally differentiating q_1^* with respect to $\underline{w}$ yields $\frac{dq_1^*(\lambda^*(Q^*(\underline{w}),\underline{w}))}{d\underline{w}} = \frac{\partial q_1^*}{\partial \lambda}\left[\frac{\partial \lambda^*}{\partial Q}\frac{\partial Q^*(\underline{w})}{\partial \underline{w}} + \frac{\partial \lambda^*}{\partial \underline{w}}\right] = \frac{\partial q_1^*}{\partial \lambda}\frac{\partial \lambda^*}{\partial \underline{w}}\left[H_3\frac{\partial Q^*(\underline{w})}{\partial \underline{w}} + 1\right]$, where the second equality follows from $\frac{\partial \lambda^*}{\partial Q} = H_3\frac{\partial \lambda^*}{\partial \underline{w}}$. Substituting $\frac{dQ^*(\underline{w})}{d\underline{w}} = \frac{H_3\frac{\partial \lambda^*}{\partial \underline{w}}}{V' - H_3\frac{\partial \lambda^*}{\partial Q}}$ into this last expression yields $\frac{dq_1^*(\lambda^*(Q^*(\underline{w}),\underline{w}))}{d\underline{w}} = \frac{\partial q_1^*}{\partial \lambda}\frac{\partial \lambda^*}{\partial \underline{w}}\left[\frac{(H_3)^2\frac{\partial \lambda^*}{\partial \underline{w}}}{V' - H_3\frac{\partial \lambda^*}{\partial Q}} + 1\right] = \frac{\partial q_1^*}{\partial \lambda}\frac{\partial \lambda^*}{\partial \underline{w}}\left[\frac{(H_3)^2\frac{\partial \lambda^*}{\partial \underline{w}} + V' - H_3\frac{\partial \lambda^*}{\partial Q}}{V' - H_3\frac{\partial \lambda^*}{\partial Q}}\right]$. Since $\frac{\partial q_1^*}{\partial \lambda} > 0$ and $\frac{\partial \lambda^*}{\partial \underline{w}} > 0$, $\frac{dq_1^*(\lambda^*(Q^*(\underline{w}),\underline{w}))}{d\underline{w}}$ has the same sign as the term in brackets. As shown above, this sign is positive if $V' < 0$ and 0 if $V' = 0$. Thus, we have $\frac{dq_1^*(\lambda^*(Q^*(\underline{w}),\underline{w}))}{d\underline{w}} \geq 0$ with strict inequality if and only if $V' < 0$ at $Q^*(\underline{w})$ as required. □

A.5 Proof of Proposition 1

Proof. We begin this proof by showing that $Q^p > Q^\ell$ holds. Note that for all $Q > 0$, we have $C'(Q) = W'(Q)Q + W(Q) > W(Q)$. Consequently, whenever $C(Q) = \underline{C}(Q)$, we have $\underline{C}'(Q) > W(Q)$. Moreover, for all $m \in \mathcal{M}$ and $Q \in (Q_1(m), Q_2(m))$, we also have $\underline{C}'(Q) = C'(Q_2(m)) > W(Q_2(m)) > W(Q)$. Combining $\underline{C}'(Q) > W(Q)$ with the optimality condition $V(Q^\ell) = \underline{C}'(Q^\ell)$ shows that $V(Q^\ell) > W(Q^\ell)$. Since V is non-increasing, W is strictly increasing and Q^p satisfies $V(Q^p) = W(Q^p)$, $Q^p > Q^\ell$ follows as required.

Now consider introducing a minimum wage of $\underline{w} = W(Q^p)$. The monopsony will then optimally hire at least Q^p workers because the marginal benefit $V(Q) \geq V(Q^p)$ of hiring $Q < Q^p$ workers always exceeds the marginal cost $\underline{w} = V(Q^p)$. Theorem 1 establishes that the marginal cost of hiring Q workers under optimal procurement with a minimum wage of $\underline{w}$ is increasing in Q and strictly exceeds $V(Q)$ for $Q > Q^p$. Consequently, the monopsony will optimally employ the efficient quantity $Q^p > Q^\ell$ of workers under a minimum wage of $\underline{w} = W(Q^p)$. Moreover, the monopsony will optimally procure these workers by setting a market-clearing wage of $W(Q^p)$ (see Theorem 1), thereby eliminating both involuntary unemployment and wage dispersion. Theorem 1 also establishes that the minimal cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity Q under a minimum wage of $\underline{w}$ is increasing in both Q and $\underline{w}$. This implies that, relative to the laissez-faire equilibrium, imposing a minimum wage of $\underline{w} = W(Q^p)$ increases workers' total pay. Since $W(Q) > V(Q)$ holds for all $Q > Q^p$, no minimum wage can induce the monopsony to hire more than the efficient quantity Q^p. Thus, setting a minimum wage of $\underline{w} = W(Q^p)$ maximizes total employment. Moreover, social surplus is maximized when the monopsony hires the efficient quantity of workers under a market-clearing wage, which is precisely what is achieved by setting $\underline{w} = W(Q^p)$. This establishes each statement of Proposition 1 concerning a minimum wage of $\underline{w} = W(Q^p)$.

Next, consider introducing a minimum wage of $\underline{w} = W(Q_2(m))$. The marginal cost of hiring $Q \leq Q_2(m)$ workers is then $W(Q_2(m))$, while the marginal cost of hiring $Q > Q_2(m)$ workers is $\underline{C}'(Q)$ (see Theorem 1). By assumption, $Q^\ell \in (Q_1(m), Q_2(m))$ holds for some $m \in \mathcal{M}$ and we therefore have $\underline{C}'(Q_2(m)) = \underline{C}'(Q^\ell) = V(Q^\ell)$. Consequently, for all $Q > Q_2(m)$ we have $Q > Q^\ell$ and $\underline{C}'(Q) > \underline{C}'(Q_2(m)) = V(Q^\ell) \geq V(Q)$. This establishes that the monopsony will not hire more than $Q_2(m)$ workers. In the first paragraph of this proof we also established that $\underline{C}'(Q_2(m)) > W(Q_2(m))$, implying that $V(Q^\ell) > W(Q_2(m))$. This implies that the monopsony will hire strictly more than Q^ℓ workers under a minimum wage of $\underline{w} = W(Q_2(m))$. Consequently, if $Q^p \geq Q_2(m)$, then the monopsony will hire precisely $Q_2(m)$ workers and involuntary unemployment is eliminated. If $Q^p < Q_2(m)$ then the monopsony will hire $D(\underline{w}) \in (Q^\ell, Q^p)$ workers, rationing these workers at the minimum wage. However, involuntary unemployment will be lower relative to the laissez-faire equilibrium since the total number of workers who participate is $Q_2(m)$ in both cases but $D(\underline{w}) > Q^\ell$ workers are hired under this minimum wage. Repeating our previous argument for a minimum wage of $\underline{w} = W(Q^p)$ shows that introducing a minimum wage of $\underline{w} = W(Q_2(m))$ also increases workers' total pay relative to the laissez-faire equilibrium.

Next, we consider introducing a minimum wage $\underline{w} \in (w_1(Q^\ell), W(Q_2(m)))$. Then combining our arguments here with the results of Theorem 2 and Theorem 3 shows that employment increases in $\underline{w}$ on $\underline{w} < W(Q^p)$, is maximized at $\underline{w} = W(Q^p)$ and decreases in $\underline{w}$ on $\underline{w} > W(Q^p)$. As just argued, relative to the laissez-faire equilibrium, employment is higher under a minimum wage of $\underline{w} = W(Q_2(m))$. This implies that employment is higher under any minimum wage $\underline{w} \in (w_1(Q^\ell), W(Q_2(m)))$. To show that introducing a minimum wage of $\underline{w} \in (w_1(Q^\ell), W(Q_2(m)))$ increases workers' total pay relative to the laissez-faire equilibrium, we can again repeat our previous argument for the case where $\underline{w} = W(Q^p)$.

It remains to show that introducing a minimum wage of $\underline{w} \in (w_1(Q^\ell), W(Q_2(m)))$ also decreases involuntary unemployment relative to the laissez-faire equilibrium. The proof of Theorem 1 shows that the optimal mechanism is either a two-wage mechanism or it involves rationing $S(\underline{w})$ workers at the minimum wage. In the latter case, involuntary unemployment decreases relative to the laissez-faire equilibrium since total employment increases and $S(\underline{w}) < Q_2(m)$ by assumption. Similarly, in the former case it suffices to show that the equilibrium mass of workers that participate in the mechanism decreases relative to the laissez-faire equilibrium. This is established in the proof of Case 1 of Theorem 2.

It now only remains to prove the final statement of the proposition. Any minimum wage $\underline{w}$ that eliminates involuntary unemployment is necessarily such that $\underline{w} \leq W(Q^p)$. Moreover, we know that employment increases in $\underline{w}$ on $\underline{w} < W(Q^p)$. Consequently, any minimum wage that eliminates involuntary unemployment necessarily increases total employment relative to the laissez-faire equilibrium, bringing it closer to the efficient level of Q^p. Such a minimum wage also eliminates the random, inefficient allocation that is associated with involuntary unemployment. Thus, any minimum wage that eliminates involuntary unemployment also increases social surplus relative to the laissez-faire equilibrium. $\square$

A.6 Proof of Proposition 2

Proof. We start by proving the first statement of the proposition. Suppose that there is involuntary unemployment and wage dispersion under a given minimum wage $\underline{w}$. Clearly, the proposition statement holds if $\underline{w}$ does not bind, so assume that a marginal increase in the minimum wage $\underline{w}$ results in a binding minimum wage. Case 1 of Theorem 2 then applies and there is wage dispersion and involuntary unemployment under a binding minimum wage of $\underline{w} + \varepsilon$, provided $\varepsilon > 0$ is sufficiently small. Since the lowest wage paid to workers is always equal to the minimum wage, this clearly increases under a marginal increase in $\underline{w}$ as required. That the high wage decreases under a marginal increase in $\underline{w}$ follows from Lemma 2. So it only remains to show that the average wage paid to workers also increases under a marginal increase in $\underline{w}$. Theorem 1 establishes that the minimal cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity Q under the minimum wage $\underline{w}$ is increasing in both $\underline{w}$ and Q and is convex in Q. Moreover, by Case 1 of Theorem 2, the equilibrium quantity $Q^*(\underline{w})$ of workers employed increases under a marginal increase in $\underline{w}$. Putting all of this together, we have

$$\frac{\underline{C}_R(Q^*(\underline{w}), \underline{w})}{Q^*(\underline{w})} \leq \frac{\underline{C}_R(Q^*(\underline{w}), \underline{w} + \varepsilon)}{Q^*(\underline{w})} \leq \frac{\underline{C}_R(Q^*(\underline{w} + \varepsilon), \underline{w} + \varepsilon)}{Q^*(\underline{w} + \varepsilon)}. \tag{15}$$

The first inequality in (15) follows from the fact that $\underline{C}_R$ is increasing in $\underline{w}$. To establish the second inequality, notice that the convexity of $\underline{C}_R$ in Q implies that, for all $\underline{w}$, the function $\frac{\underline{C}_R(Q,\underline{w}) - \underline{C}_R(0,\underline{w})}{Q}$ is increasing in Q. Combining this with the fact that $\underline{C}_R(0, \underline{w}) = 0$ holds for all $\underline{w}$, and that $Q^*(\underline{w})$ is increasing in $\underline{w}$, then yields the second inequality in (15). Thus, (15) establishes that the average wage paid to workers is increasing in $\underline{w}$ as required.

We now prove the second statement of Proposition 2. Suppose that there is no involuntary unemployment under a given minimum wage $\underline{w}$ and that $\underline{w} \neq W(Q^p)$. Then the effects of a marginal increase in $\underline{w}$ are described in Case 2 of Theorem 2, and the equilibrium quantity of employed workers increases. There are two possible subcases. First, if there is no wage dispersion or involuntary unemployment following the marginal increase in $\underline{w}$, then all workers are paid the minimum wage before and after this increase. Second, if the marginal increase in the minimum wage induces wage dispersion and involuntary unemployment, then some employed workers are paid the higher minimum wage, while others are paid an even higher efficiency wage. $\square$

Online Appendix to

"Optimal labor procurement under minimum wages and monopsony power"

by

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August 1, 2023

This online appendix to "Optimal labor procurement under minimum wages and monopsony power" extends the analysis of the paper to accommodate, in turn, quantity competition among firms and horizontal differentiation of jobs and workers. Section OA deals with quantity competition while Section OB analyze a monopsony problem for a firm that has jobs at opposite ends of the Hotelling line. Section OC we introduce a piecewise linear parameterization of the function W and explore its properties. In Section OD we relate efficiency wages to migration and unemployment and revisit the introduction of the so-called \$5-day by the Ford motor company in 1914 from this perspective.

OA Quantity competition

A natural question is to what extent the effects identified for our monopsony model generalize to more competitive environments. To address this question, we now extend the model to allow for quantity competition between firms. This extension is not only in line with David Card's call for models of wage-setting with imperfect competition (Card, 2022b) but—since it relates to a Cournot-based setup—it also generalizes a framework that has proved productive for empirical analysis of market power in labor markets (Berger, Herkenhoff, and Mongey, 2022). We first introduce the setup, derive the equilibrium and discuss its properties. Then we analyze the effects of minimum wages.

OA.1 Setup

Suppose now that there are n firms procuring labor. We index these firms by i. For each firm i, the marginal value for procuring the y_i-th unit of labor is given by a continuously decreasing function $V(y_i)$ satisfying $V(0) > W(0)$ and $V(1) < W(1)$, where we use y_i to distinguish individual firms' quantities from the quantities q_1 and q_2 that are used in the main body. The firms compete in quantities as follows. They simultaneously submit quantities y_i to a Walrasian auctioneer as in standard oligopoly and oligopsony models with quantity competition. However, rather than procuring the $Q := \sum_{i=1}^n y_i$ units at the market-clearing wage $W(Q)$, which is the standard assumption in Cournot models and leads to a procurement cost function of C, we assume that the auctioneer can use the optimal procurement mechanism and thus procures the Q units at minimal total cost $\underline{C}(Q)$. Firm i who employs y_i units has to pay the cost $\frac{y_i}{Q}\underline{C}(Q)$. Modulo replacing the cost function C with $\underline{C}$, this is the same as in standard Cournot models since $\frac{y_i}{Q}\underline{C}(Q) = y_i W(Q)$ for all $Q \notin \bigcup_{m\in\mathcal{M}}(Q_1(m), Q_2(m))$. The efficient quantity for a given n is denoted by Q_n^p and is such that

$$V\left(\frac{Q_n^p}{n}\right) = W(Q_n^p).$$

This is the quantity that would emerge if the firms were price-takers.

OA.2 Equilibrium

The analysis from the main body extends to this model, insofar as we will have involuntary unemployment and wage dispersion whenever $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. In models in which market-clearing wages are imposed, the quantity in a symmetric equilibrium, denoted Q_n^C, satisfies

$$V\left(\frac{Q_n^C}{n}\right) = W(Q_n^C) + \frac{Q_n^C}{n}W'(Q_n^C), \tag{OA.1}$$

provided a symmetric equilibrium exists. Since $W' > 0$, we have $Q_n^C < Q_n^p$. That is, with market-clearing wages the equilibrium quantity is inefficiently small.

Let Q_n^* denote the aggregate quantity in a symmetric equilibrium under quantity competition when the quantity is procured at minimal cost and denote by Q^e the equilibrium quantity under perfect competition and price-taking behavior, that is, $Q^e = S(V(0))$.

Proposition OA.1. *The quantity setting game has a unique equilibrium, and this equilibrium is symmetric. The aggregate equilibrium quantity Q_n^* is increasing in n. If $Q_n^p \leq Q_n^*$, then $n > 1$ and $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. As $n \to \infty$, if $C(Q^e) = \underline{C}(Q^e)$, then we have $Q_n^* \to Q^e$ and if $Q^e \in (Q_1(m_e), Q_2(m_e))$ for some $m_e \in \mathcal{M}$, then we have $Q_n^* \to \tilde{Q}$, where $\tilde{Q} \in (Q^e, Q_2(m_e))$.*

Proof. Firm i's first-order condition is

$$V(y_i) = \frac{Q - y_i}{Q^2}\underline{C}(Q) + \frac{y_i}{Q}\underline{C}'(Q).$$

The left-hand side is decreasing in y_i. The partial derivative of the right-hand side with respect to y_i is $-\frac{1}{Q^2}(\underline{C}(Q) - Q\underline{C}'(Q))$, which is positive because $\underline{C}$ is convex. This implies that for any aggregate quantity Q there is a unique y_i that satisfies the first-order condition. This y_i must thus be the same for all i. Hence, any equilibrium is symmetric. Given this, we can write the first-order condition as

$$V\left(\frac{Q}{n}\right) = \frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q). \tag{OA.2}$$

The left-hand side is decreasing in Q. The derivative of the right-hand side with respect to Q is

$$-\frac{n-1}{nQ^2}\left(\underline{C}(Q) - Q\underline{C}'(Q)\right) + \frac{1}{n}\underline{C}''(Q) \geq 0. \tag{OA.3}$$

Here the inequality follows from the fact that $\underline{C}$ is convex, which in turn implies that $\underline{C}'' \geq 0$ and $Q\underline{C}'(Q) \geq \underline{C}(Q)$. Because at $Q = 0$, the left-hand side is larger than the right-hand side, there is a unique Q that satisfies (OA.2). This proves that the equilibrium is unique and symmetric.

To see that Q_n^* is increasing in n, suppose to the contrary that it is not and we have $Q_n^* \geq Q_{n+1}^*$ for some n. This implies $\frac{Q_n^*}{n} > \frac{Q_{n+1}^*}{n+1}$ and therefore

$$\begin{aligned} V\left(\frac{Q_{n+1}^*}{n+1}\right) > V\left(\frac{Q_n^*}{n}\right) &= \frac{n-1}{n}\frac{\underline{C}(Q_n^*)}{Q_n^*} + \frac{1}{n}\underline{C}'(Q_n^*) \\ &\geq \frac{n-1}{n}\frac{\underline{C}(Q_{n+1}^*)}{Q_{n+1}^*} + \frac{1}{n}\underline{C}'(Q_{n+1}^*) \\ &\geq \frac{n}{n+1}\frac{\underline{C}(Q_{n+1}^*)}{Q_{n+1}^*} + \frac{1}{n+1}\underline{C}'(Q_{n+1}^*). \end{aligned}$$

Here, the first inequality is due to (OA.3) and the second follows from the fact that the derivative of $\frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q)$ with respect to n is

$$\frac{1}{n^2 Q}\left[\underline{C}(Q) - Q\underline{C}'(Q)\right] \leq 0,$$

where the inequality holds because $\underline{C}(Q)$ is convex. Since in equilibrium

$$V\left(\frac{Q_{n+1}^*}{n+1}\right) = \frac{n}{n+1}\frac{\underline{C}(Q_{n+1}^*)}{Q_{n+1}^*} + \frac{1}{n+1}\underline{C}'(Q_{n+1}^*),$$

we have the desired contradiction.

That $Q_n^p < Q_n^*$ holds for n sufficiently small follows from the discussion after the proposition by choosing $n = 1$ since $h(Q, 1) > W(Q)$ for all $Q \in (Q_1(m), Q_2(m))$. Moreover,

$Q_n^p \leq Q_n^*$ requires $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ since otherwise $h(Q,n) = W(Q) + \frac{Q}{n}W'(Q)$, which implies $Q_n^* < Q_n^p$. The arguments after the proposition imply that $h(Q,n) < W(Q)$ for some $Q \in (Q_1(m), Q_2(m))$ can only occur if n is sufficiently large.

Assume now that $\underline{C}(Q^e) = C(Q^e)$ and let $Q_\infty := \lim_{n\to\infty} Q_n^*$. Taking limits of both sides of (OA.2) yields

$$V(0) = \frac{\underline{C}(Q_\infty)}{Q_\infty}. \tag{OA.4}$$

The definition of Q^e then implies that $V(0) = \frac{\underline{C}(Q_\infty)}{Q_\infty} = W(Q^e) = \frac{\underline{C}(Q^e)}{Q^e}$. Using

$$\frac{d}{dQ}\left(\frac{\underline{C}(Q)}{Q}\right) = \frac{Q\underline{C}'(Q) - \underline{C}(Q)}{Q^2} \geq 0,$$

where the inequality holds because $\underline{C}$ is convex, we have that the solution to the equation $V(0) = \frac{\underline{C}(Q_\infty)}{Q_\infty}$ is unique. Since Q^e satisfies this equation we thus have $Q_\infty = Q^e$. Hence, if $Q^e \notin \cup_{m\in\mathcal{M}}(Q_1(m)), Q_2(m))$ then Q^e is also the aggregate quantity in the limit as claimed.

Assume now that $Q^e \in (Q_1(m_e)), Q_2(m_e))$ for some $m_e \in \mathcal{M}$. For $Q \in (Q_1(m_e)), Q_2(m_e))$, $\underline{C}(Q)$ increases linearly from $C(Q_1(m_e))$ to $C(Q_2(m_e))$ with a slope that is greater than $V(0)$. The latter follows from our observation that $\underline{C}'(Q^e) > V(0)$. Because W is increasing we have

$$\frac{C(Q_1(m_e))}{Q_1(m_e)} = W(Q_1(m_e)) < W(Q^e) = V(0) < W(Q_2(m_e)) = \frac{C(Q_2(m_e))}{Q_2(m_e)}.$$

This implies there exists a unique number $\tilde{Q} \in (Q_1(m_e)), Q_2(m_e))$ such that $\frac{\underline{C}(\tilde{Q})}{\tilde{Q}} = V(0)$. If $Q^e \in (Q_1(m_e)), Q_2(m_e))$ this is then the aggregate quantity in the limit as claimed.

We are left to show that $\tilde{Q} > Q^e$ holds whenever $Q^e \in (Q_1(m_e), Q_2(m_e))$. To see that this holds, rearrange (OA.4) to

$$Q_\infty V(0) = \underline{C}(Q_\infty)$$

and recall that $Q^e V(0) = C(Q^e)$. Since $C(Q^e) > \underline{C}(Q^e)$, $\tilde{Q} = Q_\infty > Q^e$ follows. $\square$

As Proposition OA.1 shows, in our model of quantity competition the equilibrium is always unique and symmetric and the equilibrium quantity is increasing in n. For n sufficiently large, $Q_n^p < Q_n^*$ is possible. That is, the equilibrium quantity can be excessively large. To develop an understanding of how such a reversal can occur, consider the first-order condition under symmetry,

$$V\left(\frac{Q}{n}\right) = \frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q) =: h(Q,n).$$

If $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then $h(Q,n)$ is increasing and concave in Q and, for all $n \in \mathbb{N}$, it satisfies $h(Q_i(m), n) > W(Q_i(m))$. Moreover, $h(Q, n)$ decreases in n and satisfies $h(Q, 1) > W(Q)$ for all $Q \in (Q_1(m), Q_2(m)))$. In contrast, for n sufficiently large, there exists at least one interval $(a_n, b_n) \subset (Q_1(m), Q_2(m))$ such that $h(Q, n) < W(Q)$ for all $Q \in (a_n, b_n)$, where a_n decreases in n and b_n increases in n.[1] Consequently, if $V(Q/n) = h(Q, n)$ for $Q \in (a_n, b_n)$, then $Q_n^* \in (a_n, b_n)$ and $Q_n^p < Q_n^*$. Figure 12 illustrates the relationship between the functions W and h. Intuitively, the first-order condition implies that a firm's perceived marginal cost $h(Q, n)$ of procuring the quantity Q/n when it faces $n-1$ competitors is a convex combination of $\underline{C}'(Q)$ (which is larger than $W(Q)$) and $\underline{C}(Q)/Q$ (which is less than $W(Q)$ for $Q \in (Q_1(m), Q_2(m)))$. As n increases, the weight on $\underline{C}(Q)/Q$ increases, eventually leading to $h(Q, n) < W(Q)$ for some values of Q.

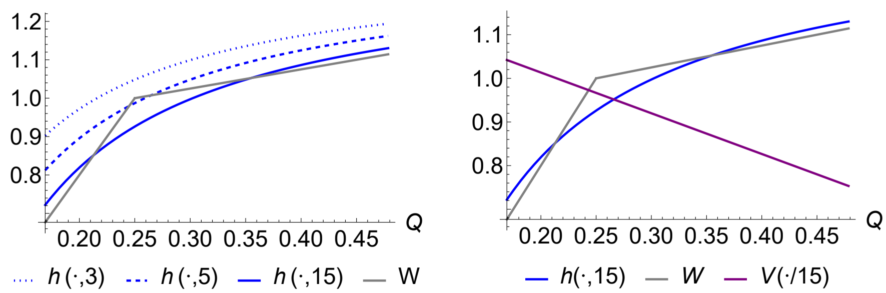

Figure 12: The left-hand panel displays W for the piecewise linear specification of W from (2), and $h(\cdot, n)$ for $n = 3$, $n = 5$ and $n = 15$. The right-hand panel focuses on the case where $n = 15$ and shows that $Q_n^p < Q_n^*$ for $V(Q/n) = 1.2 - 14Q/n$.

As $n \to \infty$, Q_n^p converges to the efficient (or Walrasian) quantity Q^e, which in turn satisfies $V(0) = W(Q^e)$. Consequently, the last statement of Proposition OA.1 distinguishes the cases where there is no $m \in \mathcal{M}$ such that $Q^e \in (Q_1(m), Q_2(m))$ and where there exists a $m_e \in \mathcal{M}$ such that $Q^e \in (Q_1(m_e), Q_2(m_e)))$. Observe that in the latter case

$$\underline{C}'(Q^e) = C'(Q_2(m_e)) > W(Q_2(m_e)) > W(Q^e) = V(0).$$

Proposition OA.1 implies that key features of the monopsony model—wage dispersion and involuntary unemployment—extend to quantity competition. Moreover, the relationship between competition and involuntary unemployment is not monotone because increasing competition can bring the equilibrium quantity into or out of an ironing interval $(Q_1(m), Q_2(m))$. Within such an interval, competition decreases wage dispersion and involuntary unemployment and increases $w_1(Q_n^*)$ and employment, while leaving the high wage $W(Q_2(m))$ fixed.

[1] If there are multiple subintervals over which $h(Q, n) < W(Q)$ for some n, index these by k. Then for each k, a_n^k is decreasing in n and b_n^k is increasing in n because h decreases in n. Of course, eventually two or more of these subintervals may collapse into one, that is if $b_n^k < a_n^{k+1}$, we may have $b_{n'}^k \geq a_{n'}^{k+1}$ for some $n' > n$. But this does not invalidate the point that the set of $Q \in (Q_1(m), Q_2(m))$ for which $h(Q, n) < W(Q)$ increases in n in the set inclusion sense.

If $\underline{C}(Q^e) < C(Q^e)$ holds, there is wage dispersion and involuntary unemployment even in the limit as $n \to \infty$. This yields a "natural" unemployment rate associated with perfect competition of $(Q_2(m_e) - \tilde{Q})/Q_2(m_e)$. In contrast to the usual notion of a natural unemployment rate, this unemployment is a result of inefficient resource allocation in the form of both random allocation and excessive economic activity (since $\tilde{Q} > Q^e$). In other words, there is the possibility of inefficient perfect competition.

Figures 13 and 14 illustrate these effects for the piecewise linear specification of W from (2). The left-hand panels of these figures are plotted using $V(y_i) = 1.1 - 8y_i$ and the right-hand panels are plotted using $V(y_i) = 1.2 - 8y_i$. This implies that for the left-hand panels we have $Q^e = 0.45 \in (Q_1, Q_2) = (0.169, 0.478)$ and $\tilde{Q} = 0.4516$, while for the right-hand panels we have $Q^e = 0.65 > Q_2$.

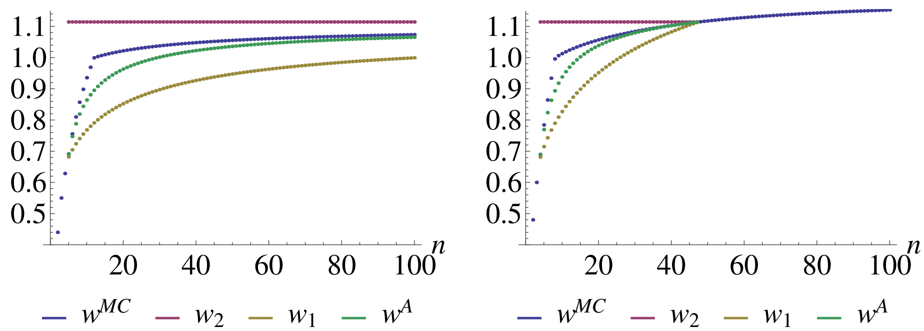

Figure 13: Equilibrium wages as a function of n, where w_1 denotes the lower equilibrium wage, w_2 denotes the higher equilibrium wage, $w^{MC} = W(Q_n^*)$ denotes the market-clearing wage and w^A the average wage $w^A = (w_1 + w_2)/2$. On the left, $W(Q^e) = 1.1 < 1.114 = w_2$ and on the right $W(Q^e) = 1.2 > w_2$.

Even though each firm's market share becomes infinitesimal as $n \to \infty$, market power is still exerted in equilibrium because the auctioneer procures the aggregate quantity at the minimal cost.

OA.3 Minimum wage effects and competition

In models with quantity competition and market-clearing wages, setting a minimum wage above $W(Q_n^C)$ (the market-clearing wage for the equilibrium quantity Q_n^C absent wage regulation) and below $W(Q_n^p)$ (the competitive wage) has a positive effect on total employment and, accordingly, workers' pay. To see this, recall that the competitive quantity Q_n^p is such that $V\left(\frac{Q_n^p}{n}\right) = W(Q_n^p)$ while the equilibrium quantity satisfies (OA.1). Together with $W' > 0$, this implies that $Q_n^C < Q_n^p$. Any minimum wage $\underline{w} \in (W(Q_n^C), W(Q_n^p)]$ then has a positive

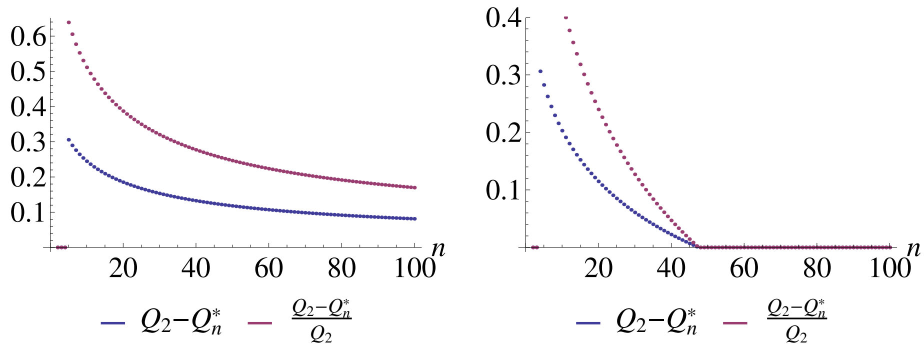

Figure 14: Involuntary unemployment and the unemployment rate as a function of n. On the left, there is involuntary unemployment of size $Q_2 - \tilde{Q} = 0.0269$ and an unemployment rate of 5.6% as $n \to \infty$.

employment effect. Since $\lim_{n\to\infty} Q_n^p = Q^e = \lim_{n\to\infty} Q_n^C$, the scope for this kind of quantity and social-surplus increasing minimum wage regulation vanishes in the limit as $n \to \infty$.[2]

Even if the symmetric equilibrium in the model with market-clearing wages is the unique equilibrium absent a minimum wage, a binding minimum wage $\underline{w} \in (W(Q_n^C), W(Q_n^p))$ inevitably gives rise to a continuum of equilibria. To see this, denote by $r_i(Q_{-i})$ the best response function of an arbitrary firm i to the aggregate quantity $Q_{-i} = \sum_{j\neq i} y_j$ demanded by its rivals. If the best response function is given by the first-order condition $V(r_i) - W(Q_{-i} + r_i) - r_i W'(Q_{-i} + r_i) = 0$, the equilibrium is unique and symmetric.[3] Denoting by $r_{\underline{w},i}(Q_{-i})$ the best response function given minimum wage $\underline{w} \in (W(Q_n^*), W(Q_n^p))$, we have

$$r_{\underline{w},i}(Q_{-i}) = \max\{r_i(Q_{-i}), \min\{S(\underline{w}) - Q_{-i}, V^{-1}(\underline{w})\}\},$$

where the term $\min\{S(\underline{w}) - Q_{-i}, V^{-1}(\underline{w})\}$ accounts for the possibility that even though the firm could procure the quantity $S(\underline{w}) - Q_{-i}$ at the minimum wage $\underline{w}$ it only wants to do

[2]Whether the differences $W(Q_n^p) - W(Q_n^C)$ and $Q_n^p - Q_n^C$ monotonically decrease in n—and the scope for this kind of minimum wage of regulation—depends on the specifics of the model. If W and V are both linear, then both $W(Q_n^p) - W(Q_n^C)$ and $Q_n^p - Q_n^C$ decrease in n.

[3]To see this, totally differentiate the first-order condition to obtain $r_i' = -\frac{W' + r_i W''}{W' + r_i W'' + W' - V'}$, which satisfies $-1 < r_i' < 0$, where we drop arguments for ease of notation. The aggregate quantity Q given Q_{-i} and i's best response satisfies $Q = Q_{-i} + r_i(Q_{-i})$. The right-hand side is increasing in Q_{-i} and hence invertible. Following Anderson, Erkal, and Piccinin (2020), we can thus write $Q_{-i} = f_i(Q)$ as a function of Q, where f_i is increasing. This allows us to construct what Anderson, Erkal, and Piccinin call the *inclusive-best response* function $\tilde{r}_i(Q) := r_i(f_i(Q))$, which gives the optimal quantity that i would choose if the aggregate quantity is Q, which includes its own quantity. We have $\tilde{r}_i' = \frac{r_i'}{1+r_i'} < 0$. The aggregate quantity Q is an equilibrium quantity if and only if $\sum_{i=1}^n \tilde{r}_i(Q) = Q$. Because the left-hand side decreases and the right-hand side increases in Q, it follows that the Q satisfying this equality is unique. Moreover, because the firms are symmetric, we have $\tilde{r}_i = \tilde{r}_j$ for all $i, j \in \{1, \dots, n\}$. Hence, the unique equilibrium is symmetric.

so if this quantity is small enough and its willingness to pay is greater than $\underline{w}$. This means that it will not procure more than $V^{-1}(\underline{w})$. Since $Q_n^C < S(\underline{w}) < Q_n^p$, we have

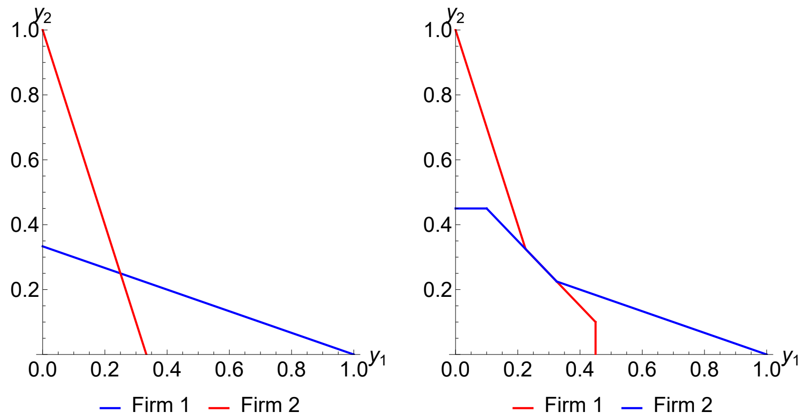

Figure 15: Standard quantity competition without a minimum wage (left panel) and with a minimum wage of $\underline{w} = 0.55$ (right panel). The minimum wage generates a continuum of equilibria. The figures assumes $V(y_i) = 1 - y_i$ and $W(Q) = Q$, which implies that $Q_n^* = 1/2$ and $Q_n^p = 2/3$.

$$r'_{\underline{w},i}\left(Q_{-i}\right)|_{Q_{-i}=\frac{n-1}{n}S(\underline{w})} = -1.$$

This implies that in the neighborhood of the symmetric equilibrium in which each firm chooses $S(\underline{w})/n$ there is a also a continuum of necessarily asymmetric equilibria as illustrated in Figure 15. Given that V is decreasing, the symmetric equilibrium is the one that maximizes social surplus and is therefore a natural selection.

To analyze the effects of introducing a minimum wage, we maintain focus on the symmetric equilibrium and study its comparative statics.[4] Similarly to the model without a minimum wage, given a minimum wage $\underline{w}$, we let

$$h_R(Q,n,\underline{w}) := \frac{n-1}{n}\frac{\underline{C}_R(Q,\underline{w})}{Q} + \frac{1}{n}\underline{C}'_R(Q,\underline{w})$$

denote the firm-level marginal cost of procurement under symmetry in the model with quantity competition. Observe that for $Q \le S(\underline{w})$ (equivalently, $\underline{w} \ge W(Q)$), we have $\underline{C}_R(Q,\underline{w}) = \underline{w}Q$ and $\underline{C}'_R(Q,\underline{w}) = \underline{w} = \frac{\underline{C}_R(Q,\underline{w})}{Q}$, which implies that $h_R(Q,n,\underline{w}) = \underline{w}$. For $Q > S(\underline{w})$, $h_R(Q,n,\underline{w})$ is larger than $\underline{w}$ and strictly increasing in Q. Moreover, $h_R(Q,n,\underline{w})$ is

[4] As stated in Proposition OA.1, without wage regulation, the symmetric equilibrium is the unique equilibrium. Whether given a minimum wage $\underline{w}$ the symmetric equilibrium is the socially optimal equilibrium when the equilibrium involves wage dispersion and involuntary unemployment is an open question. Of course, if the aggregate quantity is the same in a symmetric equilibrium and an asymmetric equilibrium, social surplus is larger in the symmetric equilibrium.

continuous in Q everywhere, except possibly at $Q = S(\underline{w})$, where it is continuous if and only if $\underline{C}'_R(Q, \underline{w})$ is continuous at that point.[5] Finally, for $\underline{w} < w_1(Q)$ (equivalently, $Q > w_1^{-1}(\underline{w})$), we have

$$h_R(Q, n, \underline{w}) = \frac{n-1}{n}\frac{\underline{C}(Q)}{Q} + \frac{1}{n}\underline{C}'(Q) = h(Q, n)$$

because $\underline{C}_R(Q, \underline{w}) = \underline{C}(Q)$ and hence $\underline{C}'_R(Q, \underline{w}) = \underline{C}'(Q)$ for $\underline{w} < w_1(Q)$. Putting all of this together, in the model with quantity competition the minimum wage binds in exactly the same instances as in the monopsony model.

Since $V(Q/n)$ is decreasing in Q and $h_R(Q, n, \underline{w})$ has the same curvature properties as $h(Q, n)$, it follows that if there exists a Q satisfying

$$V(Q/n) = h_R(Q, n, \underline{w}), \tag{OA.5}$$

then Q/n is the symmetric equilibrium of the model with quantity competition given the minimum wage $\underline{w}$. If no such quantity exists, $h_R(Q, n, \underline{w})$ must be discontinuous at Q, which implies $Q = S(\underline{w})$. In this case, the symmetric equilibrium quantity is $S(\underline{w})/n$. Summarizing, we have the following lemma:

Lemma OA.1. *The model with quantity-setting firms and a given minimum wage* $\underline{w}$ *has a symmetric equilibrium. In this equilibrium, each firm chooses the quantity* Q/n *with* Q *satisfying* (OA.5) *if such a* Q *exists and* $S(\underline{w})/n$ *otherwise.*

The characterization of the symmetric equilibrium in the quantity setting game with a minimum wage mirrors the characterization of the optimal quantity in the monopsony model with a minimum wage. Similarly to Corollary 1, the aggregate quantity in the symmetric equilibrium is given by the quantity that satisfies (OA.5) and equates firm-level marginal values and marginal costs, whenever such a quantity exists, and is otherwise given by the quantity $S(\underline{w})$ supplied at the minimum wage. As we will show next, relative to the monopsony case, a difference arises for the comparative statics associated with an increase in the minimum wage when the equilibrium quantity is characterized by (OA.5) and inside an ironing interval $(Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Recall that in the monopsony model, when there is equilibrium wage dispersion and involuntary unemployment a marginal increase in $\underline{w}$ increases the equilibrium quantity and decreases the equilibrium level of involuntary unemployment because $\underline{C}'_R(Q, \underline{w})$ decreases in $\underline{w}$. In contrast, with $n \geq 2$, $h_R(Q, n, \underline{w})$ is a convex combination of $\underline{C}'_R(Q, \underline{w})$, which decreases in $\underline{w}$, and $\underline{C}_R(Q, \underline{w})/Q$, which increases in $\underline{w}$. Thus, with quantity competition, the effect of a marginal increase in the minimum wage will

[5]This last observation follows from the fact that $\underline{C}_R(Q, \underline{w})$ and hence $\underline{C}_R(Q, \underline{w})/Q$ are continuous at $Q = S(\underline{w})$.

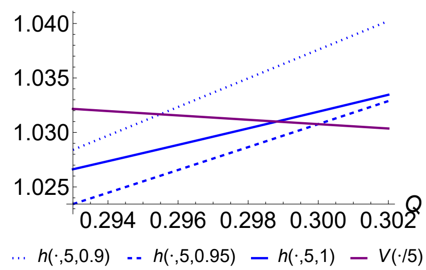

Figure 16: Illustration of non-monotone minimum wage effects with quantity competition.

not necessarily be monotone when there is wage dispersion and involuntary unemployment. This is illustrated in Figure 16 for the piecewise linear specification of W from (2) and a linear marginal value function V for $n = 5$ with $\underline{w} = 0.9$ (dotted), $\underline{w} = 0.95$ (dashed) and $\underline{w} = 1$ (solid). From $\underline{w} = 0.9$ to $\underline{w} = 0.95$, the equilibrium quantity increases, and from $\underline{w} = 0.95$ to $\underline{w} = 1$, it decreases.

However, as the following proposition shows, the marginal effect of increasing the minimum wage when the minimum wage is equal to the lower of the two wages absent wage regulation, that is at $\underline{w} = w_1(Q_n^*)$, on the equilibrium employment level $Q_n^*(\underline{w})$ is positive:

Proposition OA.2. *Suppose $n \in \mathbb{N}$ and $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Then at $\underline{w} = w_1(Q_n^*)$, the marginal effect of increasing the minimum wage on the equilibrium quantity $Q_n^*(\underline{w})$ is positive, that is, $\frac{dQ_n^*(\underline{w})}{d\underline{w}}|_{\underline{w}=w_1(Q_n^*)} > 0$.*

Proof. Note first that $\underline{C}_R'(Q, \underline{w})$ is continuous at $\underline{w} = w_1(Q)$ because discontinuities in $\underline{C}_R'(Q, \underline{w})$ only occur at $\underline{w} = W(Q)$. The equilibrium condition is thus

$$V\left(\frac{Q_n^*(\underline{w})}{n}\right) = h(Q_n^*(\underline{w}), n, \underline{w}) = \frac{n-1}{n}\frac{\underline{C}_R(Q_n^*(\underline{w}), \underline{w})}{Q_n^*(\underline{w})} + \frac{1}{n}\underline{C}_R'(Q_n^*(\underline{w}), \underline{w}).$$

Totally differentiating with respect to $\underline{w}$, dropping arguments and writing $\underline{C}_R'$ and $\underline{C}_R''$ in lieu of $\frac{\partial \underline{C}_R}{\partial Q}$ and $\frac{\partial^2 \underline{C}_R}{\partial Q^2}$ yields

$$\left[V' - (n-1)\left[\frac{Q_n^*\underline{C}_R' - \underline{C}_R}{(Q_n^*)^2}\right] - \underline{C}_R''\right]\frac{dQ_n^*}{d\underline{w}} = (n-1)\frac{\partial \underline{C}_R}{\partial \underline{w}}\frac{1}{Q_n^*} + \frac{\partial \underline{C}_R'}{\partial \underline{w}}.$$

Since the term in brackets on the left-hand side is negative, $\frac{dQ_n^*}{d\underline{w}}$ has the opposite sign of $(n-1)\frac{\partial \underline{C}_R}{\partial \underline{w}}\frac{1}{Q_n^*} + \frac{\partial \underline{C}_R'}{\partial \underline{w}}$. From the proof of Theorem 1, we know that $\frac{\partial \underline{C}_R}{\partial \underline{w}} = \lambda^* \geq 0$ and $\frac{\partial \underline{C}_R'}{\partial \underline{w}} = \frac{\partial \lambda^*}{\partial Q} \leq 0$, where λ^* is the solution value of the Lagrange multiplier associated with the minimum wage constraint. At $\underline{w} = w_1(Q)$, we have $\lambda^* = 0$ and $\frac{\partial \lambda^*}{\partial Q} < 0$. We therefore have $\frac{dQ_n^*}{d\underline{w}}|_{\underline{w}=w_1(Q_n^*)} > 0$ as required. $\square$

Proposition OA.2 shows that a minimum wage close to but above $w_1(Q_n^*)$ increases the equilibrium quantity if $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. This resonates with an insight from the monopsony model, where a marginal increase in the minimum wage increases employment whenever there is wage dispersion and involuntary unemployment. However, in the model with quantity competition increasing the equilibrium quantity is not necessarily a move in the right direction because of the possibility of excessively high employment, that is, $Q_n^* > Q_n^p$. More generally, the following theorem describes the effects of imposing a binding minimum wage when $Q_n^* \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. In its proof, we show that for $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$,

$$h_\gamma(Q, n) := \frac{n-1}{n} W(Q) + \frac{1}{n}\gamma(Q) \tag{OA.6}$$

is the limit of $h_R(Q, n, \underline{w})$ as $\underline{w}$ approaches $W(Q)$ from below. This function is continuous in Q and its role and properties are analogous to those of γ in the monopsony model. Assuming $Q_n^* \in (Q_1(m), Q_2(m))$, we let $\hat{Q}_{H,n}$ denote the largest value of Q such that $V(Q/n) = h_\gamma(Q, n)$.

Theorem OA.1. *Whenever there is involuntary unemployment and wage dispersion under a given minimum wage in the model with quantity competition, increasing the minimum wage to $\underline{w} = W(\hat{Q}_{H,n})$ increases employment and eliminates involuntary unemployment. If there is involuntary unemployment and no wage dispersion under a given minimum wage, increasing the minimum wage decreases employment and increases involuntary unemployment. Moreover, if $\underline{w} \neq W(Q_n^p)$ and there is no involuntary unemployment under a given minimum wage, a marginal increase in the minimum wage increases employment.*

Proof. As noted, the minimum wage only binds if $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$. Fixing Q, define

$$h_\gamma(Q, n) := \lim_{\underline{w} \uparrow W(Q)} h(Q, n, \underline{w}).$$

Since $\underline{C}_R(Q, \underline{w})$ is continuous, it satisfies $\underline{C}_R(Q, W(Q)) = W(Q)Q$ and $\lim_{\underline{w} \uparrow W(Q)} \frac{\underline{C}_R(Q,\underline{w})}{Q} = W(Q)$. From the monopsony model we know that $\lim_{\underline{w} \uparrow W(Q)} \underline{C}'_R(Q, \underline{w}) = \gamma(Q)$, which is continuous in Q. We thus obtain $h_\gamma(Q, n)$ as given in (OA.6). Since $h_\gamma(Q, n)$ is continuous, V is continuously decreasing and $Q_n^* \in (Q_1(m), Q_2(m))$, there exist smallest and largest values of Q such that

$$V(Q/n) = h_\gamma(Q, n).$$

We denote these values of Q by $\hat{Q}_{L,n}$ and $\hat{Q}_{H,n}$, respectively. Since V is decreasing and $h_\gamma(Q, n) < C'(Q_2(m))$ holds for $Q \in (Q_1(m), Q_2(m))$, we have

$$Q_n^* < \hat{Q}_{L,n} \quad \text{and} \quad \hat{Q}_{H,n} \leq Q_2(m).$$

Moreover, since $h_\gamma(Q,n) > W(Q)$ holds unless $\underline{C}'_R(Q,\underline{w})$ is continuous at $\underline{w} = W(Q)$, we have

$$\hat{Q}_{H,n} \leq Q^p_n. \tag{OA.7}$$

This last inequality is strict unless $h_\gamma(\hat{Q}_{H,n},n) = W(\hat{Q}_{H,n})$. Since $h_\gamma(Q,n)$ converges to $W(Q)$ as $n \to \infty$, provided $Q^e \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, we have $\lim_{n\to\infty} \hat{Q}_{H,n} = \lim_{n\to\infty} Q^p_n = Q^e$.

It follows that for $\underline{w} \leq W(\hat{Q}_{L,n})$, the equilibrium given the minimum wage $\underline{w}$ involves wage dispersion and involuntary unemployment. Moreover, for $\underline{w} \in [W(\hat{Q}_{H,n}), W(Q_2(m)]$, there is no wage dispersion in equilibrium. Minimum wages $\underline{w} \in [W(\hat{Q}_{H,n}), W(Q^p_n)]$ correspond to the *pure Robinson oligopsony* region, where increases in $\underline{w}$ increase equilibrium employment without inducing involuntary unemployment. $\square$

Note that because $h_\gamma(Q,n) \geq W(Q)$, the aggregate equilibrium quantity in the presence of a minimum wage $\underline{w} = W(Q^*_n)$ is never larger than Q^p_n. Therefore, when $Q^*_n > Q^p_n$, one effect of imposing a minimum wage equal to the market-clearing wage for the equilibrium quantity absent wage regulation is that it prevents excessively high levels of employment. Since the ordering $\hat{Q}_{H,n}(m) \leq Q^p_n$ (see (OA.7)) does not depend on the ordering of Q^*_n and Q^p_n, this also implies that even when $Q^*_n > Q^p_n$ holds under the laissez-faire equilibrium, total employment increases in $\underline{w}$ for $\underline{w} \in [W(\hat{Q}_{H,n}(m)), W(Q^p_n)]$ without inducing involuntary unemployment. Since we know from Proposition OA.2 that increasing the minimum wage at $w_1(Q^*_n)$ increases employment, if $Q^*_n > Q^p_n$, then the effects of the minimum wage on total employment must be non-monotone on $[w_1(Q^*_n), W(\hat{Q}_{H,n}(m))]$. Furthermore, if the Walrasian quantity Q^e is inside some ironing interval (i.e. if $Q^e \in (Q_1(m_e), Q_2(m_e))$ for some $m_e \in \mathcal{M}$), then there is scope for social-surplus increasing minimum wage regulation even in the perfectly competitive limit. Setting $\underline{w} = W(\hat{Q}_{H,n}(m_e))$ will eliminate involuntary unemployment and we have $\underline{w} \to W(Q^e)$ as $n \to \infty$ because $\lim_{n\to\infty} \hat{Q}_{H,n}(m) = Q^e$.

OB Horizontally differentiated jobs

We now return to a monopsony setting but allow for horizontal differentiation of workers, with the monopsony offering horizontally differentiated jobs. For this setting, we show that in addition to involuntary unemployment, the optimal mechanism may involve inefficient matching of workers to jobs, both of which can be remedied by an appropriately chosen minimum wage.

OB.1 Setup

Consider a variant of the Hotelling model in which a monopsony with jobs at locations 0 and 1 has a willingness to pay of $V(Q_\ell)$ for the Q_ℓ-th worker employed at a given location $\ell \in \{0,1\}$. As before, $V(Q_\ell)$ is assumed to be continuous and strictly decreasing. There is a continuum of workers with linear transportation costs whose locations, which are the private information of each worker, are uniformly distributed between 0 and 1. The total mass of workers is 1. The value of the outside option of each worker is normalized to 0.[6] The payoff of a worker at location z that works at 0 for a wage of w is $w - z$, while this worker's payoff of working at 1 for a wage of w is $w - (1 - z)$. Observe that this implies that the market-clearing wage to hire Q_ℓ workers at a given location is $W(Q_\ell) = Q_\ell$, which in turn means that the cost of procurement at each location under market-clearing wages is $C(Q_\ell) = Q_\ell^2$. Of course, the monopsony can hire Q_ℓ workers at $\ell = 0, 1$ if and only if $\sum_\ell Q_\ell \leq 1$.

OB.2 Equilibrium

We first derive the minimum cost of procuring the quantity $Q_\ell \in [0, 1/2]$ at a given location, assuming that the same quantity is procured at the other location. First, notice that conditional on being employed, the expected transportation cost of a worker at any location $z \in [0,1]$, who is equally likely to work at each location, is $1/2$. To satisfy the individual rationality constraint of such a worker, they must be paid a wage of at least $1/2$. Consequently, by offering a wage of $1/2$ to workers who agree to enter a lottery which allocates them to work at location 0 or location 1, each with probability $1/2$, or who multi-task by spending half their time at each location, the monopsony can procure any quantity $Q_\ell \in [0, 1/2]$ at both locations at a marginal procurement cost of $1/2$. Since the marginal cost of procuring Q_ℓ workers at a market-clearing wage is $2Q_\ell$, the monopsony can procure the quantity $Q_\ell \in [0, 1/2]$ at each location at a cost of

$$\underline{C}_H(Q_\ell) := \begin{cases} Q_\ell^2, & Q_\ell \in [0, 1/4] \\ (Q_\ell - 1/4)/2 + 1/16, & Q_\ell \in (1/4, 1/2] \end{cases}$$

by offering a wage of $1/2$ to attract "universalists" (workers who are willing to do either job) and a wage of $1/4$ to attract "specialists" (workers with locations no further away from 0 and 1 than $1/4$, who are guaranteed the job closest to their location). Notice that the indi-

[6] This is without loss of generality within the domain of problems in which the value of the outside option and the willingness to pay per worker are independent of the workers' locations since all that matters for these problems is the difference between the latter and the former.

vidual rationality constraint will bind for all employed workers with locations $z \in (1/4, 3/4)$. Consequently, for the marginal workers at 1/4 and 3/4, the incentive compatibility constraints, which require that these workers are indifferent between working as a specialist or as a universalist, coincide with their individual rationality constraints.

The preceding arguments establish that this scheme with wage dispersion and random worker-job matches results in lower procurement costs, relative to market-clearing wages for any $Q_\ell \in (1/4, 1/2]$. Arguments along the lines of those in Balestrieri, Izmalkov, and Leao (2021) and Loertscher and Muir (2023), who study optimal selling mechanisms on the Hotelling line, can be used to establish that $\underline{C}(Q_\ell)$ is in fact the minimal cost of procurement, subject to workers' incentive compatibility and individual rationality constraints.[7]

The equilibrium level of employment Q_ℓ^* at each location $\ell \in \{0, 1\}$ is given by the unique number satisfying $V(Q_\ell^*) = \underline{C}'(Q_\ell^*)$. We say that the equilibrium involves *involuntary unemployment* if there is a positive mass of workers who would be willing to work but are not employed at the equilibrium wages, and we say that it involves *worker-job mismatches* if workers with $z < 1/2$ work at location 1 or workers with $z > 1/2$ work at location 0 in equilibrium.[8] The following proposition summarizes characteristics of the equilibrium. As it follows directly from the preceding arguments, we do not provide a separate proof.

Proposition OB.1. *If $V(1/4) \leq 1/2$, then $Q_\ell^* \leq 1/4$ and worker-job mismatches and involuntary unemployment do not occur in equilibrium. If $V(1/4) > 1/2 > V(1/2)$, then $Q_\ell^* \in (1/4, 1/2)$ and worker-job mismatches and involuntary unemployment do occur in equilibrium. If $V(1/2) \geq 1/2$, then $Q_\ell^* = 1/2$ and the equilibrium involves worker-job mismatches but no involuntary unemployment.*

Figure 17 illustrates the case $V(1/4) > 1/2 > V(1/2)$ in Proposition OB.1 for the linear specification $V(Q_\ell) = v - Q_\ell$ with $v = 7/8$. For this linear specification, $V(1/4) > 1/2 >$

[7] An outline of the argument, adapted from the monopoly screening problem in Loertscher and Muir (2023) to the procurement setting and assuming, for now, that all workers are employed, is as follows. Let $x_\ell(z)$ denote the probability that the worker who reports type $z \in [0, 1]$ works at location $\ell \in \{0, 1\}$. Incentive compatibility implies that $x_1(z) - x_0(z)$ is non-decreasing. Type $\hat{z}$ is the worst-off type if $x_1(\hat{z}) = x_0(\hat{z})$. Because all workers are employed, we have $x_0(z) + x_1(z) = 1$, implying $x(z) \equiv x_0(z)$ is sufficient, and incentive compatibility becomes equivalent to $x(z)$ being non-increasing, and $\hat{z}$ is worst-off if $x(\hat{z}) = 1/2$. Given any worst-off type $\hat{z} \in [0, 1]$, incentive compatibility yields the designer's objective in terms of virtual costs and values. Because its pointwise minimizer is not monotone, one needs to iron the virtual types. (Put differently, the cost of procurement is not convex in Q_0, the number of units procured at location 0.) The pointwise minimizer given the ironed virtual type function must assign a worker in the ironing interval with equal probability to jobs at 0 and 1. Consequently, the value of the ironed virtual type function over the ironing interval must be 0. Moreover, this also means that not employing some of these workers is also optimal. Thus, the assumption that all workers are employed can easily be relaxed.

[8] If worker-job mismatches are optimal, workers who work at the high wage of 1/2 are indifferent between working and not. Thus, those—if any—who are involuntarily unemployed are also indifferent between being unemployed and working.

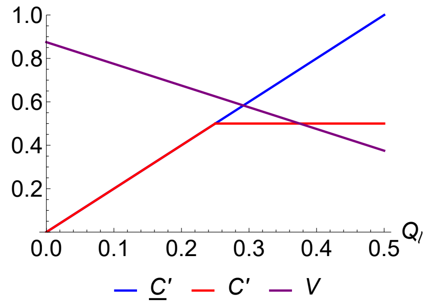

Figure 17: Illustration of Proposition OB.1 for $V(Q_\ell) = v - Q_\ell$ with $v = 7/8$.

$V(1/2)$ is equivalent to $v \in (3/4, 1)$.

OB.3 Minimum wage effects

If a minimum wage of $\underline{w} = 1/2$ is imposed, then provided $V(1/4) > 1/2$, the strict profitability of worker-job mismatches vanishes without any negative effects on the equilibrium level of employment. More generally, the minimum cost of procuring $Q_\ell \in [0, 1/2]$ at location $\ell \in \{0, 1\}$ given the regulated minimum wage $\underline{w} \in [0, 1/2]$, denoted $\underline{C}_{H,R}(Q_\ell, \underline{w})$, is

$$\underline{C}_{H,R}(Q_\ell, \underline{w}) = \begin{cases} \underline{w} Q_\ell, & Q_\ell \in [0, S_\ell(\underline{w})] \\ (Q_\ell - S_\ell(\underline{w}))/2 + 1/16, & Q_\ell \in (S_\ell(\underline{w}), 1/2], \end{cases}$$

where $S_\ell(w) = w$ is the labor supply function at location ℓ. Consequently, the marginal cost of procuring labor is $\underline{C}'_{H,R}(Q_\ell, \underline{w}) = \underline{w}$ for $Q_\ell \leq \underline{w}$ and $\underline{C}'_{H,R}(Q_\ell, \underline{w}) = 1/2$ for $Q_\ell \in (\underline{w}, 1/2]$.[9]

Denoting by Q^p_ℓ the quantity the monopsony would procure under price-taking behavior at location ℓ, which is the unique number satisfying $V(Q^p_\ell) = Q^p_\ell$ if $V(1/2) \leq 1/2$ (and otherwise $Q^p_\ell = 1/2$) and by Q^*_ℓ the equilibrium quantity employed at ℓ under the laissez-faire equilibrium, the effects of minimum wages $\underline{w} \in [0, 1/2]$ are as follows.

If $Q^*_\ell \leq 1/4$, then for $\underline{w} \in (Q^*_\ell, Q^p_\ell)$, a marginal increase in $\underline{w}$ increases the equilibrium quantity employed and workers' pay without inducing involuntary unemployment. This corresponds to a standard Robinson region. For $Q^*_\ell \in (1/4, 1/2)$, minimum wages $\underline{w} < 1/4$ have no effect. Under a minimum wage $\underline{w} \in (1/4, Q^*_\ell)$, the monopsony hires $\underline{w}$ workers at the minimum wage and the $Q^*_\ell - \underline{w}$ workers at a wage of $1/2$. A marginal increase in the minimum wage leaves total employment and involuntary unemployment unaffected and increases the number of workers employed at the minimum wage. Social surplus increases because the number of worker-job mismatches decreases. All workers are weakly better off with the minimum wage increase. For $\underline{w} \in (Q^*_\ell, Q^p_\ell)$, employment is $\underline{w}$ and all workers

[9]Note that there is no scope for ironing because $\underline{C}'_{H,R}(Q_\ell, \underline{w})$ is already monotone.

who are employed are paid the minimum wage. There is no involuntary unemployment. A marginal increase in the minimum wage increases employment and social surplus. All workers are weakly better off with the minimum wage increase. For $\underline{w} > Q_\ell^p$ with $Q_\ell^p < 1/2$, a marginal increase in $\underline{w}$ reduces the equilibrium quantity employed and increases involuntary unemployment, corresponding to the neoclassical effects of minimum wages.

The guidance for policy makers contemplating a marginal increase in $\underline{w}$, presented in the following proposition, is similar to that provided in Theorem 2. As it follows immediately from the preceding discussion and results, we do not provide a separate proof.

Proposition OB.2. *If there is no involuntary unemployment under a given minimum wage $\underline{w} < 1/2$ and $\underline{w} \neq Q_\ell^p$, a marginal increase in $\underline{w}$ increases total employment without inducing involuntary unemployment. If there is involuntary unemployment and wage dispersion under a given minimum wage $\underline{w} < 1/2$, then a marginal increase in $\underline{w}$ increases social surplus and the surplus of all workers (with a strict increase for workers employed at the minimum wage after the increase in $\underline{w}$), without affecting total employment. If there is involuntary unemployment and no wage dispersion, then a marginal increase in $\underline{w}$ decreases total employment and increases involuntary unemployment.*

OC Piecewise linear parameterization

Throughout this section we consider the piecewise linear specification,

$$W(Q) = \begin{cases} aQ, & Q \in \left[0, \underline{q}\right) \\ bQ + (a-b)\underline{q}, & Q \in \left[\underline{q}, 1\right], \end{cases} \tag{OC.8}$$

where $a > b > 0$ and $\underline{q} \in (0,1)$. Setting $a = 4$, $b = 0.5$ and $\underline{q} = 0.25$ we obtain (2) as a special case. As we will see, this is a very tractable parameterization that permits closed-form expressions for the functions $\underline{C}_R$ and γ.

Before proceeding with the analysis, we start by stating and proving a lemma that—given a minimum wage $\underline{w}$—generally holds for the low-wage function w_1 under a general two-wage mechanism parameterized by (Q, q_1, q_2). Applying this lemma to our piecewise linear specification then reveals that the slope of the low-wage function does not vary with $\underline{w}$. This property, which is illustrated in Figure 3, will prove extremely useful when it comes to deriving close-form expressions for $\underline{C}_R$ and γ.

Lemma OC.1. *For any $Q \in (Q_1(m), Q_2(m))$ and $\underline{w} \in [w_1(Q), W(Q))$, the optimal values* *of* q_1 *and* q_2 *are such that*

$$\left(\frac{W(q_2)-W(q_1)}{q_2-q_1}\right)^2 = W'(q_1)W'(q_2). \tag{OC.9}$$

Proof. The optimal mechanism given $\underline{w}$ being a two-wage mechanism implies that q_1 and q_2 are such that

$$W(q_1)+(Q-q_1)\frac{W(q_2)-W(q_1)}{q_2-q_1} = \underline{w}. \tag{OC.10}$$

The cost of procurement as a function q_1 and q_2, denoted $K(q_1, q_2)$, is

$$\begin{aligned} K(q_1,q_2) &= q_1\underline{w} + (Q-q_1)W(q_2) \\ &= \underline{w}Q + (Q-q_1)(W(q_2)-\underline{w}), \end{aligned}$$

where the second equality follows by adding and subtracting $Q\underline{w}$. Subtracting (OC.10) from $W(q_2)$, one obtains

$$\begin{aligned} W(q_2)-\underline{w} &= W(q_2)-W(q_1)-(Q-q_1)\frac{W(q_2)-W(q_1)}{q_2-q_1} \\ &= (q_2-Q)\frac{W(q_2)-W(q_1)}{q_2-q_1}. \end{aligned}$$

Using this expression to replace $W(q_2)-\underline{w}$ in $K(q_1, q_2)$ yields

$$K(q_1,q_2) = Q\underline{w} + (Q-q_1)(q_2-Q)\frac{W(q_2)-W(q_1)}{q_2-q_1}. \tag{OC.11}$$

To simplify notation in what follows, we use the short-hand notation $B = \frac{W(q_2)-W(q_1)}{q_2-q_1}$, $W_i' = W'(q_i)$ for $i = 1, 2$ and $\beta = \frac{Q-q_1}{q_2-q_1}$, bearing in mind that W_1' is *not* the derivative of $w_1(Q)$ nor directly related to this function in any other way. The objective is thus to minimize $K(q_1, q_2)$ over q_1 and q_2 subject to the constraint (OC.10). We arbitrarily choose q_2 as the control variable and let q_1 be an implicit function of q_2 given by (OC.10). Totally differentiating (OC.10) yields

$$\frac{dq_1}{dq_2} = -\frac{\beta}{1-\beta}\frac{W_2'-B}{W_1'-B}. \tag{OC.12}$$

Partially differentiating B with respect to q_1 and q_2 gives

$$\frac{\partial B}{\partial q_1} = \frac{B-W_1'}{q_2-q_1} \quad \text{and} \quad \frac{\partial B}{\partial q_2} = \frac{W_2'-B}{q_2-q_1}.$$

This implies

$$(Q - q_1)\frac{\partial B}{\partial q_1} = \beta(B - W_1')$$

and

$$(q_2 - Q)\frac{\partial B}{\partial q_2} = (1-\beta)(W_2' - B) \quad \text{and} \quad (Q - q_1)\frac{\partial B}{\partial q_2} = \beta(W_2' - B),$$

which will prove useful below.

Letting $k(q_2) = K(q_1(q_2), q_2)$, we have

$$k'(q_2) = (Q - q_1)[\beta B + (1-\beta)W_2'] - \frac{dq_1}{dq_2}(q_2 - Q)[(1-\beta)B + \beta W_1'],$$

which at an optimum is 0. Substituting in $\frac{dq_1}{dq_2}$ from (OC.12), using the fact that $\frac{Q-q_1}{q_2-Q} = \frac{\beta}{1-\beta}$ and somewhat tedious algebra reveals that $k'(q_2) = 0$ is equivalent to

$$B^2 = W_1' W_2',$$

which is what was to be shown. □

For our piecewise linear specification, Lemma OC.1 implies $\frac{W(q_2)-W(q_1)}{q_2-q_1} = \frac{W(Q_2)-W(Q_1)}{Q_2-Q_1}$ for any $\underline{w} \in [w_1(Q), W(Q))$ because $W'(q_1)$ does not vary with $q_1 < \underline{q}$ and $W'(q_2)$ does not vary with q_2 for $q_2 > \underline{q}$. For example, for the specification in (2), we have $W'(q_1) = 4$ and $W'(q_2) = 1/2$ and hence $\frac{W(q_2)-W(q_1)}{q_2-q_1} = \sqrt{2}$. Observe also that $B^2 = W'(q_1)W'(q_2)$ with $B = \frac{W(q_2)-W(q_1)}{q_2-q_1}$ holds for $q_i = Q_i$ with $i = 1, 2$. To see this, rearrange the first-order conditions $C'(Q_i) = \frac{C(Q_2)-C(Q_1)}{Q_2-Q_1}$ to obtain, with $B = \frac{W(Q_2)-W(Q_1)}{Q_2-Q_1}$,

$$Q_1 W'(Q_1) = BQ_2 \quad \text{and} \quad Q_2 W'(Q_2) = BQ_1.$$

This implies $W'(Q_1)W'(Q_2) = B^2$.

For our piecewise linear specification, (OC.9) also implies that q_1 and q_2 are such that $\frac{W(q_2)-W(q_1)}{q_2-q_1} = \frac{W(Q_2(m))-W(Q_1(m))}{Q_2(m)-Q_1(m)}$. For $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ and $y \in (S(\underline{w}), w_1^{-1}(\underline{w}))$, the low-wage function $w_1(y, Q, \underline{w}) = W(q_1) + (y - q_1)\frac{W(q_2)-W(q_1)}{q_2-q_1}$ is therefore a parallel shift of the low-wage function

$$w_1(y) = W(Q_1(m)) + (y - Q_1(m))\frac{W(Q_2(m)) - W(Q_1(m))}{Q_2(m) - Q_1(m)}$$

satisfying $w_1(q_1, Q, \underline{w}) = W(q_1)$, $w_1(Q, Q, \underline{w}) = \underline{w}$ and $w_1(q_2, Q, \underline{w}) = W(Q_2(m))$ whose derivative with respect to y is $\frac{W(Q_2(m))-W(Q_1(m))}{Q_2(m)-Q_1(m)}$.

The identify $W(q_1) + (Q - q_1)\frac{W(Q_2(m))-W(Q_1(m))}{Q_2(m)-Q_1(m)} = \underline{w}$ also makes it easy to see that, as stated in Lemma 1, q_1 increases in $\underline{w}$, for a fixed Q, since the left-hand side increase in q_1. Intuitively, as the minimum wage increase, more units are procured at the minimum wage and fewer at the high wage. For the same reason, as Q increases, keeping $\underline{w}$ fixed, q_1 decreases.

Closed-form solutions for $\underline{C}_R$ and γ For $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$—that is, for quantities and minimum wages such that a two-wage mechanism is optimal—our piecewise linear specification permits a closed-form solutions for $\underline{C}_R$ and γ. As above, letting $B = \frac{W(Q_2)-W(Q_1)}{Q_2-Q_1}$ be short-hand for the slope of $w_1(Q)$ for $Q \in (Q_1, Q_2)$ and using the observation above about the parallel shift properties of the low-wage function in the presence of a minimum $\underline{w}$, q_1 for $Q \in (S(\underline{w}), w_1^{-1}(\underline{w}))$ is such that

$$W(q_1) + B(Q - q_1) = \underline{w}.$$

Using $W(q_1) = aq_1$, we thus have

$$q_1 = \frac{\underline{w} - BQ}{a - B}.$$

Given q_1, the parallel shift property then implies that q_2 is such that

$$W(q_2) = W(q_1) + B(q_2 - q_1).$$

Substituting q_1 and using $W(q_2) = bq_2 + (a-b)\underline{q}$ then yields

$$q_2 = \frac{\underline{w} - BQ}{b - B} - \frac{a-b}{b-B}\underline{q}.$$

Note that both q_1 and q_2 are linear in Q and $\underline{w}$ and their cross partials with respect to $\underline{w}$ and Q are 0. (Because $b < B < a$ holds, q_1 increases increases in $\underline{w}$ and decreases in Q and q_2 decreases in $\underline{w}$ and increases in Q, reflecting the properties stated in Lemma 1.) The minimum cost of procurement is thus

$$\underline{C}_R(Q, \underline{w}) = \underline{w}q_1 + (Q - q_1)W(q_2).$$

Because $W(q_2)$ is linear in q_2 and both q_1 and q_2 are, as noted, linear in Q, $\underline{C}_R(Q, \underline{w})$ is quadratic in Q, implying that $\underline{C}'_R(Q, \underline{w})$ is linear in Q and, because the cross partials of q_1 and q_2 with respect to $\underline{w}$ and Q are 0, linear in $\underline{w}$. Some straightforward algebra reveals that

$$\underline{C}'_R(Q, \underline{w}) = \frac{aB(a\underline{q} - b\underline{q} + 2bQ) - 2B^2\underline{w}}{(a-B)(B-b)}.$$

Because $\underline{C}'_R(Q, \underline{w})$ is linear in both Q and $\underline{w}$ with cross partials of 0, it follows that $\gamma(Q) = \lim_{\underline{w} \uparrow W(Q)} \underline{C}'_R(Q, \underline{w})$ will be piecewise linear in Q. Moreover, at $Q = \underline{q}$, $\gamma(\underline{q}) = aq = W(\underline{q})$, implying that at the minimum wage equal to $W(\underline{q})$, marginal cost of procurement will be continuous; see Figure 9 for illustrations.

OD Efficiency wages, migration and unemployment

Efficiency wage theory is customarily associated with the so-called Five-Dollar Day introduced by the Ford Motor Company in 1914.[10] A pervasive feature of that wage increase was that it caused workers to migrate to Detroit (see, for example, Sward, 1948, p.53). As we now show, when workers face a fixed cost of moving or participating in the labor market, this gives rise to a procurement cost function that is non-convex and consequently may make the use of an efficiency wage and involuntary unemployment optimal.

Specifically, consider a model with a monopsony firm that operates in a market in which the inverse labor supply function is W_A. We assume that this function is increasing and differential. For ease of exposition, we also assume that it is convex. This implies that absent any migration, the cost $QW_A(Q)$ of procuring Q units of labor is convex in Q, which in turn implies that without migration the firm optimally sets a market-clearing wage. To model migration, we assume that there is another pool of workers whose opportunity costs of working after migrating are described by the inverse supply function W_B, which we also assume to be convex, differentiable and increasing. Each worker in this pool has the same fixed cost $k > 0$ of moving. For $i \in \{A, B\}$, let $S_i(w) = W_i^{-1}(w)$ and, for $w > W_B(0) + k$, let $S_{AB}(w) = S_A(w) + S_B(w - k)$ denote the supply function that the firm faces. Moreover, for $Q > S_A(W_B(0) + k) =: \check{Q}$, we let $W_{AB}(Q) = S_{AB}^{-1}(Q)$. Then the inverse labor supply function the firm faces is $W(Q) = W_A(Q)$ for $Q \leq \check{Q}$ and $W(Q) = W_{AB}(Q)$ for $Q > \check{Q}$, yielding the cost of procurement function $C(Q) = W(Q)Q$ that accounts for migration.[11] The key implication of this is that $C(Q)$ is not convex. As shown below, we have

$$\lim_{Q \uparrow \check{Q}} C'(Q) > \lim_{Q \downarrow \check{Q}} C'(Q), \tag{OD.13}$$

[10] Contrary to perceived wisdom, a wage of five dollars per day was not uniformly applied across all workers from the time of its introduction in 1914. See, for example, Sward (1948) who notes that according to the company's financial statement 30% of the overall workforce were paid less than that in 1916.

[11] For example, for $W_A(Q) = 4Q$, $W_B(Q) = \frac{4}{7}Q + \frac{1}{2}$ and $k = 1/2$, we obtain the specification in (2). To see this, note that $W_A(Q) = 4Q$ and $W_B(Q) = 4Q/7 + 1/2$ imply $S_A(w) = w/4$ and $S_B(w) = 7(w - 1/2)/4$ and hence using $k = 1/2$ for $w \geq \hat{w}$ we have $S_{AB}(w) = S_A(w) + S_B(w - k) = 2w - 7/4$. Inverting S_{AB} yields $W_{AB}(Q) = Q/2 + 7/8$, which is the second line in (2). It remains to verify that $\hat{Q} = 1/4$, which is the case since $S_A(W_B(0) + k) = (1/2 + 1/2)/4 = 1/4$.

and the marginal cost of procurement decreases at $\check{Q}$. Geographical migration is only one possible interpretation of problems involving fixed costs. One can also think of workers moving from one industry to another or as workers joining the labor force at some fixed cost.

This perspective resonates with the prevalent view that migration is a cause of unemployment in the region to which workers migrate. However, here involuntary unemployment occurs not because of frictions such as costly search or costly wage adjustment, but rather as a consequence of optimal pricing on the part of the firm. It also offers a novel interpretation of the episode at the Ford Motor Company in the mid 1910s. According to this interpretation, with high enough wages, workers were willing to bear the fixed cost of moving, making the cost of procurement non-convex in the short run and efficiency wages optimal: "the greatest cost cutting measure" according to the dictum often attributed to Henry Ford. As the demand for its cars and its demand for labor continued increasing, eventually it became optimal to set market-clearing wages again. More broadly, the model with fixed costs of migration or labor market participation and an optimal mechanism used by the firm offers a framework in which economic expansion may be a cause of involuntary unemployment.

To see that (OD.13) holds, let $\breve{w} = W_B(0) + k$ (which is the same as $W_A(\check{Q})$). We then have

$$\lim_{Q \uparrow \check{Q}} C'(Q) = W_A(\check{Q}) + \check{Q}(S'_A)^{-1}(\check{Q}) > W_{AB}(\check{Q}) + \check{Q}(S'_{AB})^{-1}(\check{Q}) = \lim_{Q \downarrow \check{Q}} C'(Q).$$

Here, the inequality holds because $W_A(\check{Q}) = W_{AB}(\check{Q}) = \breve{w}$ and, for $w \geq \breve{w}$, $S_{AB}(w) = S_A(w) + S_B(w-k)$. This implies that $S'_{AB}(w) = S'_A(w) + S'_B(w-k) > S'_A(w)$, which in turn implies that $(S'_{AB})^{-1}(\check{Q}) = \frac{1}{S'_{AB}(\breve{w})} < \frac{1}{S'_A(\hat{w})} = (S'_A)^{-1}(\hat{Q})$. Consequently, the function C is not convex as required.

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