Optimal Hotelling Auctions*

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Abstract

We derive the optimal mechanism for a designer with products at each end of the Hotelling line for sale. Buyers have linear transportation costs and private information about their locations. These are independent draws from a commonly known distribution. Two independent auctions are optimal if and only if two independent auctions are efficient. Otherwise, the problem exhibits countervailing incentives and worst-off types that are endogenous to the allocation rule. Combining a saddle point property with an appropriate ironing procedure allows us to characterize the optimal selling mechanism as a function of a single parameter and to derive associated comparative statics. The optimal mechanism is always ex post inefficient; it involves entering a set of types into a lottery with positive probability. This set and the associated ex post lotteries vary non-trivially with the problem parameters. A two-stage clock auction involving coarse bidding implements the optimal selling mechanism in dominant strategies.

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1 Introduction

From radio spectrum licenses to airport landing slots, search keywords and cloud computing, sellers frequently sell multiple assets simultaneously, some of which may be homogeneous, while others are horizontally differentiated. This raises two questions: Under what conditions is it optimal to sell the homogeneous goods in independent auctions? If independent auctions are not optimal, then what is the optimal selling mechanism?

In this paper, we answer these two questions for a general, yet conceptually simple multi-product model based on Hotelling (1929). Specifically, we derive the mechanism that maximizes the seller's expected revenue subject to the buyers' incentive compatibility and interim individual rationality constraints, assuming that the seller has goods at each end of the unit interval for sale. Buyers have single-unit demand, the same commonly known gross valuation, linear transportation costs and are privately informed about their locations that are independent draws from the same, commonly known distribution.

The Hotelling setup captures horizontal differentiation—agents disagree about which product is the best even if their prices are the same—in a general way. For example, telecommunications firms may have different values for two different radio spectrum licences depending on their business model and on the portfolio of licenses they already hold. Likewise, airlines' and users' preferences over airport landing slots and cloud computing capacity may differ as a function of the time of day, with some preferring mornings, some evenings and others being largely indifferent. In search keyword auctions, airlines may have a strong preference for the keyword "flight" and hotels for the keyword "accommodation" while travel agents' preferences may lie somewhere in between. Each of these situations can be captured in a reduced-form fashion by varying the buyers' locations on the unit interval in the Hotelling model. From a mechanism design perspective, the model remains tractable because each agent's private information is one-dimensional.

We first show that the optimal selling mechanism reduces to running two independent optimal auctions if and only if no type of buyer has a positive valuation for both goods (and, consequently, horizontal differentiation plays no role). In terms of model parameters, normalizing the maximum transportation cost to one, this occurs if and only if the buyers' gross valuation is less than one half. This condition is independent of the distribution from which buyers' locations are drawn, the number of goods the seller has to sell at each location and the number of buyers. An equivalent way of stating this result is that running two independent auctions is optimal if and only if running two independent auctions is efficient.[1]

[1]This result has potentially important implications for market definitions in antitrust. For example, if buyers' locations are uniformly distributed, then two independent sellers that sell products from either end of the Hotelling line (and have a constant marginal cost of production of zero) do not compete with each other

We show that the optimal selling mechanism always involves ex post inefficiency. That is, the final allocation is inefficient with positive probability even if all of the goods are always sold or if all of the buyers are always served. The inefficiency arises from the randomization that is part of the optimal selling mechanism when two independent auctions are not optimal and from the positive reserve prices set under optimal auctions otherwise.

Notwithstanding the challenges involved in its derivation, the optimal selling mechanism permits a simple two-stage clock auction implementation that endows the agents with dominant strategies. The first stage of this auction provides buyers with a coarse bidding language that allows them to express whether they have a strong preference for one of the goods for sale, or whether they are more flexible in their preferences. The second stage is initiated if and only if a good is over-demanded, in which case an ascending clock auction is used to reduce demand. Agents who indicated that they have flexible preferences in the first stage never take any subsequent actions; their final allocation is contingent on other agents' bids (as are their prices) and may involve a lottery. Intuitively, the "threat" of this lottery allows the seller to extract higher payments from agents with strong preferences.

What makes the mechanism design problem challenging is that the Hotelling setup gives rise to what has become known as *countervailing incentives*.[2] Depending on their own location, as well as those of the other agents, a buyer's incentive compatibility constraint may be upward or downward binding. For example, the dominant strategy payment for a buyer allocated good 0 is determined by the largest location it could have reported without changing its allocation, while the dominant strategy payment for a buyer who is allocated good 1 is determined by the smallest location it could have reported without changing its allocation. A consequence of this is that the interim worst-off type for each buyer is endogenous to the allocation rule and typically in the interior of the type space. This contrasts with *standard* mechanism design settings, defined as mechanism design problems in which it is a priori known for which types the individual rationality constraint binds because the monotonicity incentive compatibility imposes on the allocation rule pins these down.Standard mechanism design settings include sales and procurement auctions, public good problems and two-sided allocations problems à la Myerson and Satterthwaite (1983).

We handle the aforementioned challenges by developing an ironing procedure that pins down the allocation rule as a function of a single parameter: a *critical type* for each agent. We show that the optimal mechanisms satisfies a saddle point property: given the agents' critical

if the gross valuation is less than one. Consequently, if the gross valuation is between a half and one, then an outsider observing two independent sellers would conclude that the two products or markets are independent. In contrast, the two markets are connected if the two products are sold by a single multi-product seller, which would, for example, be the case if the two independent sellers merged.

[2]This is "countervailing incentives" in the sense of Lewis and Sappington (1989), who coined the term.

types, the allocation rule maximizes the designer's objective function (i.e. the virtual surplus function) and given the allocation rule, the agents' critical types are also worst-off types that minimize the designer's objective function. The saddle point property uniquely determines the *critical worst-off type* that parameterizes our ironing procedure. The optimal mechanism involves an interim allocation rule that assigns an equal probability of obtaining a good from either end of the Hotelling line to all buyers located within an ironing interval that contains the worst-off type. Consequently, all types within the ironing interval are entered into an ex post lottery with positive probability. Another implication of countervailing incentives and endogenous worst-off types is that, as we show, there is no equivalence between interim and ex post individually rational mechanisms even with independent private values. This contrasts with standard mechanism settings for which such an equivalence exists.[3]

The saddle point property of optimal mechanisms also permits us to derive a number of intuitive comparative statics results. If there are fewer goods than agents, then an increase in the number of goods changes the ironing interval of the optimal mechanism because the mechanism is entirely pinned down by the feasibility constraints. If there are more goods than agents but the supply of at least one good is less than the number of agents, then the supply of at most one good constrains the ironing interval, and increasing the supply of the other good does not change the ironing interval. Once there is a sufficient supply of both goods, increasing the supply of either good no longer affects the ironing interval. This always happens before the supply of both goods is equal to the number of agents. (Of course, the allocation probabilities always vary with the number of goods.) In contrast to a standard optimal auction with identical distributions, where the reserve price depends neither on the number of units for sale nor on the number of bidders, the optimal mechanism in our setting varies non-trivially with these parameters. In the case of scarcity, this is true even when the ratios of bidders and the supply of each good are kept fixed.

The present paper relates to three strands of literature. First, starting with Cournot (1838), the study of optimal selling strategies for a monopoly has a long tradition in economics. The ironing procedure that is part of the optimal mechanism in our model is reminiscent of the ironing that is required in settings such as Hotelling (1931), Mussa and Rosen (1978), Myerson (1981), Bulow and Roberts (1989), Condorelli (2012), or Loertscher and Muir (2022) when the underlying distributions or revenue functions fail to satisfy what Myerson referred to as the regularity condition.[4] However, in our setting the failure of the pointwise maximizer to be monotone, which makes ironing necessary, is tied to the primitives

[3]See, for example, Manelli and Vincent (2011) and Gershkov et al. (2013).

[4]Similarly, in the problems analyzed by Dworczak et al. (2021) and Akbarpour et al. (2022), the optimality of rationing hinges on the strength of the designer's preference for redistribution and on properties of the type distribution.

of the model. It derives from the countervailing incentives of the agents and holds irrespective of the properties of the type distribution. Earlier work on problems with countervailing incentives includes Lewis and Sappington (1989) and Jullien (2000), who study single-agent problems, and Lu and Robert (2001) and Loertscher and Wasser (2019), who derive optimal mechanisms for variants of partnership models.[5]

Second, our paper relates to papers that study multi-product monopoly pricing in the Hotelling model (Hotelling, 1929) such as Jiang (2007), Fay and Xie (2008) and Balestrieri et al. (2021), which all assume a uniform distribution and a single agent. Balestrieri et al. show that the optimal selling mechanism involves lotteries regardless of whether the agent's transportation cost is linear or strictly concave or convex. Relative to these papers, our analysis of the single-agent case represents a generalization with respect to the type distribution and thereby clarifies that the lottery price is independent of the type distribution while the prices of the pure goods and the lottery interval depend on the distribution. More substantively, our paper generalizes the analysis beyond the single-agent case and solves for the optimal selling mechanism on the Hotelling line when there is non-trivial interaction, i.e. competition, between the agents.[6]

Last, the paper relates to a literature in auction design that emphasizes the importance of simplified bidding languages along the lines of Milgrom (2009) and Klemperer (2010). In particular, our two-stage clock auction in which the buyers submit coarse bids in the first stage implements the allocation rule of the optimal mechanism in dominant strategies. Moreover, the paper provides a simple test for a revenue-maximizing auctioneer who has multiple differentiated assets for sale and wonders whether these assets should be auctioned off independently or in an integrated auction: They should be auctioned off independently if and only if independent auctions are efficient.

The remainder of this paper is organized as follows. Section 2 introduces the model, mechanisms and constraints. In Section 3, we provide key mechanism design results, including the saddle point characterization of optimal mechanisms and the ironing procedure. Building on this, Section 4 then derives the optimal selling mechanism and its comparative statics. Section 5 shows that the optimal mechanism has a two-stage clock auction implementation. The paper concludes with a brief discussion in Section 6. Proofs of the main results are in the Appendix and the proofs of all other results can be found in the Online

[5] Much of the partnership literature, initiated by Cramton et al. (1987), has focused on ex post efficiency, where countervailing incentives are less of an issue as the allocation rule is fixed.

[6] In this regard, the paper also shows that there is a fundamental difference in mechanism design between models of vertical differentiation à la Mussa and Rosen (1978) and horizontal differentiation. With vertical differentiation, the worst-off type is pinned down by incentive compatibility alone whereas with horizontal differentiation it varies with the allocation rule. This contrasts with oligopoly models, for which Cremer and Thisse (1991) established a strong equivalence result.

Appendix. Online Appendix B shows that lotteries remain optimal if the designer maximizes a convex combination of revenue and social surplus, if goods are optimally placed rather than being exogenously placed at 0 and 1 and if transportation costs are not linear.

2 Setup

We study a variation of the Hotelling model in which a single seller (or designer) has $K_\ell \in \{1, \ldots, N\}$ identical goods for sale at two locations $\ell \in \{0, 1\}$. The seller faces N buyers (or agents), which we index by the set $\mathcal{N} := \{1, \ldots, N\}$. Each buyer has demand for at most one unit and an outside option of value 0. Given the single-unit demand assumption, the restriction that $K_\ell \leq N$ is without loss of generality. We assume that the seller's commonly known opportunity cost of selling any good is 0. Accordingly, the case with $K_0 = K_1 = N$ captures a multi-product *monopoly pricing problem* with constant marginal costs of zero. In contrast, if $K_\ell < N$ for some $\ell \in \{0, 1\}$, then there is non-trivial strategic interaction—or competition—between the buyers. In this case, we refer to the seller's problem as an *auction design problem*.

Each buyer $n \in \mathcal{N}$ independently draws its location $x_n \in [0, 1]$ from a commonly known absolutely continuous distribution F whose corresponding density f has full support on $[0, 1]$. We assume that buyers are privately informed about their realized locations and have a gross valuation of $v > 0$ for each of the goods. Buyers have linear transportation costs, so that the willingness to pay of a buyer at location $x \in [0, 1]$ for the good at 0 is $v - x$ while its willingness to pay for the good at 1 is $v - (1 - x)$.

We assume that all agents are risk-neutral and have quasi-linear utility. The expected payoff of an agent of type $x \in [0, 1]$ that receives a single unit of good 0 with probability $q_0 \in [0, 1]$ and receives a single unit of good 1 with probability $q_1 \in [0, 1]$, where $q_0 + q_1 \in [0, 1]$, and makes a payment of $t \in \mathbb{R}$ to the seller is then

$$(v - x)q_0 + (v - (1 - x))q_1 - t.$$

For ease of exposition, throughout the paper we assume that the virtual type functions

$$\psi_B(x) := x - \frac{1 - F(x)}{f(x)} \quad \text{and} \quad \psi_S(x) := x + \frac{F(x)}{f(x)} \tag{1}$$

are *regular* in the sense that they are strictly increasing in x. The virtual type function ψ_B—which is associated with buyers in the setting of Myerson (1981)—arises when agents' downward local incentive compatibility constraints are used to compute virtual surplus and the virtual type function ψ_S—which is associated with sellers in procurement auctions or bilateral trade settings à la Myerson and Satterthwaite (1983)—arises when agents' upward local incentive compatibility constraints are used to compute virtual surplus. Notice that for all $x \in (0,1)$ we have $\psi_B(x) < x < \psi_S(x)$; this property will play an important role in the analysis.

The regularity assumption simplifies the exposition by allowing us to proceed using standard inverses.[7] It also provides conceptual clarity in the sense that it ensures that any randomization that arises under the optimal mechanism is a general feature that is a direct result of countervailing incentives and does not hinge on non-monotonicity of the functions ψ_B and ψ_S (i.e. specific curvature properties of the distribution F). Similarly, assuming that all agents draw their locations from the same distribution rules out the possibility that allocative inefficiencies of the optimal mechanism derive from discrimination based on agent-specific distributions.

3 Mechanisms, saddle points and monotonicity

We now formally introduce direct mechanisms and the feasibility, incentive compatibility and individual rationality constraints. We then analyze the implications of countervailing incentives and show that any optimal mechanism satisfies a saddle point property. Finally, we establish a strong monotonicity property and develop an associated iron procedure for our multidimensional allocation rules.

3.1 Mechanisms and constraints

By the revelation principle, focusing on incentive compatible direct mechanisms is without loss of generality. Since agents have single-unit demand, it is also without loss of generality to focus on allocation rules that randomize over the set of outcomes $\{(0,0),(0,1),(1,0)\}$ for each agent, where the outcome (a,b) represents the allocation that gives a given agent a

[7] To dispense with the regularity assumption, simply replace the virtual type functions ψ_B and ψ_S with their *ironed* counterparts $\overline{\psi}_B$ and $\overline{\psi}_S$ and utilize the generalized inverses $\overline{\psi}_B^{-1}(z) = \max\{x \in [0,1] : \overline{\psi}_B(x) = z\}$ and $\overline{\psi}_S^{-1}(z) = \min\{x \in [0,1] : \overline{\psi}_S(x) = z\}$. The ironed virtual type functions can be computed as follows. For $q \in [0,1]$, introduce a revenue function $H_B(q) = qF^{-1}(1-q)$ and a cost function $H_S(q) = qF^{-1}(q)$ associated with the distribution F (so that, as observed by Bulow and Roberts (1989), $\psi_B(x) = H_B'(q)|_{q=1-F(x)}$ and $\psi_S(x) = H_S'(q)|_{q=F(x)}$). Letting $\overline{H}_B$ denote the *concavification* of the revenue function H_B (i.e. the smallest concave function that is weakly greater than H_B at every point) and $\overline{H}_S$ denote the *convexification* of the cost function H_S (i.e. the largest convex function that is weakly less than H_S at every point), the ironed virtual type functions are then given by $\overline{\psi}_B(x) = \overline{H}_B'(q)|_{q=1-F(x)}$ and $\overline{\psi}_S(x) = \overline{H}_S'(q)|_{q=F(x)}$.

units of good 0 and b units of good 1.[8] Moreover, because the buyers' locations are independent and identically distributed, it is also without loss of generality to restrict attention to direct mechanisms that are symmetric across the buyers. We therefore focus on incentive compatible direct mechanisms $\langle \boldsymbol{Q}, T\rangle$, where $\boldsymbol{Q} = (Q_0, Q_1)$ denotes the allocation rule and T denotes the transfer rule. The allocation rule

$$\boldsymbol{Q} : [0,1]^N \to \Delta\left(\{(0,0),(0,1),(1,0)\}\right)$$

maps the vector of agent reports to the set of probability measures over the set of outcomes $\{(0,0),(0,1),(1,0)\}$, so that $Q_\ell(x_n, \boldsymbol{x}_{-n})$ denotes the probability that agent $n \in \mathcal{N}$ is given a unit of good $\ell \in \{0,1\}$ upon reporting location $x_n \in [0,1]$, when the other buyers report a vector of locations $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$. Accordingly, the probability of being allocated no good at the reported type profile $(x_n, \boldsymbol{x}_{-n})$ is $1 - Q_0(x_n, \boldsymbol{x}_{-n}) - Q_1(x_n, \boldsymbol{x}_{-n})$. The transfer rule

$$T : [0,1]^N \to \mathbb{R}$$

maps the vector of reports to payments made to the designer, where $T(x_n, \boldsymbol{x}_{-n})$ is the payment made by agent $n \in \mathcal{N}$ upon reporting location $x_n \in [0,1]$, when the other buyers report a vector of locations $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$. By the Birkhoff–von Neumann theorem, a direct allocation rule $\boldsymbol{Q}$ is feasible if and only if, for all $\ell \in \{0,1\}$ and $\boldsymbol{x} \in [0,1]^N$, we have

$$\sum_{n \in \mathcal{N}} Q_\ell(x_n, \boldsymbol{x}_{-n}) \leq K_\ell. \tag{F}$$

Given a direct mechanism $\langle \boldsymbol{Q}, T\rangle$, we let

$$q_\ell(x_n) := \mathbb{E}_{\boldsymbol{x}_{-n}}[Q_\ell(x_n, \boldsymbol{x}_{-n})]$$

denote the interim probability that agent $n \in \mathcal{N}$ obtains a unit of good $\ell \in \{0,1\}$ upon reporting location $x_n \in [0,1]$, assuming that all other agents report truthfully for all possible realizations of their vector of types $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$. Similarly, we let

$$t(x_n) := \mathbb{E}_{\boldsymbol{x}_{-n}}[T(x_n, \boldsymbol{x}_{-n})]$$

denote the interim expected payment made by agent $n \in \mathcal{N}$ upon reporting location $x_n \in [0,1]$, assuming that all other agents report truthfully for all possible realizations of their

[8] This follows from the fact that an agent located at x who receives the allocation $(1,1)$ obtains the same utility from the allocation $(1,0)$ if $x \leq \frac{1}{2}$ and the allocation $(0,1)$ if $x \geq \frac{1}{2}$.

vector of types $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$. We also let $U(x,\hat{x}) := q_0(\hat{x})(v-x) + q_1(\hat{x})(v-1+x) - t(\hat{x})$ denote the interim expected payoff of a buyer located at $x \in [0,1]$ who reports $\hat{x} \in [0,1]$, assuming that all other agents report truthfully. Letting $U(x) := U(x,x)$ denote the corresponding interim expected payoff under truthful reporting, the mechanism satisfies (Bayesian) incentive compatibility (IC) if and only if, for all $n \in \mathcal{N}$ and all $x, \hat{x} \in [0,1]$,

$$U(x) \geq U(x,\hat{x}). \tag{IC}$$

The mechanism satisfies (interim) individual rationality (IR) if and only if, for all $x \in [0,1]$,

$$U(x) \geq 0. \tag{IR}$$

3.2 Implications of countervailing incentives

Our task is to derive the direct selling mechanism that maximizes the designer's ex ante expected revenue, subject to the incentive compatibility, individual rationality and feasibility constraints. We start by applying the envelope theorem (Milgrom and Segal, 2002), which provides the second part of the following convenient characterization of the class of incentive compatible direct mechanisms.

Lemma 1. *A direct mechanism $\langle \boldsymbol{Q}, T\rangle$ is incentive compatible if and only if, for all $x, \hat{x} \in [0,1]$,*

$$q_1(x) - q_0(x) \text{ is increasing in } x \tag{M}$$

and

$$U(x) = U(\hat{x}) + \int_{\hat{x}}^{x} (q_1(y) - q_0(y))\, dy. \tag{ICFOC}$$

Given an incentive compatible direct mechanism $\langle \boldsymbol{Q}, T\rangle$, we can combine (ICFOC) with $U(x) = q_0(x)(v-x) + q_1(v-(1-x)) - t(x)$ and solve for $t(x)$. For all $x, \hat{x} \in [0,1]$, this yields

$$t(x) = q_0(x)(v-x) + q_1(x)(v-(1-x)) - U(\hat{x}) - \int_{\hat{x}}^{x} (q_1(y) - q_0(y))\, dy. \tag{2}$$

Using (2) we can then compute the designer's ex ante expected revenue $R(\boldsymbol{Q}, T)$ via

$$R(\boldsymbol{Q}, T) = N\mathbb{E}[t(x)] = N \int_0^1 t(x)\, dF(x).$$

Letting the virtual type functions Ψ_0 and Ψ_1 be given by

$$\Psi_0(x,\hat{x}) := \begin{cases} v - \psi_S(x), & x < \hat{x} \\ v - \psi_B(x), & x \geq \hat{x} \end{cases} \quad \text{and} \quad \Psi_1(x,\hat{x}) := \begin{cases} v - (1-\psi_S(x)), & x \leq \hat{x} \\ v - (1-\psi_B(x)), & x > \hat{x} \end{cases}, \tag{3}$$

this then yields the following proposition.

Proposition 1. *For every* critical type $\hat{x} \in [0,1]$*, the designer's ex ante expected revenue under any incentive compatible direct mechanism* $\langle \boldsymbol{Q}, T\rangle$ *is given by*

$$R(\boldsymbol{Q},T) = N\left(\int_0^1 [q_0(x)\Psi_0(x,\hat{x}) + q_1(x)\Psi_1(x,\hat{x})]\,dF(x) - U(\hat{x})\right).$$

Given an arbitrarily chosen critical type $\hat{x} \in [0,1]$ and following Bulow and Roberts (1989) the virtual type $\Psi_0(x,\hat{x})$ can be interpreted as the marginal revenue associated with selling a buyer at $x \in [0,1]$ a unit of the good at location 0 and the virtual type $\Psi_1(x,\hat{x})$ as the marginal revenue associated with selling a buyer at $x \in [0,1]$ a unit of the good at location 1. These virtual type functions are piecewise-defined because the downward local incentive compatibility constraints are used to compute the virtual type function for buyers located to the right of the critical type $\hat{x}$ and the upward local incentive compatibility constraints are used to compute the virtual type function for buyers located to the left of the critical type $\hat{x}$.

The seller's problem is to determine the direct mechanism $\langle \boldsymbol{Q}, T\rangle$ that maximizes its ex ante expected revenue $R(\boldsymbol{Q},T)$ subject to the individual rationality constraints given in (IR), the feasibility constraints (F), and the monotonicity condition (M). This problem is complicated by the fact the agents' worst-off types are endogenous to the allocation rule, which sets the problem apart from standard mechanism design problems. Formally, for an incentive compatible direct mechanism $\langle \boldsymbol{Q}, T\rangle$, denoting by $\Omega(\boldsymbol{Q}) := \operatorname{argmin}_{x\in[0,1]}\{U(x)\}$ the set of worst-off types, we have the following result:

Lemma 2. *Suppose that* $\langle \boldsymbol{Q}, T\rangle$ *is an incentive compatible direct mechanism. Then we have the following.*

(i) If there exists an $x \in [0,1]$ *such that* $q_1(x) - q_0(x) = 0$*, then* $\Omega(\boldsymbol{Q}) = \{x \in [0,1] : q_1(x) - q_0(x) = 0\}$.

(ii) If there does not exist an $x \in [0,1]$ *such that* $q_1(x) - q_0(x) = 0$ *but there exists an* $x \in [0,1]$ *such that* $q_1(x) - q_0(x) > 0$*, then* $\Omega(\boldsymbol{Q}) = \inf_{x\in[0,1]}\{q_1(x) - q_0(x) > 0\}$.

(iii) If there does not exist an $x \in [0,1]$ *such that* $q_1(x) - q_0(x) = 0$ *but there exists an* $x \in [0,1]$ *such that* $q_1(x) - q_0(x) < 0$*, then* $\Omega(\boldsymbol{Q}) = \sup_{x\in[0,1]}\{q_1(x) - q_0(x) < 0\}$.

Note that parts (ii) and (iii) of Lemma 2 imply that $\Omega(\boldsymbol{Q}) = y$ if $q_1(x) - q_0(x) < 0$ for all $x \leq y$ and $q_1(x) - q_0(x) > 0$ for all $x > y$. A consequence of Lemma 2 is that the seller's ex ante expected revenue $R(\boldsymbol{Q}, T)$ depends on both the allocation rule $\boldsymbol{Q}$ *and* the transfer rule T, which implicitly appears in the term $U(\hat{x})$ in Proposition 1. It is thus not a priori clear that, as in standard mechanism design problems, one can first solve for the allocation rule of the optimal mechanism and then determine the transfer rule by using (2) and making the constraint (IR) bind for the worst-off types. We next show that this standard procedure can still be applied after first establishing that optimal mechanisms satisfy a saddle point property that is independent of the transfer rule.

3.3 Saddle point theorem

We first show that $R(\boldsymbol{Q}, T)$ can be decomposed into two components: a virtual surplus function $\tilde{R}(\boldsymbol{Q}, \hat{x})$ that only depends on the allocation rule $\boldsymbol{Q}$ and the critical type $\hat{x}$ and a term $U(\hat{x})$ that also implicitly depends on the transfer rule T. As we will see, the optimal mechanism is a saddle point of $\tilde{R}(\boldsymbol{Q}, \hat{x})$. Specifically, we let

$$\tilde{R}(\boldsymbol{Q}, \hat{x}) := \int_0^1 \left[q_0(x)\Psi_0(x, \hat{x}) + q_1(x)\Psi_1(x, \hat{x})\right] dF(x)$$

denote the sum of the terms in the seller's ex ante expected revenue that only depend on the virtual type functions Ψ_0 and Ψ_1 and the allocation rule $\boldsymbol{Q}$ and are independent of the transfer rule T. We refer to this function as the seller's *virtual surplus function*. By Proposition 1, the seller's ex ante expected revenue can be rewritten as

$$R(\boldsymbol{Q}, T) = N\left(\tilde{R}(\boldsymbol{Q}, \hat{x}) - U(\hat{x})\right). \tag{4}$$

Given an incentive compatible direct mechanism $\langle \boldsymbol{Q}, T\rangle$, the following lemma identifies the set of worst-off types for each buyer as the set of critical types that *minimize* the seller's virtual surplus function $\tilde{R}(\boldsymbol{Q}, \cdot)$.

Lemma 3. *Given any incentive compatible direct mechanism $\langle \boldsymbol{Q}, T\rangle$, we have*

$$\Omega(\boldsymbol{Q}) = \underset{\hat{x}\in[0,1]}{\arg\min}\, \tilde{R}(\boldsymbol{Q}, \hat{x}).$$

While Lemma 3 is not required to solve standard mechanism design problems, the result still holds in these settings.[9] As stated, under the optimal selling mechanism $\langle \boldsymbol{Q}^*, T^*\rangle$ we

[9] Consider, for example, an optimal auction problem à la Myerson (1981) in which bidders' values are

must have $U(\omega) = 0$ for all $\omega \in \Omega(\boldsymbol{Q^*})$. Moreover, if $U(\omega) = 0$ holds for all $\omega \in \Omega(\boldsymbol{Q^*})$, then all the individual rationality constraints (IR) are satisfied. Using the decomposition from (4), the designer can therefore focus on maximizing

$$R(\boldsymbol{Q}, T) = N\left(\tilde{R}(\boldsymbol{Q}, \boldsymbol{\omega}) - U(\omega)\right) = N\tilde{R}(\boldsymbol{Q}, \omega), \tag{5}$$

subject to the constraint that $\omega \in \Omega(\boldsymbol{Q})$, the feasibility constraints (F), and the monotonicity condition (M). Let $\mathcal{Q}$ denote the set of feasible allocation rules such that the monotonicity constraint (M) holds. Combining these observations with Lemma 3 then shows that the revenue-maximizing allocation rule $\boldsymbol{Q}^*$ satisfies

$$\boldsymbol{Q}^* \in \arg\max_{\boldsymbol{Q} \in \mathcal{Q}} \min_{\hat{x} \in [0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}).$$

However, instead of solving this maximin problem directly, the following theorem provides a *saddle point* characterization of the optimal mechanism $\boldsymbol{Q}^*$ and the corresponding *critical worst-off type* ω^*.

Theorem 1. *A* saddle point $(\boldsymbol{Q}^*, \omega^*) \in \mathcal{Q} \times [0,1]$ *of the virtual surplus function $\tilde{R}$ satisfies*

$$\boldsymbol{Q}^* \in \arg\max_{\boldsymbol{Q} \in \mathcal{Q}} \tilde{R}(\boldsymbol{Q}, \omega^*), \tag{6}$$

$$\omega^* \in \arg\min_{\hat{x} \in [0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x}). \tag{7}$$

A saddle point $(\boldsymbol{Q}^*, \omega^*)$ that characterizes the optimal selling mechanism exists and is unique in the following sense: another allocation rule $\boldsymbol{Q'} \neq \boldsymbol{Q^*}$ is optimal and satisfies

$$\boldsymbol{Q'} \in \arg\max_{\boldsymbol{Q} \in \mathcal{Q}} \min_{\hat{x} \in [0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}) \tag{8}$$

if and only if $(\boldsymbol{Q'}, \omega^*)$ is a saddle point.

In words, a saddle point $(\boldsymbol{Q}^*, \omega^*)$ satisfies a consistency condition whereby the allocation rule $\boldsymbol{Q}^*$ maximizes the designer's objective given the critical type ω^*, and the critical type ω^* is a worst-off type with respect to the allocation rule $\boldsymbol{Q}^*$. An analogous result to Theorem

distributed on $[0,1]$ according to a distribution F. This gives rise to the virtual type functions defined in (1). For a given critical type $\hat{x} \in [0,1]$, we have $\Psi(x, \hat{x}) = \psi_S(x)$ for $x < \hat{x}$ and $\Psi(x, \hat{x}) = \psi_B(x)$ otherwise. Consequently, the virtual surplus is $\tilde{R}(\boldsymbol{Q}, \hat{x}) = \mathbb{E}_{\boldsymbol{x}}[\sum_{n=1}^N \Psi(x_n, \hat{x}) Q(x_n, \boldsymbol{x}_{-n})]$, where $Q(x_n, \boldsymbol{x}_{-n})$ is the probability buyer n is allocated the good at type profile $(x_n, \boldsymbol{x}_{-n})$. By IC, $Q(x_n, \boldsymbol{x}_{-n})$ is increasing in x_n. Since $\psi_S(x) > \psi_B(x)$ holds for all $x \in [0,1]$, this implies that if $Q(x_n, \boldsymbol{x}_{-n})$ increasing in x_n, then $\tilde{R}(\boldsymbol{Q}, \hat{x})$ is increasing in $\hat{x}$ and is therefore minimized at $\hat{x} = 0$. Since $x = 0$ is also the worst-off type for any increasing allocation rule, the lemma then follows.

1 also characterizes the solution of the optimal partnership dissolution problem studied by Loertscher and Wasser (2019). As mentioned, the procedure to solve standard mechanism design problems relies on the separability of the allocation and the transfer rule. One first solves for the allocation rule of the optimal mechanism and then uses the envelope formula to derive the associated transfers. In problems with countervailing incentives, it is not a priori clear whether this standard procedure is valid because the allocation rule and worst-off types are interdependent. In this regard, the virtue of the saddle point theorem is that it shows that the aforementioned separability still applies because the allocation rule of the optimal mechanism is independent of the transfer rule.[10]

3.4 Strong monotonicity and ironing

Theorem 1 simplifies our search for an optimal selling mechanism to a search for a saddle point of the designer's virtual surplus function $\tilde{R}$. However, for a fixed critical type $\hat{x} \in [0,1]$, we still need to characterize the feasible, monotone allocation rules that maximize the virtual surplus function

$$\tilde{R}(\boldsymbol{Q}, \hat{x}) := \int_0^1 \left[q_0(x)\Psi_0(x,\hat{x}) + q_1(x)\Psi_1(x,\hat{x}) \right] dF(x). \tag{9}$$

Recall that $\psi_B(x) < x < \psi_S(x)$ holds for all $x \in (0,1)$ (with $\psi_B(0) < \psi_S(0) = 0$ and $\psi_S(1) > \psi_B(1) = 1$). The definitions given in (3) then imply that, for all $\hat{x} \in (0,1)$, the virtual type function $\Psi_0(x,\hat{x})$ increases discontinuously at $x = \hat{x}$ and the virtual type function $\Psi_1(x,\hat{x})$ decreases discontinuously at $x = \hat{x}$. Consequently, we cannot provide a general characterization of the optimal allocation rule by simply pointwise maximizing the virtual surplus function, as this procedure may violate the monotonicity constraint. That is, as a result of countervailing incentives, the underlying mechanism design problem is *always* non-regular.

Suppose that *full market coverage* is feasible and optimal. That is, suppose that $q_0(x) + q_1(x) = 1$ holds for all $x \in [0,1]$. Then (9) becomes $\tilde{R}(\boldsymbol{Q}, \hat{x}) = \int_0^1 q_0(x)(\Psi_0(x,\hat{x}) - \Psi_1(x,\hat{x}))\, dF(x) + v - 1 + \hat{x}$ and (M) is equivalent to requiring that q_0 is non-increasing. Ironing the difference in the virtual type functions $\Psi_0(x,\hat{x}) - \Psi_1(x,\hat{x})$ then permits pointwise maximization of the seller's objective function—with $\Psi_0(x,\hat{x}) - \Psi_1(x,\hat{x})$ replaced by its ironed counterpart—without violating (M). However, the conditions under which this approach is always applicable are somewhat stringent as they require that both v and the

[10] Theorem 1 does not depend on the distributions being identical, the transportation costs being linear or on the designer's objective being profit. Any convex combination of social surplus and profit yields the same result.

total supply of goods $K_0 + K_1$ are sufficiently large. In particular, a sufficient condition for full market coverage is $K_0 + K_1 \geq N$ and $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$.[11] To solve the problem in general an alternative approach is therefore required. We now formulate a more general ironing procedure, which we then apply in Section 4 to explicitly characterize the optimal selling mechanism.

As mentioned, (M) only requires that $q_0 - q_1$ is decreasing. Although one may intuitively expect q_0 to be decreasing and q_1 to be increasing under any optimal mechanism, (M) may of course hold even if q_0 and q_1 are not independently monotone. We now show that it is nevertheless without loss of generality to focus on allocation rules that satisfy this *strong* monotonicity property and are such that

$$q_1(x) \text{ and } -q_0(x) \text{ are increasing in } x. \tag{SM}$$

The significance of this result is that it allows us to independently iron the virtual type functions $\Psi_0(\cdot, \hat{x})$ and $\Psi_1(\cdot, \hat{x})$, and pointwise maximize the seller's objective without violating (M). Lemma 4, where $\mathcal{Q}^{\mathrm{SM}} \subset \mathcal{Q}$ denotes the set of feasible allocation rules such that (SM) holds, formally states that the focus on allocation rules satisfying strong monotonicity is without loss of generality.

Lemma 4. *Given any feasible and monotone allocation rule $\boldsymbol{Q} \in \mathcal{Q}$, there exists a feasible and strongly monotone allocation rule $\hat{\boldsymbol{Q}} \in \mathcal{Q}^{SM}$ such that: (i) $q_1 - q_0 = \hat{q}_1 - \hat{q}_0$, (ii) $\int_0^1 (q_0(x) - \hat{q}_0(x))\, dF(x) = 0$ and (iii) $\int_0^1 (q_1(x) - \hat{q}_1(x))\, dF(x) = 0$. Consequently, we have $\Omega(\boldsymbol{Q}) = \Omega(\hat{\boldsymbol{Q}})$. Moreover, if we take any $\omega \in \Omega(\boldsymbol{Q})$ and set $U(\omega) = 0$, then the designer's revenue and the interim expected payoff of each buyer are invariant under the transformation that replaces the allocation rule $\boldsymbol{Q}$ with the allocation rule $\hat{\boldsymbol{Q}}$.*

With Lemma 4 at hand and given any critical type $\hat{x} \in (0, 1)$, we can now independently iron each of the virtual type functions Ψ_0 and Ψ_1. The appropriate ironed virtual type functions are given by

$$\overline{\Psi}_0(x, \hat{x}) = \begin{cases} v - \psi_S(x), & x \in [0, \underline{x}(\hat{x})) \\ z_0(\hat{x}), & x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})] \\ v - \psi_B(x), & x \in (\overline{x}(\hat{x}), 1] \end{cases} \tag{10}$$

[11] If $v < 1 + \frac{1}{f(0)}$ ($v < 1 + \frac{1}{f(1)}$) then there is a positive mass of types that would never be served by a monopoly seller with units for sale at location 1 (0).

and

$$\overline{\Psi}_1(x,\hat{x}) = \begin{cases} v-(1-\psi_S(x)), & x\in[0,\underline{x}(\hat{x})) \\ z_1(\hat{x}), & x\in[\underline{x}(\hat{x}),\overline{x}(\hat{x})] \\ v-(1-\psi_B(x)), & x\in(\overline{x}(\hat{x}),1] \end{cases}. \tag{11}$$

Here, each choice of critical type $\hat{x}\in(0,1)$ uniquely determines the ironing parameters $z_0(\hat{x})\in(v-1,v)$ and $z_1(\hat{x})\in(v-1,v)$, as well as the corresponding ironing interval $[\underline{x}(\hat{x}),\overline{x}(\hat{x})]$. The ironing parameters $z_0(\hat{x})$ and $z_1(\hat{z})$ and the ironing interval $[\underline{x}(\hat{x}),\overline{x}(\hat{x})]$ are pinned down by the conditions

$$\int_{\underline{x}(\hat{x})}^{\hat{x}} (v-\psi_S(x)-z_0(\hat{x}))\,dF(x) = \int_{\hat{x}}^{\overline{x}(\hat{x})} (z_0(\hat{x})-(v-\psi_B(x)))\,dF(x), \tag{12}$$

$$\int_{\underline{x}(\hat{x})}^{\hat{x}} (v-(1-\psi_S(x))-z_1(\hat{x}))\,dF(x) = \int_{\hat{x}}^{\overline{x}(\hat{x})} (z_1(\hat{x})-(v-(1-\psi_B(x))))\,dF(x), \tag{13}$$

$$\underline{x}(\hat{x}) = \min\left\{0,\psi_S^{-1}\left(v-z_0(\hat{x})\right)\right\} = \min\left\{0,\psi_S^{-1}\left(1-v+z_1(\hat{x})\right)\right\}, \tag{14}$$

$$\overline{x}(\hat{x}) = \max\left\{\psi_B^{-1}\left(v-z_0(\hat{x})\right),1\right\} = \max\left\{\psi_B^{-1}\left(1-v+z_1(\hat{x})\right),1\right\}. \tag{15}$$

Summing (12) and (13) we see that $z_0(\hat{x})+z_1(\hat{x})=2v-1$ holds. This shows why the ironing intervals for the function $\overline{\Psi}_0(\cdot,\hat{x})$ and $\overline{\Psi}_1(\cdot,\hat{x})$ coincide and we have (14) and (15). Notice that for all $\hat{x}\in(0,1)$, we also have $\underline{x}(\hat{x})<\hat{x}<\overline{x}(\hat{x})$. Moreover, we have the following comparative statics concerning the ironing parameters, which will prove useful for establishing uniqueness of the critical worst-off type (Theorem 2) and comparative statics concerning the optimal selling mechanism (Proposition 4).

Lemma 5. *The ironing parameters $z_0(\hat{x})$ and $z_1(\hat{x})$ are continuous in $\hat{x}$ and are decreasing and increasing in $\hat{x}$, respectively. Moreover, the endpoints $\underline{x}(\hat{x})$ and $\overline{x}(\hat{x})$ of the ironing interval are continuous and increasing in $\hat{x}$.*

Combining the insights from this subsection with Theorem 1, for a given specification of the problem laid out in Section 2 we can now characterize the optimal selling mechanism by finding a saddle point of the ironed virtual surplus function

$$\overline{R}(\boldsymbol{Q},\hat{x}) := \int_0^1 \left[q_0(x)\overline{\Psi}_0(x,\hat{x})+q_1(x)\overline{\Psi}_1(x,\hat{x})\right]dF(x). \tag{16}$$

4 Optimal selling mechanisms

We now provide an explicit characterization of the optimal selling mechanisms. We first show that $v > \frac{1}{2}$ is necessary and sufficient for the optimal selling mechanism *not* to consist of simply running two independent optimal auctions. Assuming $v > \frac{1}{2}$, we then characterize the optimal selling mechanisms, which we refer to as *lottery-augmented auctions.* The section concludes with comparative statics of these lottery-augmented auctions.

4.1 (Non)-optimality of independent auctions

Combining our ironing procedure with the saddle point condition allows us to provide the necessary and sufficient condition for when the seller optimally runs independent auctions at 0 and 1. Formally, we have the following proposition.

Proposition 2. *The optimal selling mechanism involves running independent auctions for goods* 0 *and* 1 *if and only if* $v \leq \frac{1}{2}$*. If it is optimal for the seller to run two independent auctions, then the seller optimally sets a reserve price of* $v - \psi_S^{-1}(v)$ *for good* 0 *and a reserve price of* $v - (1 - \psi_B^{-1}(1 - v))$ *for good* 1*. Consequently, if independent auctions are optimal, then buyers with locations* $x \in [\psi_S^{-1}(v), 1 - \psi_B^{-1}(1 - v)]$ *are never served.*

Proposition 2 shows that whether or not the optimal mechanism reduces to simply running independent optimal auctions at 0 and 1 (and we essentially end up with two copies of the problem of Myerson (1981)) only depends on v. In particular, it is independent of the distribution F and of the endowment (K_0, K_1). As noted, the necessary and sufficient condition for two independent auctions to be revenue maximizing, i.e. $v \leq \frac{1}{2}$, is the same as the necessary and sufficient condition for two independent auctions, with reserve prices of zero, to induce an efficient allocation.

4.2 Lottery-augmented auctions

We now turn to the more challenging case where $v > \frac{1}{2}$. As Proposition 2 shows, in this case the optimal selling mechanism does not simply reduce to running an independent optimal auction for each good. As we will see, if $v > \frac{1}{2}$, then all types are served with positive probability under the optimal selling mechanism. Moreover, types within the ironing interval $[\underline{x}(\omega^*), \overline{x}(\omega^*)]$, where $\omega^* \in (0, 1)$ is the critical worst-off type, participate in an ex post lottery with positive probability under the optimal selling mechanism.

This subsection is structured as follows. We first state two fundamental properties (lemmas 6 and 7) that the optimal mechanism has to satisfy whenever $v > \frac{1}{2}$. We then apply these properties to characterize the optimal mechanism, distinguishing between parameterizations with *scarcity*, defined as $K_0 + K_1 \leq N$, and *abundance*, defined as $K_0 + K_1 > N$.

We characterize the optimal selling mechanism by computing, for every critical type $\hat{x} \in [0,1]$, the ex post allocation rules $\overline{\boldsymbol{Q}}(\cdot, \hat{x}; \gamma)$ that pointwise maximize the ironed virtual surplus function $\overline{R}(\boldsymbol{Q}, \hat{x})$ defined in (16). Note that these pointwise maximizing ex post allocation rules are not necessarily unique and we therefore introduce an arbitrary index $\gamma \in \Gamma$ in order to distinguish them. We then determine the corresponding interim allocation rules $\overline{\boldsymbol{q}}(\cdot, \hat{x}; \gamma)$ and finally characterize the optimal selling mechanism by checking the saddle point condition (i.e. determining which critical type $\hat{x} \in (0,1)$ is also a worst-off type under one of the interim allocation rules that it generates). Abusing notation, we let $\overline{Q}_\ell(i, j, \hat{x}; \gamma)$ denote the probability that a given buyer is allocated a unit of good $\ell \in \{0,1\}$ upon reporting $x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ when $i \geq 0$ other buyers report locations below $\underline{x}(\hat{x})$ and $j \geq 0$ other buyers report locations above $\overline{x}(\hat{x})$ under the pointwise-maximizing ex post allocation rule $\overline{\boldsymbol{Q}}(\cdot, \hat{x}; \gamma)$.[12] Letting $p(i, j, \hat{x})$ denote the probability of any feasible state $(i, j) \in \{0, 1, \ldots, N-1\}^2$ with $i + j \leq N - 1$, we have

$$p(i, j, \hat{x}) = \binom{N-1}{i, N-1-i-j, j} \left(F(\underline{x}(\hat{x}))\right)^i \left(F(\overline{x}(\hat{x})) - F(\underline{x}(\hat{x}))\right)^{N-1-i-j} \left(1 - F(\overline{x}(\hat{x}))\right)^j,$$

where $\binom{N-1}{i,N-1-i-j,j} = \frac{(N-1)!}{i!(N-1-i-j)!j!}$ is a multinomial coefficient.[13] The interim probability that any buyer that reports a type $x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ is allocated a unit of good $\ell \in \{0,1\}$ is then given by

$$\overline{q}_\ell(\hat{x}; \gamma) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-i} p(i, j, \hat{x}) \overline{Q}_\ell(i, j, \hat{x}; \gamma). \tag{17}$$

By construction, $\hat{x} \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ holds and all buyers within the ironing interval receive the same allocation under any ex post allocation rule $\overline{\boldsymbol{Q}}(\cdot, \hat{x}; \gamma)$ that pointwise maximizes (16). It follows that $(\overline{\boldsymbol{Q}}(\cdot, \hat{x}; \gamma), \hat{x})$ is a saddle point if and only if it satisfies the condition stated in the following lemma.

Lemma 6. *Whenever $v > \frac{1}{2}$, there exists a critical worst-off type $\omega^* \in (0,1)$ that satisfies $\overline{q}_0(\omega^*; \gamma) = \overline{q}_1(\omega^*; \gamma)$ for some index $\gamma \in \Gamma$.*

The critical worst-off ω^* identified in Lemma 6 parameterizes the optimal selling mechanism insofar as ω^* pins down the ex post allocation rule as a pointwise maximizer of the

[12] Restricting attention to allocation rules that offer all types in the ironing interval the same ex post allocation is without loss of generality.

[13] Adopting the standard combinatorial convention that $0^0 = 1$ means that the expression for $p(i, j, \hat{x})$ is still valid when $\hat{x} = 0, 1$ and for critical types $\hat{x} \in (0,1)$ such that $\underline{x}(\hat{x}) = 0$ or $\overline{x}(\hat{x}) = 1$.

ironed virtual surplus function $\overline{R}(\cdot, \omega^*)$ and the corresponding transfer rule can then be computed by substituting the allocation rule, $\hat{x} = \omega^*$ and $U(\omega^*) = 0$ into (2).

For every critical type $\hat{x} \in [0,1]$ that is a candidate for a saddle point, we now systematically compute the corresponding pointwise maximizing ex post allocation rules $\overline{Q}_\ell(i, j, \hat{x}; \gamma)$ for buyers in the ironing interval $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$. The following lemma shows that without loss of generality attention can be restricted to to critical types $\hat{x} \in (0,1)$ that fall within two cases. Here and below, we use the notation $-\ell$ for "not ℓ".

Lemma 7. *Whenever $v > \frac{1}{2}$ a critical worst-off type $\omega^* \in (0,1)$ is either such that: (i) $z_0(\omega^*), z_1(\omega^*) > 0$, or (ii) $z_\ell(\omega^*) > 0$ and $z_{-\ell}(\omega^*) = 0$ for some $\ell \in \{0,1\}$.*

Letting $\hat{x}_0 := \min\left\{0, z_1^{-1}(0)\right\}$ and $\hat{x}_1 := \max\left\{z_0^{-1}(0), 1\right\}$, Lemma 7 immediately implies that $\omega^* \in [\hat{x}_0, \hat{x}_1]$.[14] Another critical type that will play an important role in the analysis is $\hat{x}_A \in [0,1]$ such that $z_0(\hat{x}_A) = z_1(\hat{x}_A)$.[15] Note that for $v > \frac{1}{2}$, we have $\hat{x}_A \in (\hat{x}_0, \hat{x}_1)$. Figure 1 provides an illustration of the ironed virtual type functions under the critical types $\hat{x} \in \{\hat{x}_0, \hat{x}_A, \hat{x}_1\}$.

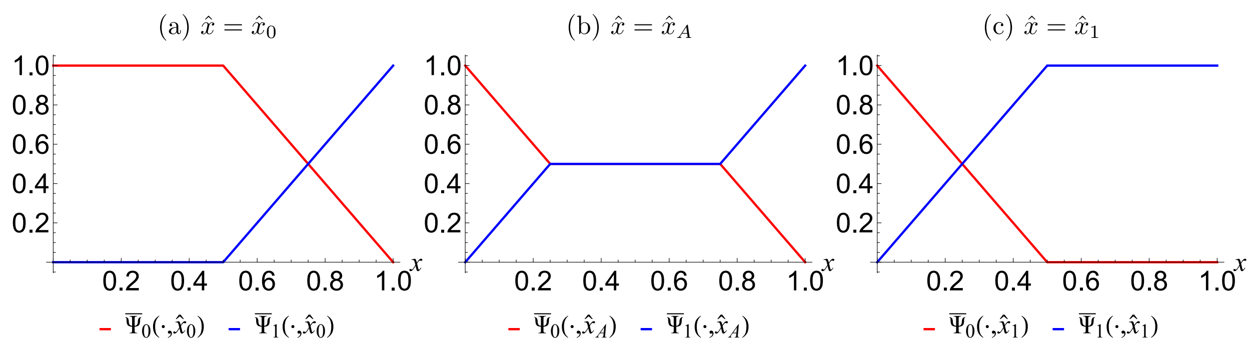

Figure 1: Assuming that buyer locations are uniformly distributed and $v = 1$, this figure displays the ironed virtual type functions $\overline{\Psi}_0(\cdot, \hat{x})$ (red) and $\overline{\Psi}_1(\cdot, \hat{x})$ (blue) for three cases: $\hat{x} = \hat{x}_0$, $\hat{x} = \hat{x}_A$ and $\hat{x} = \hat{x}_1$.

With these fundamental properties at hand, we now turn to the characterization of the optimal mechanisms, explicitly computing the pointwise maximizing ex post allocation rules for cases involving scarcity, before then dealing with the cases involving abundance. Fixing a critical type $\hat{x} \in [\hat{x}_0, \hat{x}_1]$, we proceed by essentially reading the pointwise maximizing ex post allocation rules off the ironed virtual type functions given any realization of reports $\boldsymbol{x}$, and

[14] Note that if $v > \frac{1}{2}$ then we have $\hat{x}_1 > \hat{x}_0$. Moreover, if $v \geq 1 + 1/f(0)$, then $\hat{x}_0 = 0$ and if $v \geq 1 + 1/f(1)$, then $\hat{x}_1 = 1$.

[15] Note that such a critical type always exists and corresponds to an ironing interval with $\underline{x}(\hat{x}_A) > 0$ and $\overline{x}(\hat{x}_A) < 1$ since, for all $\hat{x} \in [0,1]$, we have $\overline{\Psi}_0(0, \hat{x}) > \overline{\Psi}_1(0, \hat{x})$ and $\overline{\Psi}_1(1, \hat{x}) > \overline{\Psi}_0(1, \hat{x})$.

thereby determining the ex post allocation for buyers in the ironing interval for every feasible state (i, j). Interestingly, the problem with scarcity is slightly simpler because competition among the buyers and the feasibility constraints (F) provide tighter constraints on the optimal selling mechanism. The additional degrees of freedom associated with abundance make the analysis slightly more involved.

Scarcity: $K_0 + K_1 \leq N$. We begin with Case 1 from Lemma 7 and consider critical types $\hat{x} \in (\hat{x}_0, \hat{x}_1)$ (since this is the set of critical types such that $z_0(\hat{x}), z_1(\hat{x}) > 0$). In each feasible state (i, j), the seller's preferences, as encoded by the ironed virtual type functions $\overline{\Psi}_0$ and $\overline{\Psi}_1$, together with the feasibility constraints then uniquely pin down the ex post lottery offered to buyers that report a location in the ironing interval. In this case, the seller's preferences are such that it strictly prefers to serve buyers in the ironing interval at either location whenever this is feasible, i.e. if any goods remain after first allocating the units of good 0 to the i buyers that reported a location $x < \underline{x}(\hat{x})$ and the units of good 1 to the j buyers that reported a location $x > \overline{x}(\hat{x})$. Moreover, whenever the buyers in the ironing interval are entered into a non-trivial lottery (i.e. one involving goods from both locations), there are always weakly fewer goods than buyers involved in the lottery. Consequently, the pointwise maximizing ex post allocation rules are unique (up to a set of measure zero) and we have $\overline{Q}_0(i, j, \hat{x}) = \frac{K_0 - i}{N-i-j}\mathbb{1}(i < K_0,\, j < N - K_0) + \mathbb{1}(j \geq N - K_0)$ and $\overline{Q}_1(i, j, \hat{x}) = \frac{K_1 - j}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(i \geq N - K_1)$ (see Appendix B.10 for a derivation).

Next, we deal with Case 2 from Lemma 7. For $\ell \in \{0, 1\}$, we consider the critical type $\hat{x} = \hat{x}_\ell$ (which is such that $z_\ell(\hat{x}) > 0$ and $z_{-\ell}(\hat{x}) = 0$). Here, the seller allocates the good at location ℓ to buyers in the ironing interval whenever this is feasible. However, the seller is now indifferent between serving the buyers in the ironing interval at location $-\ell$ and not serving these buyers. For these cases the pointwise maximizing ex post allocation rules are not unique. However, any ex post lottery offered by the seller to buyers in the ironing interval under pointwise maximization can be characterized as a convex combination of two extremal lotteries: one which allocates the good at location ℓ to as many buyers in the ironing interval as possible and then serves none of the remaining buyers and one which allocates the good at location ℓ to as many buyers in the ironing interval as possible and then allocates the good at location $-\ell$ to as many of the remaining buyers as possible. Utilizing the expressions from the previous case, it is straightforward to compute the corresponding set of pointwise maximizing ex post allocation rules. Since these ex post allocation rules correspond to binary convex combinations, the index set Γ can be taken to be the set $[0, 1]$ without loss of generality.

Putting all of this together, the following functions specify the set of pointwise maximizing ex post allocation rules for each critical type $\hat{x}$ that is a candidate for the critical worst-off type ω^* under scarcity. In particular, for all feasible states $(i,j) \in \{0,1,\ldots,N-1\}^2$ such that $i+j \leq N-1$, critical types $\hat{x} \in [\hat{x}_0, \hat{x}_1]$ and $\gamma \in [0,1]$, we have

$$\overline{Q}_0(i,j,\hat{x};\gamma) = \begin{cases} \frac{K_0-i}{N-i-j}\mathbb{1}(i<K_0,\, j<N-K_0) + \mathbb{1}(j \geq N-K_0), & \hat{x} \in [\hat{x}_0, \hat{x}_1) \\ \gamma\left(\frac{K_0-i}{N-i-j}\mathbb{1}(i<K_0,\, j<N-K_0) + \mathbb{1}(j \geq N-K_0)\right), & \hat{x} = \hat{x}_1 \end{cases}, \quad (18)$$

$$\overline{Q}_1(i,j,\hat{x};\gamma) = \begin{cases} \gamma\left(\frac{K_1-j}{N-i-j}\mathbb{1}(i<N-K_1,\, j<K_1) + \mathbb{1}(i \geq N-K_1)\right), & \hat{x} = \hat{x}_0 \\ \frac{K_1-j}{N-i-j}\mathbb{1}(i<N-K_1,\, j<K_1) + \mathbb{1}(i \geq N-K_1), & \hat{x} \in (\hat{x}_0, \hat{x}_1] \end{cases}. \quad (19)$$

Abundance: $K_0 + K_1 > N$. As with scarcity, we begin with Case 1 from Lemma 7 and consider critical types $\hat{x} \in (\hat{x}_0, \hat{x}_1)$ (which is the set of critical types such that $z_0(\hat{x}), z_1(\hat{x}) > 0$). Combining Case 1 from Lemma 7 with the abundance condition $K_0 + K_1 > N$ means that the seller will optimally serve any buyer that reports a location within the ironing interval. However, because there are more total units than agents, the feasibility constraints do not immediately pin down the ex post lottery offered to these buyers. Therefore, we must consider several further subcases. First, if $z_0(\hat{x}) > z_1(\hat{x}) > 0$ (or, equivalently, if $\hat{x} \in (\hat{x}_0, \hat{x}_A)$), then the seller allocates the units of good 0 to as many buyers in the ironing interval as possible (i.e. after first giving the i buyers that reported $x < \underline{x}(\hat{x})$ a unit of good 0), before then allocating units of good 1 to any remaining buyers. Consequently, there is a unique ex post allocation rule that pointwise maximizes (16) subject to the feasibility constraints, and we have $\overline{Q}_0(i,j,\hat{x}) = \frac{K_0-i}{N-i-j}\mathbb{1}(i<K_0,\, j<N-K_0) + \mathbb{1}(j \geq N-K_0)$ and $\overline{Q}_1(i,j,\hat{x}) = 1 - \overline{Q}_0(i,j,\hat{x})$ (see Appendix B.10 for a derivation). Second, if $z_1(\hat{x}) > z_0(\hat{x}) > 0$ (or, equivalently, if $\hat{x} \in (\hat{x}_A, \hat{x}_1)$), then the seller allocates the units of good 1 to as many buyers in the ironing interval as possible (i.e. after first giving the j buyers that reported $x > \overline{x}(\hat{x})$ a unit of good 1), before then allocating units of good 0 to any remaining buyers. Consequently, there is again a unique ex post allocation rule that pointwise maximizes (16) subject to the feasibility constraints, and we have $\overline{Q}_0(i,j,\hat{x}) = \frac{N-K_1-i}{N-i-j}\mathbb{1}(i < N-K_1,\, j < K_1) + \mathbb{1}(j \geq K_1)$ and $\overline{Q}_1(i,j,\hat{x}) = 1 - \overline{Q}_0(i,j,\hat{x})$ (see Appendix B.10 for a derivation). Finally, if $z_0(\hat{x}) = z_1(\hat{x})$ (or, equivalently, if $\hat{x} = \hat{x}_A$), then the seller is indifferent between giving buyers in the ironing interval a unit of good 0 and a unit of good 1. In contrast to the two previous subcases, the seller's preferences (as encoded by the ironed virtual type functions $\overline{\Psi}_0$ and $\overline{\Psi}_1$) together with the feasibility constraints do not immediately pin down a unique ex post allocation for buyers in the ironing interval. Nevertheless, without loss of generality we can parameterize the continuum of ex post allocation rules that pointwise maximize (16) by taking convex combinations of two "extremal" lotteries: one where the seller allocates the units of good 0 to as many buyers in the ironing interval as possible (which corresponds to the lottery the seller constructs whenever $z_0(\hat{x}) > z_1(\hat{x}) > 0$) and one where the seller allocates the units of good 1 to as many buyers in the ironing interval as possible (which corresponds to the lottery the seller constructs whenever $z_1(\hat{x}) > z_0(\hat{x}) > 0$). Utilizing the expressions from the previous subcases, we can compute the corresponding set of pointwise maximizing ex post allocation rules by setting $\Gamma = [0, 1]$. For all $\gamma \in [0, 1]$, we have

$$\begin{aligned} \overline{Q}_0(i, j, \hat{x}_A; \gamma) = \gamma \min\left\{\tfrac{K_0 - i}{N - i - j}, 1\right\} \mathbb{1}(i < K_0) \\ + (1 - \gamma) \min\left\{\tfrac{N - K_1 - i}{N - i - j}, 1\right\} \mathbb{1}(i < N - K_1) \end{aligned} \tag{20}$$

(see Appendix B.10 for a derivation).[16] Next, we deal with Case 2 from Lemma 7. Again, for $\ell \in \{0, 1\}$, we consider the critical type $\hat{x} = \hat{x}_\ell$ (which implies that $z_\ell(\hat{x}) > 0$ and $z_{-\ell}(\hat{x}) = 0$). Here, the analysis is analogous to the case involving scarcity and any ex post lottery offered by the seller to buyers in the ironing interval under pointwise maximization can, again, be characterized as a convex combination of two extremal lotteries: one which allocates the units of good ℓ to as many buyers in the ironing interval as possible and then serves none of the remaining buyers and one which allocates the units of good ℓ to as many buyers in the ironing interval as possible and then allocates units of the good $-\ell$ to all remaining buyers.

Putting all of this together, the following functions specify the set of pointwise maximizing ex post allocation rules for each critical type $\hat{x}$ that is a candidate for the critical worst-off type ω^* under abundance. In particular, for all feasible states $(i, j) \in \{0, 1, \ldots, N-1\}^2$ such

[16] Note that the seller can implement the ex post allocation rule from (20) without violating the feasibility constraints by randomizing over the two extremal lotteries and implementing the one that allocates a unit of the good at location 0 to buyers in the ironing interval wherever possible with probability γ and the one that allocates a unit of the good at location 1 to buyers in the ironing interval wherever possible with probability $1 - \gamma$.

that $i+j \leq N-1$, critical types $\hat{x} \in [\hat{x}_0, \hat{x}_1]$ and $\gamma \in [0,1]$, we have

$$\overline{Q}_0(i,j,\hat{x};\gamma) = \begin{cases} \frac{K_0-i}{N-i-j}\mathbb{1}(i < K_0,\, j < N-K_0) + \mathbb{1}(j \geq N-K_0), & \hat{x} \in [\hat{x}_0, \hat{x}_A) \\ \overline{Q}_0(i,j,\hat{x}_A;\gamma), & \hat{x} = \hat{x}_A \\ \frac{N-K_1-i}{N-i-j}\mathbb{1}(i < N-K_1,\, j < K_1) + \mathbb{1}(j \geq K_1), & \hat{x} \in (\hat{x}_A, \hat{x}_1) \\ \gamma\left(\frac{N-K_1-i}{N-i-j}\mathbb{1}(i < N-K_1,\, j < K_1) + \mathbb{1}(j \geq K_1)\right), & \hat{x} = \hat{x}_1 \end{cases}, \tag{21}$$

$$\overline{Q}_1(i,j,\hat{x};\gamma) = \begin{cases} \gamma\left(1-\overline{Q}_0(i,j,\hat{x}_0;\gamma)\right), & \hat{x} = \hat{x}_0 \\ 1-\overline{Q}_0(i,j,\hat{x};\gamma), & \hat{x} \in (\hat{x}_0, \hat{x}_A) \\ 1-\overline{Q}_0(i,j,\hat{x}_A;\gamma), & \hat{x} = \hat{x}_A \\ 1-\overline{Q}_0(i,j,\hat{x};1), & \hat{x} \in (\hat{x}_A, \hat{x}_1] \end{cases}, \tag{22}$$

where $\overline{Q}_0(i,j,\hat{x}_A;\gamma)$ is as defined in (20).

Having exhausted all possible cases, we are now in a position to summarize this analysis. In particular, for all $\ell \in \{0,1\}$, feasible states $(i,j) \in \{0,1,\ldots,N-1\}^2$ such that $i+j \leq N-1$, critical types $\hat{x} \in (0,1)$ and $\gamma \in [0,1]$, using the pointwise maximizing ex post allocation rules $\overline{Q}_\ell(i,j,\hat{x};\gamma)$ computed in (18), (19), (21) and (22), we can calculate the associated interim allocations $\overline{q}_\ell(\hat{x};\gamma)$ via (17). The following theorem then characterizes the optimal selling mechanism and associated unique critical worst-off type ω^*.

Theorem 2. *Suppose that $v > \frac{1}{2}$. Then the optimal selling mechanism is characterized by the unique critical worst-off type $\omega^* \in [\hat{x}_0, \hat{x}_1]$ such that $\overline{q}_0(\omega^*;\gamma) = \overline{q}_1(\omega^*;\gamma)$ holds for some $\gamma \in [0,1]$. If $\omega^* \notin \{\hat{x}_0, \hat{x}_A, \hat{x}_1\}$, then $\overline{q}_0(\omega^*;\gamma) = \overline{q}_1(\omega^*;\gamma)$ holds for all $\gamma \in [0,1]$. If $\omega^* \in \{\hat{x}_0, \hat{x}_A, \hat{x}_1\}$, then there also exists a unique γ^* such that $\overline{q}_0(\omega^*;\gamma^*) = \overline{q}_1(\omega^*;\gamma^*)$.*

4.2.1 Monopoly pricing problems

We conclude this subsection by considering the special case of *monopoly pricing problems* such that $v > \frac{1}{2}$ and $K_0 = K_1 = N$. Under monopoly pricing problems competition among the buyers and the feasibility constraints (F) play no role in determining the optimal selling mechanism. Consequently, for all $x_n \in [0,1]$, $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$ and $\ell \in \{0,1\}$, we have $Q_\ell(x_n, \boldsymbol{x}_{-m}) = q_\ell(x_n)$ and the derivation of the functions $\overline{q}_0$ and $\overline{q}_1$ simplifies substantially. This allows us to both compute the correspondence $\Delta\overline{q}(\hat{x}) := \{\overline{q}_0(\hat{x},\hat{x};\gamma) - \overline{q}_1(\hat{x},\hat{x};\gamma) : \gamma \in [0,1]\}$ and characterize the critical worst-off type $\hat{x}_A$ more explicitly.

First, suppose that $\hat{x} \in [\hat{x}_0, \hat{x}_A)$, which implies that $z_0(\hat{x}) > z_1(\hat{x}) \geq 0$. The allocation rules $\overline{q}_0$ and $\overline{q}_1$ that pointwise maximize (16) are then simply $\overline{q}_0(x,\hat{x};\gamma) = \mathbb{1}(\overline{\Psi}_0(x,\hat{x}) \geq \overline{\Psi}_1(x,\hat{x})) = \mathbb{1}(x \leq \overline{x}(\hat{x}_A))$ and $\overline{q}_1(x,\hat{x};\gamma) = 1 - \overline{q}_0(x,\hat{x})$ (see Panel (a) of Figure 2). Second,

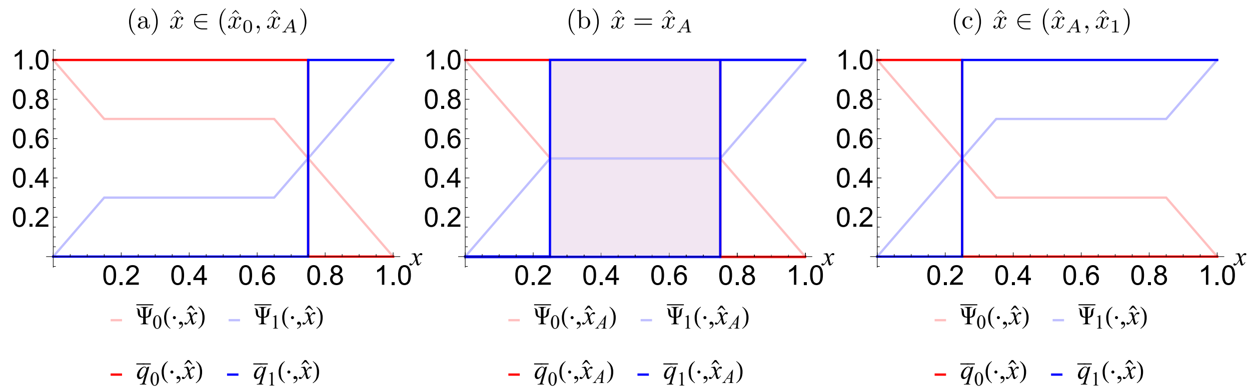

Figure 2: Assuming that buyer locations are uniformly distributed and $v = 1$, this figure illustrates the allocation rules $\overline{q}_0(\cdot, \hat{x})$ (red) and $\overline{q}_1(\cdot, \hat{x})$ (blue) that pointwise maximizes the ironed virtual type functions and the ironed virtual type functions $\overline{\Psi}_0(\cdot, \hat{x})$ (red) and $\overline{\Psi}_1(\cdot, \hat{x})$ (blue) for three cases: $\hat{x} \in (\hat{x}_0, \hat{x}_A)$, $\hat{x} = \hat{x}_A$ and $\hat{x} \in (\hat{x}_A, \hat{x}_1)$.

suppose that $\hat{x} \in (\hat{x}_A, \hat{x}_1]$, which implies that $z_1(\hat{x}) > z_0(\hat{z}) \geq 0$. In this case the unique pointwise maximizers are given by $\overline{q}(x, \hat{x}; \gamma) = \mathbb{1}(\overline{\Psi}_1(x, \hat{x}) \geq \overline{\Psi}_0(x, \hat{x})) = \mathbb{1}(x \leq \underline{x}(\hat{x}_A))$ and $\overline{q}_1(x, \hat{x}) = 1 - \overline{q}_0(x, \hat{x})$ (see Panel (c) of Figure 2). Finally, suppose that $\hat{x} = \hat{x}_A$, which implies that $z_0(\hat{x}) = z_1(\hat{x}) > 0$. In this case there are a continuum of allocation rules $\overline{q}_0$ and $\overline{q}_1$ that pointwise maximize (16). For $\gamma \in [0, 1]$ we have $\overline{q}_0(x, \hat{x}_A; \gamma) = \mathbb{1}(x < \underline{x}(\hat{x}_A)) + \gamma \mathbb{1}(x \in [\underline{x}(\hat{x}_A), \overline{x}(\hat{x}_A)])$ and $\overline{q}_1(x, \hat{x}_A, \gamma) = 1 - \overline{q}_0(x, \hat{x}_A; \gamma)$ (see Panel (b) of Figure 2). Putting all of this together, for all $\hat{x} \in [\hat{x}_0, \hat{x}_1]$, the correspondence $\Delta\overline{q}$ that we previously defined is given by

$$\Delta\overline{q}(\hat{x}) = \begin{cases} 1, & \hat{x} \in [\hat{x}_0, \hat{x}_A) \\ [-1, 1], & \hat{x} = \hat{x}_A \\ -1, & \hat{x} \in (\hat{x}_A, \hat{x}_1] \end{cases}.$$

The unique critical worst-off type that satisfies $0 \in \Delta\overline{q}(\omega^*)$ and characterizes the optimal selling mechanism corresponds to $\omega^* = \hat{x}_A$ and $\gamma^* = \frac{1}{2}$. Intuitively, an ironing parameter of $z(\hat{x}_A) = 0$ makes the seller indifferent between giving types in the ironing interval $[\underline{x}(\hat{x}_A), \overline{x}(\hat{x}_A)]$ good 0 and good 1. Absent any binding feasibility constraints, this must hold for any types that the seller enters in a non-trivial lottery. Summarizing, we have the following proposition.

Proposition 3. *If $v > \frac{1}{2}$ and $K_0 = K_1 = N$, then the critical worst-off type is given by*

$\omega^* = \hat{x}_A$ *and the optimal selling mechanism is characterized by*

$$q^*(x) = \begin{cases} 1, & x < \underline{x}(\hat{x}_A) \\ \frac{1}{2}, & x \in [\underline{x}(\hat{x}_A), \overline{x}(\hat{x}_A)], \\ 0, & x > \overline{x}(\hat{x}_A) \end{cases} \quad t^*(x) = \begin{cases} v - \psi_S^{-1}\left(\frac{1}{2}\right), & x < \underline{x}(\hat{x}_A) \\ v - \frac{1}{2}, & x \in [\underline{x}(\hat{x}_A), \overline{x}(\hat{x}_A)] \\ v - \left(1 - \psi_B^{-1}\left(\frac{1}{2}\right)\right), & x > \overline{x}(\hat{x}_A) \end{cases}.$$

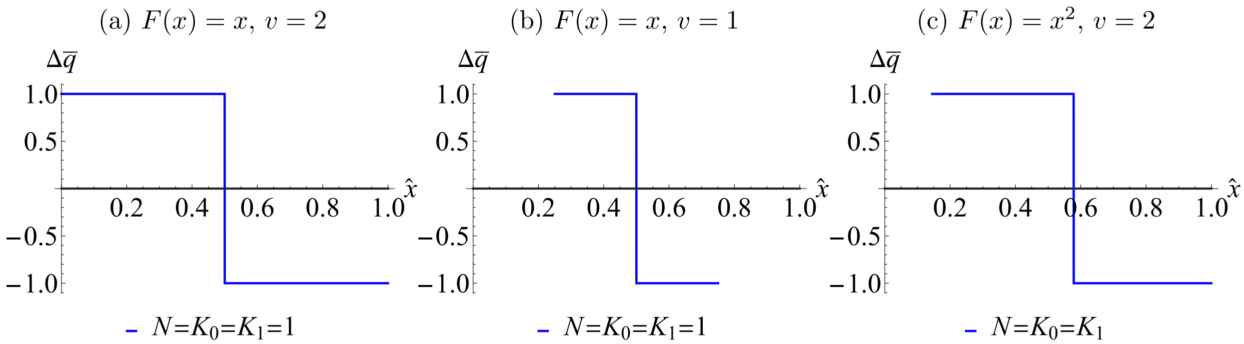

Figure 3: Panels (a) and (b) illustrate the correspondence $\Delta\overline{q}$ for monopoly pricing problems with $F(x) = x$ and $\hat{x}_A = \frac{1}{2}$. In Panel (a) we have $v = 2$, which implies that $\hat{x}_0 = 0$ and $\hat{x}_1 = 1$ and the correspondence $\Delta\overline{q}$ is defined for all $x \in [0, 1]$. In Panel (b) we have $v = 1$, which implies that $\hat{x}_0 = \frac{1}{4}$ and $\hat{x}_1 = \frac{3}{4}$ and the correspondence $\Delta\overline{q}$ is only defined for $\hat{x} \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Panel (c) illustrates this correspondence for any monopoly pricing problem with $F(x) = x^2$ and $v = 2$. Here, $\hat{x}_0 \approx 0.145$, $\hat{x}_1 = 1$ and $\hat{x}_A \approx 0.578$.

As Proposition 3 shows, the optimal selling mechanism for monopoly pricing problems can be implemented by setting three prices: prices p_0 and p_1 for the "pure" goods 0 and 1, respectively, and a price $p_L = v - \frac{1}{2}$ for the lottery that gives buyers a unit of goods 0 and 1 with probability $\frac{1}{2}$ each. Notice that the lottery price is independent of the type distribution. However, the "pure" good prices p_0 and p_1 and the subset of types that participate in the lottery do depend on the type distribution. If buyer locations are uniformly distributed, then we have a critical worst-off type of $\omega^* = \frac{1}{2}$ (see panels (a) and (b) of Figure 3). The interval of types that participate in the lottery is given by $\left[\frac{1}{4}, \frac{3}{4}\right]$ and we have prices of $p_0 = p_1 = v - \frac{1}{4}$ and $p_L = v - \frac{1}{2}$. If $F(x) = x^2$, then we have a critical worst-off type of $\omega^* \approx 0.578$ (see Panel (b) of Figure 3). The interval of types that participate in the lottery is approximately $\left[\frac{1}{3}, 0.768\right]$ with prices $p_0 = v - \frac{1}{3}$, $p_1 \approx v - 0.232$ and $p_L = v - \frac{1}{2}$. Compared to the case with the uniform distribution, there is a relatively high concentration of "captive" buyers close to location 1 when $F(x) = x^2$. Consequently, relative to the uniform distribution, the seller sets a higher price for good 1 and serves a smaller interval of types at this location, while still selling a larger quantity of $1 - F(0.768) \approx 0.411$ of good 1 as a pure good.

Beyond the monopoly pricing problems considered in this subsection, the interim allocations for buyers located left and right of the ironing interval will generally vary with the reports of other agents. Consequently, the seller cannot implement the optimal selling mechanism by simply posting a menu of three prices. However, as we will see in Section 5.2, the seller can implement the optimal selling mechanism in dominant strategies by running a two-stage clock auction that involves setting at most three prices.

4.3 Comparative statics

In this section we shed further light on the structure of the optimal selling mechanisms by deriving a number of comparative statics. We begin with graphical illustrations of key properties and then provide formal results.

4.3.1 Graphical illustrations

We begin by illustrating the correspondence $\Delta\overline{q}(\hat{x}) = \{\overline{q}_0(\hat{x}, \hat{x}; \gamma) - \overline{q}_1(\hat{x}, \hat{x}; \gamma)\}$ for a variety of cases. Exploiting the fact that the critical worst-off type is uniquely pinned down by the condition $0 \in \Delta\overline{q}(\omega^*)$, we will then highlight a number of comparative statics relating to the critical worst-off type ω^*. Section 4.3.2 then contains the formal analysis.

Balanced markets. We start by considering *balanced markets*, which are such that $K_0 + K_1 = N$. If $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$ also holds, then we have full market coverage under the optimal selling mechanism (i.e. it is both feasible and optimal to serve all buyer types with probability 1). Here, the feasibility constraints uniquely pin down the interim allocations for buyers in the ironing interval and the correspondence $\Delta\overline{q}$ is always a function (see Panel (a) of Figure 4). If $v \in \left(\frac{1}{2}, 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}\right)$, then the feasibility constraints still uniquely pin down the interim allocations for agents in the ironing interval, except for at the boundaries $\hat{x} = \hat{x}_0$ and $\hat{x} = \hat{x}_1$, where the correspondence $\Delta\overline{q}$ exhibits a vertical segment.[17] Panels (b), (c) and (d) of Figure 4 provide an illustration. This figure also contains examples where $\omega^* \in \{\hat{x}_0, \hat{x}_1\}$: this arises when v is sufficiently small and the endowment is sufficiently asymmetric. As Figure 4 illustrates—and as we formally show in Section 4.3.2—under balanced markets ω^* increases monotonically in K_0 (and, consequently, decreases monotonically in K_1). Intuitively, if K_0 increases (and K_1 decreases), then good 1 becomes relatively more scarce. Consequently, ω^* increases and the ironing interval moves toward 1 in order to maintain $0 \in \Delta\overline{q}(\omega^*)$. When the seller faces a highly asymmetric endowment it may also only

[17] Recall from Section 4.2 that when $\omega^* = \hat{x}_\ell$ we have $z_{-\ell}(\omega^*) = 0$. Consequently, the seller is indifferent between allocating units good $-\ell$ to agents in the ironing interval and not serving them. As a result, there are a continuum of pointwise maximizing ex post allocation rules.

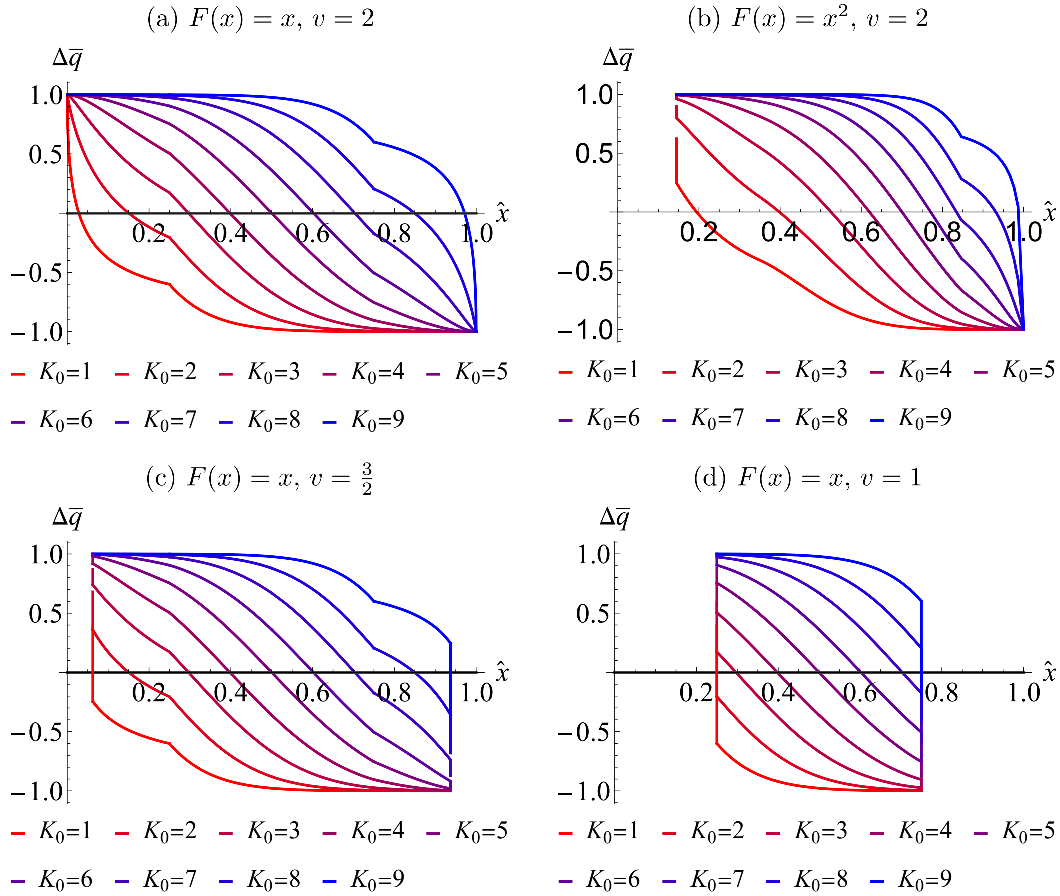

Figure 4: Panels (a), (c) and (d) illustrate the correspondence $\Delta\overline{q}$ for a series of balanced market with $N = 10$, $F(x) = x$ and $\hat{x}_S = \frac{1}{2}$. In Panel (a) $v = 2$, in Panel (c) $v = \frac{3}{2}$ and in Panel (d) $v = 1$. Panel (b) illustrates the correspondence $\Delta\overline{q}$ for a series of balanced markets with $N = 10$, $F(x) = x^2$, $v = 2$ and $\hat{x}_S \approx 0.700$.

sell the abundant good as a "pure" good, and sell the entire endowment of the scarce good via a lottery. For example, in Panel (a) of Figure 4, we have $\omega^* \leq \frac{1}{4}$ and $\underline{x}(\omega^*) = 0$ when $K_0 \in \{1, 2\}$ and $\omega^* \geq \frac{3}{4}$ and $\overline{x}(\omega^*) = 1$ when $K_0 \in \{8, 9\}$.

Symmetric endowments. We now consider parameterizations involving *symmetric endowments* such that $K_0 = K_1 = K \in \{1, \ldots, N\}$. The following lemma will prove useful.

Lemma 8. *There exists a unique $\hat{x}_S \in [0, 1]$ satisfying $F(\underline{x}(\hat{x}_S)) = 1 - F(\overline{x}(\hat{x}_S))$.*

Under the critical type $\hat{x}_S$ introduced in Lemma 8, buyers are equally likely to be located left and right of the ironing interval. Consequently, if v is sufficiently large (so that $\hat{x}_S \in (\hat{x}_0, \hat{x}_1)$) and if we have scarcity and a symmetric endowment, then under the critical type $\hat{x}_S$

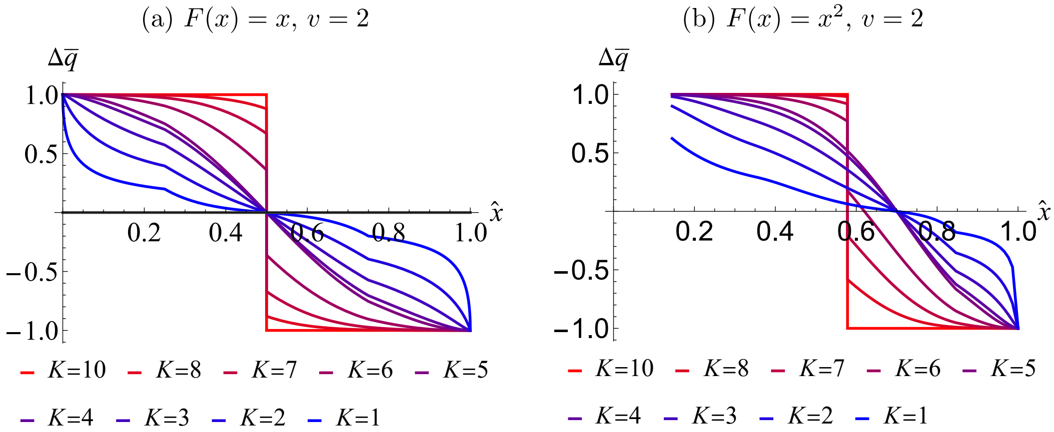

Figure 5: This figure illustrates comparative statics under symmetric endowments. Panel (a) illustrates the correspondence $\Delta\overline{q}$ for $F(x) = x$, $v = 2$ and $N = 10$ and a series of markets with symmetric endowments such that $K_0 = K_1 = K$. Panel (b) sets $F(x) = x^2$, $v = 2$ and $N = 10$ and considers the same set of symmetric endowments.

buyers are equally likely to be allocated either good upon reporting a location in the ironing interval, which implies that $\omega^* = \hat{x}_S$.

Panel (a) in Figure 5 illustrates a series of examples involving symmetric endowments for $F(x) = x$ and $v = 2$ (which implies that $\hat{x}_0 = 0$ and $\hat{x}_1 = 1$). For the parameterizations involving scarcity (i.e. $K \leq \lfloor \frac{N}{2} \rfloor = 5$) we see that the correspondence $\Delta\overline{q}$ is always a function as the feasibility constraints uniquely pin down the allocation for agents in the ironing interval and we have $\omega^* = \hat{x}_S = \frac{1}{2}$. However, in the region involving abundance (i.e. $K > \lfloor \frac{N}{2} \rfloor$) the feasibility constraints do not necessarily uniquely pin down the allocation for buyers in the ironing interval and the correspondence $\Delta\overline{q}$ exhibits a vertical section at $\hat{x} = \hat{x}_A = \frac{1}{2}$. Panel (a) of Figure 5 also shows that as the number of goods available at each $\ell \in \{0, 1\}$ increases from $K = 1$ to $K = N$, the correspondence $\Delta\overline{q}$ converges to the step function illustrated in Panel (a) of Figure 3. Panel (b) of Figure 5 exhibits similar features. However, this panel, which assumes $F(x) = x^2$ and $v = 2$, shows that $\omega^* = \hat{x}_S \approx 0.700$ holds for $K \leq \lfloor \frac{N}{2} \rfloor$ and that ω^* monotonically converges from $\hat{x}_S$ to $\hat{x}_A \approx 0.578$ as K increases from 1 to N.[18]

Asymmetric endowments. Figure 6 provides additional parameterizations that illustrate the convergence of ω^* to $\hat{x}_A$ (the critical worst-off type associated with monopoly pricing problems) once the feasibility constraints are sufficiently slack. In fact, figures 5 and 6 illustrate a more general phenomenon: as K_0 and K_1 increase, ω^* typically converges to $\hat{x}_A$

[18]This convergence is trivial for the uniform distribution where $\hat{x}_S = \hat{x}_A$.

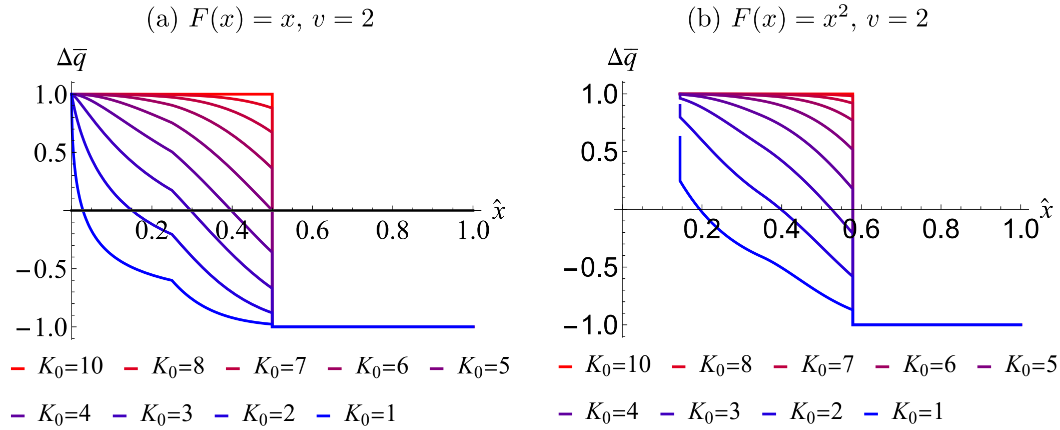

Figure 6: This figure illustrates behavior under asymmetric endowments. Panel (a) illustrates the correspondence $\Delta\bar{q}$ for a series of markets with asymmetric endowments and $N = 10$ and $K_0 = K_1 = K$. Panel (b) illustrates this correspondence for a series of markets with asymmetric endowments and $N = K_1 = 10$.

before we reach the monopoly pricing case of $K_0 = K_1 = N$. The analysis in the following subsection provides a tight theoretical characterization of this phenomenon.

4.3.2 Formal results

We now show to what extent the properties illustrated above are general properties of the optimal mechanism. We begin with the case $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$, which implies that the critical worst-off type—which we denote by $\hat{x}(K_0, K_1)$ for this case—is independent of v. Throughout the analysis, we keep $N \geq 2$ fixed.[19] Note that while the results in this section are true as stated when $N = 2$, some of the statements are vacuous in this case as (among other things) there is no scarcity region with $K_0 + K_1 < N$ and only a single balanced market parameterization such that $K_0 + K_1 = N$.

Proposition 4. *Suppose that $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$ and $N \geq 2$. In the scarcity region with $K_0 + K_1 < N$ we always have $\hat{x}(K_0, K_1) < \hat{x}(K_0 + 1, K_1)$ and $\hat{x}(K_0, K_1) > \hat{x}(K_0, K_1 + 1)$. We now restrict attention to the abundance region and assume that $K_0 + K_1 \geq N$. In this region there exists a point (K_0^A, K_1^A) with $K_0^A + K_1^A \in \{N, N+1\}$ such that $\hat{x}(K_0, K_1) = \hat{x}_A$ if and only if $K_0 \geq K_0^A$ and $K_1 \geq K_1^A$. Moreover, we have $\hat{x}(K_0, K_1) < \hat{x}(K_0 + 1, K_1) \leq \hat{x}_A$ if $K_0 < K_0^A$, $\hat{x}(K_0, K_1) = \hat{x}(K_0 + 1, K_1)$ if $K_0 \geq K_0^A$, $\hat{x}(K_0, K_1) > \hat{x}(K_0, K_1 + 1) \geq \hat{x}_A$ if $K_1 < K_1^A$ and $\hat{x}(K_0, K_1) = \hat{x}(K_0, K_1 + 1)$ if $K_1 \geq K_1^A$.*

Proposition 4 immediately implies the following additional comparative statics.

[19] If $K_0 = K_1 = N = 1$ then the monopoly pricing analysis from Section 4.2.1 applies.

(a) $F(x) = x$ (b) $F(x) = x^2$

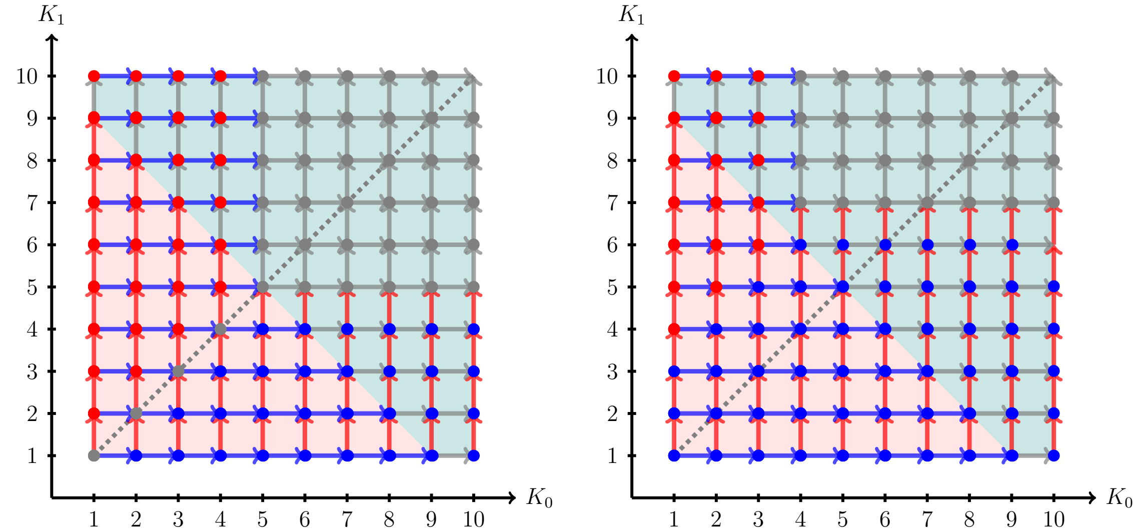

Figure 7: This figure illustrates the comparative statics stated in Proposition 4 for $N = 10$ with $F(x) = x$ in Panel (a) and $F(x) = x^2$ in Panel (b). In both panels the scarcity region is shaded in pink and the abundance region is shaded in blue. Gray dots indicate that $\hat{x}(K_0, K_1) = \hat{x}_A$, red dots indicate that $\hat{x}(K_0, K_1) < \hat{x}_A$ and blue dots indicate that $\hat{x}(K_0, K_1) > \hat{x}_A$. Similarly, gray arrows indicate steps in the parameter space such that $\hat{x}(K_0, K_1)$ is constant, red arrows indicate steps such that $\hat{x}(K_0, K_1)$ decreases and blue arrow indicate steps such that $\hat{x}(K_0, K_1)$ increases.

Corollary 1. *Suppose that* $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$ *and* $N \geq 2$. *Then the critical worst-off type* $\hat{x}(K_0, K_1)$ *increases monotonically in the ratio* $\frac{K_0}{K_1}$. *Moreover, if* $K_0 = K_1 = K \in \{1, \ldots, N\}$, *then the critical worst-off type* $\hat{x}(K, K)$ *varies monotonically in* K, *and is such that* $\hat{x}(K, K) = \hat{x}_S$ *for* $K \leq \lfloor \frac{N}{2} \rfloor$ *and* $\hat{x}(K, K) = \hat{x}_A$ *for* $K \geq \max\{K_0^A, K_1^A\}$.

As shown by Corollary 2 below, studying balanced markets allows us to identify the point (K_0^A, K_1^A) from Proposition 4. Once we compute (K_0^A, K_1^A) for a given value of N and type distribution F, then the results of Proposition 4 provide us with sufficient information to fully construct the abundance region in figures such as those displayed in Figure 7.

Corollary 2. *Suppose that* $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$ *and* $N \geq 2$ *and consider balanced markets such that* $K_0 + K_1 = N$. *Then one of four cases applies:*

(i) If there exists $K \in \{1, \ldots, N-1\}$ *with* $K_0 = K$, $K_1 = N - K$ *and* $\hat{x}(K, N-K) = \hat{x}_A$, *then we have* $K_0^A = K$ *and* $K_1^A = N - K$.

(ii) If there exists $K \in \{1, \ldots, N-2\}$ *with* $K_0 = K$, $K_1 = N - K$ *and* $\hat{x}_A \in (\hat{x}(K, N-K), \hat{x}(K+1, N-K-1))$, *then we have* $K_0^A = K + 1$ *and* $K_1^A = N - K$.

(iii) If $\hat{x}(1, N-1) > \hat{x}_A$, then we have $K_0^A = 1$ and $K_1^A = N$.

(iv) If $\hat{x}(N-1, 1) < \hat{x}_A$, then we have $K_0^A = N$ and $K_1^A = 1$.

Figure 7 provides a graphical illustration of our comparative statics results. In the scarcity region the feasibility constraints for both goods are binding and uniquely pin down the critical worst-off type, which mechanically adjusts in response to changes in the supply of each good in order to maintain the saddle point condition. However, the feasibility constraints are less tight and bind for at most one of the goods once we are in the abundance region. Our analysis of monopoly pricing problems showed that we must have $\hat{x}(N, N) = \hat{x}_A$. Moreover, in the previous section, we saw that $\hat{x}(K_0, K_1) = \hat{x}_A$ applies in the abundance region for numerous other parameterizations where the feasibility constraints are sufficiently slack. Proposition 4 and Corollary 2 together provide a sharp characterization of when there is sufficient supply of each good so that the designer can achieve its "ideal" critical worst-off type $\hat{x}_A$. In particular, there is a "rectangle" in the abundance region with $K_0 + K_1 \geq N$ such that $\hat{x}(K_0, K_1)$ if and only if (K_0, K_1) lies within the rectangle. Moreover, the bottom-left corner of this rectangle—(K_0^A, K_1^A)—is either such that $K_0^A + K_1^A = N$ or such that $K_0^A + K_1^A = N + 1$.

With all of these comparative statics at hand, it is straightforward to generalize the analysis and allow for the possibility that $v \in \left(\frac{1}{2}, 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}\right)$. Fixing $N \geq 2$, we now denote the critical worst-off type by $\omega^*(K_0, K_1, v)$. Naturally, if $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$, then $\omega^*(K_0, K_1, v) = \hat{x}(K_0, K_1)$. Moreover, since $\hat{x}_A \in (\hat{x}_0, \hat{x}_1)$ holds for all $v > \frac{1}{2}$, if $\hat{x}(K_0, K_1) = \hat{x}_A$, then we also have $\omega^*(K_0, K_1, v) = \hat{x}_A$. However, in general $\omega^*(K_0, K_1, v) = \max\{\min\{\hat{x}(K_0, K_1), \hat{x}_1\}, \hat{x}_0\}$, where $\hat{x}_0$ is decreasing in v and $\hat{x}_1$ is increasing in v. Using this expression for $\omega^*(K_0, K_1, v)$ we can generalize our comparative statics results. Specifically, if we replace $\hat{x}(K_0, K_1, v)$ with $\omega^*(K_0, K_1, v)$ and $\hat{x}_S$ with $\max\{\min\{\hat{x}_S, \hat{x}_1\}, \hat{x}_0\}$, then corollaries 1 and 2 are true as stated for any $v > \frac{1}{2}$. Proposition 4 is also true as stated for any $v > \frac{1}{2}$ if we additionally replace any strict inequalities involving critical types with weak inequalities.

In Section 4.1 we provided a necessary and sufficient condition ($v \leq \frac{1}{2}$) for the optimal selling mechanism to reduce to running two independent auctions. When $v > \frac{1}{2}$ we asymptotically recover this case, as well as the optimality of a single auction (since a critical worst-off type of $\omega^* = 0$ corresponds to running a single optimal auction at location 1 and a critical worst-off type of $\omega^* = 1$ corresponds to running a single optimal auction at location 0).

Proposition 5. *If K_0 and K_1 vary with N in such a way that $\frac{K_0(N)}{N} \to 0$ and $\frac{K_1(N)}{N} \to 1$ as $N \to \infty$, then $\lim_{N\to\infty} \omega^*(K_0, K_1, v) = 0$. Similarly, if K_0 and K_1 vary with N in such a way that $\frac{K_1(N)}{N} \to 0$ and $\frac{K_0(N)}{N} \to 1$ as $N \to \infty$, then $\lim_{N\to\infty} \omega^*(K_0, K_1, v) = 1$. Moreover,* *suppose that K_0 and K_1 vary with N in such a way that $\frac{K_0(N)}{N} \to \mu_0$ and $\frac{K_1(N)}{N} \to \mu_1$ and let $\hat{x} = \lim_{N\to\infty} \omega^*(K_0(N), K_1(N), v)$. Then if $\mu_0 \leq F(\underline{x}(\hat{x}))$ and $\mu_1 \leq 1 - F(\overline{x}(\hat{x}))$, the optimal selling mechanism converges to running two independent auctions as $N \to \infty$.*

5 Clock auction implementation

In this section, we first study the implementation of the optimal selling mechanisms in dominant strategies, with and without ex post individual rationality constraints. Then we describe a clock auction that implements the optimal mechanisms in dominant strategies.

5.1 Dominant strategies

We first introduce the formal definitions of dominant strategy incentive compatibility (DIC) and ex post individual rationality (EIR) and complete the description of the ex post allocation rules of the optimal mechanism. We then derive the dominant strategy prices that satisfy EIR with equality for the ex post worst-off types and show that the countervailing incentives of our problem give rise to a revenue difference between mechanisms satisfying IR and those satisfying EIR.

5.1.1 DIC and EIR

Given a direct mechanism $\langle \boldsymbol{Q}, T \rangle$, we let

$$U(x_n, y_n, \boldsymbol{x}_{-n}) := Q_0(y_n, \boldsymbol{x}_{-n})(v - x_n) + Q_1(y_n, \boldsymbol{x}_{-n})(v - (1 - x_n)) - T(y_n, \boldsymbol{x}_{-n})$$

denote the ex post payoff of agent n upon reporting y_n at type profile $\boldsymbol{x} = (x_n, \boldsymbol{x}_{-n})$, where ex post means that the agent's payoff is evaluated after all reports are submitted but before any randomization performed by the designer (as a function of the agents' reports) has occurred. We also let $U(x_n, \boldsymbol{x}_{-n}) := U(x_n, x_n, \boldsymbol{x}_{-n})$ denote agent n's ex post payoff upon truthfully reporting x_n at type profile $\boldsymbol{x} = (x_n, \boldsymbol{x}_{-n})$. A direct mechanism $\langle \boldsymbol{Q}, T \rangle$ then satisfies DIC if and only if, for all $y_n \in [0, 1]$ and all $\boldsymbol{x} \in [0, 1]^N$, we have

$$U(x_n, \boldsymbol{x}_{-n}) \geq U(x_n, y_n, \boldsymbol{x}_{-n}). \tag{DIC}$$

It satisfies EIR if and only if, for all $\boldsymbol{x} \in [0, 1]^N$, we have

$$U(x_n, \boldsymbol{x}_{-n}) \geq 0. \tag{EIR}$$

5.1.2 Optimal ex post allocation rules

We now complete the description of the ex post allocation rule under the optimal mechanism (in the previous section this was only done for agents within the ironing interval). Given a type profile $\boldsymbol{x} \in [0,1]^N$, we let $x_{(i)}$ denote its i-th highest element and $x_{[i]}$ denote its i-th lowest element. For example, $x_{[K_0]}$ is then the K_0-th lowest reported location and $x_{(K_1)}$ is the K_1-th highest reported location. For ease of exposition we abstract from the possibility of ties by assuming that no two elements of $\boldsymbol{x}$ are the same.[20] For $i > N$ we employ the convention of setting $x_{[i]} = 1$ and $x_{(i)} = 0$. We will also use the notation $x_{[i]}^{-n}$ and $x_{(i)}^{-n}$ to denote, respectively, the i-th lowest and i-th highest element of the vector $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$, and we similarly set $x_{[N]}^{-n} = 1$ and $x_{(N)}^{-n} = 0$.

We first consider the case $v \leq \frac{1}{2}$. Here, the optimal mechanism consists of two independent, optimal auctions, and the corresponding optimal allocation rule is $Q_0(x_n, \boldsymbol{x}_{-n}) = \mathbb{1}(x_n \leq \min\{x_{[K_0]}, \psi_S^{-1}(v)\})$ and $Q_1(x_n, \boldsymbol{x}_{-n}) = \mathbb{1}(x_n \geq \max\{x_{(K_1)}, \psi_B^{-1}(1-v)\})$. Next, we consider the case where $v > \frac{1}{2}$ and lotteries are part of the optimal mechanism, which is characterized by the parameters (ω^*, γ^*).[21]

For $\omega^* \neq \hat{x}_A$, let $\tilde{x} \in (0,1)$ denote the unique point of intersection of $\Psi_0(x, \omega^*)$ and $\overline{\Psi}_1(x, \omega^*)$ so that $\tilde{x}$ is such that $\Psi_0(\tilde{x}, \omega^*) = \overline{\Psi}_1(\tilde{x}, \omega^*)$. Either (a) $\tilde{x} < \underline{x}$ or (b) $\tilde{x} > \overline{x}$ will hold, corresponding to $\omega^* \in (\hat{x}_A, \hat{x}_1)$ and $\omega^* \in (\hat{x}_0, \hat{x}_A)$, respectively. The designer then prioritizes allocating good 0 for $x < \tilde{x}$ and good 1 for $x > \tilde{x}$.

Assume first that $\omega^* \neq \hat{x}_A$. For $x_n < \min\{\underline{x}(\omega^*), \tilde{x}\}$, we have

$$\begin{aligned} Q_0(x_n, \boldsymbol{x}_{-n}) &= \mathbb{1}(x_n \leq x_{[K_0]}), \\ Q_1(x_n, \boldsymbol{x}_{-n}) &= \mathbb{1}(x_n > x_{[K_0]}, x_n \geq \max\{x_{(K_1)}, \psi_S^{-1}(1-v)\}), \end{aligned} \tag{23}$$

where we let $\psi_S^{-1}(1-v) = 0$ for $v > 1$.[22] If $\tilde{x} < x_n < \underline{x}(\omega^*)$, then we have $Q_0(x_n, \boldsymbol{x}_{-n}) = \mathbb{1}(x_n < x_{(K_1)}, x_n \leq x_{[K_0]})$ and $Q_1(x_n, \boldsymbol{x}_{-n}) = \mathbb{1}(x_n \geq x_{(K_1)})$. If $x_n > \max\{\overline{x}(\omega^*), \tilde{x}\}$, then similarly

$$\begin{aligned} Q_0(x_n, \boldsymbol{x}_{-n}) &= \mathbb{1}(x_n < x_{(K_1)}, x_n \leq \min\{x_{[K_0]}, \psi_B^{-1}(v)\}), \\ Q_1(x_n, \boldsymbol{x}_{-n}) &= \mathbb{1}(x_n \geq x_{(K_1)}), \end{aligned} \tag{24}$$

where we let $\psi_B^{-1}(v) = 1$ for $v > 1$,[23] while for $\overline{x}(\omega^*) < x_n < \tilde{x}$, the ex post allocation is

[20] Of course, ties can always be accommodated by augmenting the allocation rule with an arbitrary tie-breaking rule. However, the exposition is simpler if we preclude these zero probability events.

[21] By Theorem 2, whenever γ is not uniquely pinned down, without loss of generality we can set $\gamma = 0$.

[22] Note that when $x_n < \underline{x}(\omega^*)$ we necessarily have $\overline{\Psi}_0(x_n, \omega^*) > 0$. However, we may have $\overline{\Psi}_1(x_n, \omega^*) = v - (1 - \psi_S(x_n)) < 0$, in which case the designer will never allocate good 1 to agent n.

[23] When $x_n > \overline{x}(\omega^*)$, $\overline{\Psi}_1(x_n, \omega^*) > 0$ always holds but we may have $\overline{\Psi}_0(x_n, \omega^*) = v - \psi_B(x_n) < 0$, in which case the designer will never allocation good 0 to agent n.

$Q_0(x_n, \boldsymbol{x}_{-n}) = \mathbb{1}(x_n \leq x_{[K_0]})$ and $Q_1(x_n, \boldsymbol{x}_{-n}) = \mathbb{1}(x_n > x_{[K_0]}, x_n \geq x_{(K_1)})$. If $\omega^* = \hat{x}_A$, the ex post allocation is given by (23) for $x_n < \underline{x}(\omega^*)$ and (24) for $x_n > \overline{x}(\omega^*)$.

Finally, for $x_n \in [\underline{x}(\omega^*), \overline{x}(\omega^*)]$, the ex post allocation $Q_\ell(x_n, \boldsymbol{x}_{-n})$ is given by applying the expressions for $\overline{Q}_\ell$ derived in Section 4.2. In particular, for all $\boldsymbol{x}$ satisfying $x_n \in [\underline{x}(\omega^*), \overline{x}(\omega^*)]$, $x^{-n}_{[i]} < \underline{x}(\omega^*) \leq x^{-n}_{[i+1]}$ and $x^{-n}_{(j+1)} \leq \overline{x}(\omega^*) < x^{-n}_{(j)}$, we have $Q_\ell(x_n, \boldsymbol{x}_{-n}) = \overline{Q}_\ell(i, j, \omega^*; \gamma^*)$, where $\overline{Q}_\ell(i, j, \omega^*; \gamma^*)$ is defined in (18) and (19) when $K_0 + K_1 \leq N$ and in (21) and (22) when $K_0 + K_1 > N$.

5.1.3 Dominant strategy prices

We are now in a position to derive the dominant strategy prices that implement the optimal allocation rule and satisfy EIR with equality for the ex post worst-off types.[24]

Throughout the rest of this section we let $s_0 := v - \underline{x}(\omega^*)$ and $s_1 := v - (1 - \overline{x}(\omega^*))$. These are the prices offered to an agent for goods 0 and 1 under monopoly pricing problems, that is $K_0 = K_1 = N$, and will serve as starting prices in the clock auction implementation in Section 5.2. For $\ell \in \{0, 1\}$, we let also $p_\ell(\boldsymbol{x}_{-n})$ denote the price agent $n \in \mathcal{N}$ has to pay to obtain good ℓ with certainty at type profile $\boldsymbol{x}_{-n} \in [0, 1]^{N-1}$ and $p_L(\boldsymbol{x}_{-n})$ denote the price agent n has to pay when participating in the lottery. In slight abuse of notation, we let $\overline{Q}_\ell(\boldsymbol{x}_{-n})$ denote the probability that agent $n \in \mathcal{N}$ is allocated good $\ell \in \{0, 1\}$ upon reporting $x_n \in [\underline{x}(\omega^*), \overline{x}(\omega^*)]$ at type profile $\boldsymbol{x}_{-n} \in [0, 1]^{N-1}$.[25] The following lemma then specifies the prices that provide the DIC-EIR implementation of the allocation rule of the optimal mechanism and satisfy (EIR) with equality for the ex post worst-off types. For the sake of notational brevity, we write $\underline{x}$ and $\overline{x}$ instead of $\underline{x}(\omega^*)$ and $\overline{x}(\omega^*)$ in the lemma statement. The lemma also uses our convention that $\psi_S^{-1}(1 - v) = 0$ and $\psi_B^{-1}(v) = 1$ for $v > 1$.

Lemma 9. *If $v \leq \frac{1}{2}$, then the dominant strategy prices for agent n are $p_0(\boldsymbol{x}_{-n}) = \max\{v - x^{-n}_{[K_0]}, v - \psi_S^{-1}(v)\}$ and $p_1(\boldsymbol{x}_{-n}) = \max\{v - (1 - x^{-n}_{(K_1)}), v - (1 - \psi_B^{-1}(1 - v))\}$.*

If $v > \frac{1}{2}$ and $\overline{Q}_\ell(\boldsymbol{x}_{-n}) \in [0, 1)$ holds for all $\ell \in \{0, 1\}$, then the dominant strategy prices

[24] A well-known property of DIC is that agent n's transfer $T(x_n, \boldsymbol{x}_{-n})$ must not vary with its reported type whenever its allocation remains the same. Consequently, any DIC allocation rule can be implemented by posting a menu of prices with associated consumption choices for each agent.

[25] Here, $\overline{Q}_\ell(\boldsymbol{x}_{-n}) = \overline{Q}_\ell(i, j, \omega^*; \gamma^*)$ where i and j are the indices such that $x^{-n}_{[i]} < \underline{x}(\omega^*) \leq x^{-n}_{[i+1]}$ and $x^{-n}_{(j+1)} \leq \overline{x}(\omega^*) < x^{-n}_{(j)}$ and $\overline{Q}_\ell(i, j, \omega^*; \gamma^*)$ is defined in (18) and (19) when $K_0 + K_1 \leq N$ and in (21) and (22) when $K_0 + K_1 > N$.

for agent n are

$$p_0(\boldsymbol{x}_{-n}) = \max\{s_0, v - x_{[K_0]}^{-n}\} - \max\{\overline{Q}_0(\boldsymbol{x}_{-n}) - \overline{Q}_1(\boldsymbol{x}_{-n}), 0\}(\overline{x} - \underline{x}),$$
$$p_1(\boldsymbol{x}_{-n}) = \max\{s_1, v - (1 - x_{(K_1)}^{-n})\} - \max\{\overline{Q}_1(\boldsymbol{x}_{-n}) - \overline{Q}_0(\boldsymbol{x}_{-n}), 0\}(\overline{x} - \underline{x}),$$
$$p_L(\boldsymbol{x}_{-n}) = \overline{Q}_0(\boldsymbol{x}_{-n})(v - \underline{x}) + \overline{Q}_1(\boldsymbol{x}_{-n})(v - (1 - \overline{x})) - \max\{\overline{Q}_0(\boldsymbol{x}_{-n}), \overline{Q}_1(\boldsymbol{x}_{-n})\}(\overline{x} - \underline{x}).$$

If $v > \frac{1}{2}$ and $\overline{Q}_\ell(\boldsymbol{x}_{-n}) = 1$ holds for some $\ell \in \{0, 1\}$, then several cases are possible. First, suppose that $K_0 + K_1 < N$. The dominant strategy prices are then $p_0(\boldsymbol{x}_{-n}) = \max\{v - x_{[K_0]}^{-n}, v - \psi_B^{-1}(v)\}$ and $p_1(\boldsymbol{x}_{-n}) = \max\{v - (1 - x_{(K_1)}^{-n}), v - (1 - \psi_S^{-1}(1 - v))\}$. Second, suppose that $K_0 + K_1 \geq N$. Then several further subcases are possible. If $x_{[K_0]}^{-n} < \min\{\tilde{x}, \underline{x}\}$ or if $\overline{Q}_0 = 1$ and if $x_{[K_0]}^{-n} < \tilde{x}$, then the dominant strategy prices are $p_0 = v - x_{[K_0]}^{-n}$ and $p_1 = \max\{v - (1 - x_{[K_0]}^{-n}), v - (1 - \psi_S^{-1}(1 - v)\}$. If $x_{(K_1)}^{-n} > \max\{\overline{x}, \tilde{x}\}$ or if $\overline{Q}_1 = 1$ and $x_{(K_1)}^{-n} > \tilde{x}$, then $p_0(\boldsymbol{x}_{-n}) = v - x_{(K_1)}^{-n}$ and $p_1(\boldsymbol{x}_{-n}) = v - (1 - x_{(K_1)}^{-n})$. Finally, if $x_{(K_1)}^{-n} \leq \tilde{x} \leq x_{[K_0]}^{-n}$, then $p_0(\boldsymbol{x}_{-n}) = v - \tilde{x}$ and $p_1(\boldsymbol{x}_{-n}) = v - (1 - \tilde{x})$, where we set $\tilde{x} = \underline{x}$ if $\omega^* = \hat{x}_A$ and $\overline{Q}_1 = 1$ and $\tilde{x} = \overline{x}$ if $\omega^* = \hat{x}_A$ and $\overline{Q}_0 = 1$.

If the ironing interval is extremal (i.e. if either $\underline{x}(\omega^*) = 0$ or $\overline{x}(\omega^*) = 1$ holds), then only one pure price is required whenever $\overline{Q}_\ell(\boldsymbol{x}_{-n}) \in [0, 1)$ for $\ell \in \{0, 1\}$. The dominant strategy prices in Lemma 9 still implement the desired allocation as no agent would elect to purchase good 1 at the price p_1 when $\overline{x}(\omega^*) = 1$ or good 0 at the price p_0 when $\underline{x}(\omega^*) = 0$.

With Lemma 9 at hand, we are now in a position to state the necessary and sufficient conditions for the expected revenue of the seller to remain the same irrespective of whether the mechanism has to respect IR or EIR. As we will see, there is no difference if and only if $v \leq \frac{1}{2}$ or $K_0 = K_1 = N$. Otherwise, the expected revenue under IR is strictly larger than under EIR. This divergence is due to the countervailing incentives inherent to our setting. It does not arise in standard settings.

Proposition 6. *The DIC-EIR implementation of the optimal allocation rule generates strictly less expected revenue than its IC-IR implementation unless $v \leq \frac{1}{2}$ or $K_0 = K_1 = N$. In contrast, there is a DIC-IR implementation of the optimal allocation rule that generates the same expected revenue as the IC-IR implementation.*

The first part of Proposition 6 highlights a difference in expected revenue that alternative notions of individual rationality make in our setting with countervailing incentives. The proof consists of showing that the interim expected payoff of the type ω^* under EIR, denoted $u^{EIR}(\omega^*)$, is strictly positive whenever the optimal mechanism involves using a lottery, unless $K_0 = K_1 = N$ (i.e. we have a monopoly pricing problem). Adding the constant $u^{EIR}(\omega^*)$ to all the DIC-EIR prices from Lemma 9 yields a DIC-IR implementation that generates the same expected revenue as its IC-IR implementation, which proves the second part of Proposition 6. By construction, that implementation violates EIR. The divergence arises precisely because of countervailing incentives: If $K_0 < N$ and the K_0-lowest element of $\boldsymbol{x}_{-n}$ is less than $\underline{x}(\omega^*)$ or if $K_1 < N$ and the K_1-highest element is larger than $\overline{x}(\omega^*)$, then any DIC-EIR implementation gives the interim worst-off type ω^* a positive payoff. This cannot occur in mechanism design settings without countervailing incentives because there any interim worst-off type is then also always ex post worst-off.[26]

5.2 Two-stage clock auction

Clock auctions have long been recognized as having a variety of advantages over static (direct) allocation mechanisms, including the preservation of winner privacy; see, for example, Ausubel (2004), Milgrom (2017), Milgrom and Segal (2020) and Loertscher and Marx (2020). With that in mind, we now construct a clock auction that implements the optimal mechanism in dominant strategies.

If $v \leq \frac{1}{2}$, then the optimal mechanism never uses a lottery, and two standard clock auctions with reserve prices $v - \underline{x}(\hat{x}_a)$ and $v - (1 - \overline{x}(\hat{x}_a))$ for goods 0 and 1, respectively, implement the optimal mechanism.[27] Similarly, if $K_0 = K_1 = N$, then there are no strategic interactions between the agents and the optimal mechanism can be implemented with the prices $p_L(\boldsymbol{x}_{-n}) = v - 1/2$, $p_0(\boldsymbol{x}_{-n}) = v - \underline{x}(\omega^*)$ and $p_1(\boldsymbol{x}_{-n}) = v - (1 - \overline{x}(\omega^*))$. So from here onward, we assume that $v > \frac{1}{2}$ and $\min\{K_0, K_1\} < N$.

Consider first the case where the ironing interval is interior, that is, $\underline{x}(\omega^*) > 0$ and $\overline{x}(\omega^*) < 1$. This implies that both goods are sold as pure goods for some type realizations. The two-stage clock auction posts reserve or—more accurately—starting prices s_ℓ for each good $\ell \in \{0, 1\}$. In the first stage, called the *coarse bidding* stage, the action set for each agent is to bid on good 0, bid on good 1 or not to bid on either. Let $D_\ell(p_\ell^{CA})$ be the number of bidders who demand good ℓ at the clock-auction price p_ℓ^{CA}. The number of bidders who choose not to bid on any good is then $N - D_0(s_0) - D_1(s_1)$. Agents who choose not to bid subsequently take no action. Their allocation probabilities are given by $\overline{Q}_0(D_0(s_0), D_1(s_1), \omega^*)$ and $\overline{Q}_1(D_0(s_0), D_1(s_1), \omega^*)$ and they pay the relevant dominant strategy price.

[26] Manelli and Vincent (2011) and Gershkov et al. (2013) establish the equivalence of IC-IR and DIC-EIR implementation for mechanism design settings involving independent private values. However, the environments studied in these papers rule out the possibility of countervailing incentives. Interestingly, in partnership models à la Cramton et al. (1987), which exhibit countervailing incentives, the result that ex post efficiency is possible with an appropriate ownership structure also requires individual rationality to hold at the interim stage. Under EIR, their mechanism would almost surely run a deficit.

[27] See Milgrom and Segal (2020) for a formal definition of clock auctions.

Consider now agents who bid on good 0 or good 1 in the first stage. If $D_\ell(s_\ell) \le K_\ell$, then all agents who bid on good $\ell \in \{0, 1\}$ are immediately allocated good ℓ. If $D_\ell(s_\ell) = K_\ell$, then they pay s_ℓ and if $D_\ell(s_\ell) < K_\ell$, then they pay the relevant dominant strategy price p_ℓ. At this point, the two-stage clock auction for good ℓ ends (that is, the second stage is not activated).

In contrast, if $D_\ell(s_\ell) > K_\ell$, the second stage, called the *ascending auction* stage, is activated for good $\ell \in \{0, 1\}$. In this case, the clock auction price p_ℓ^{CA} for good ℓ increases continuously. At each point in the auction, the bidders who initially bid on good ℓ indicate whether they continue to demand that good or not. Once a bidder stops demanding the good, the bidder becomes irreversibly inactive and takes no further action. For $\ell = 0$, if $v - \overline{p}_0^{CA} < \psi_S^{-1}(1 - v)$, then the clock auction stops at the price $\overline{p}_0^{CA}$ at which demand D_0 decreases from $K_0 + 1$ to K_0 and if $v - \overline{p}_0^{CA} \ge \psi_S^{-1}(1 - v)$, then it continues to the price $\overline{\overline{p}}_0^{CA}$ at which demand decreases from K_0 to $K_0 - 1$. Analogously, for $\ell = 1$, if $v - (1 - \overline{p}_1^{CA}) > \psi_B^{-1}(v)$, then the clock auction stops at the price $\overline{p}_1^{CA}$ at which demand D_1 decreases from $K_1 + 1$ to K_1 and if $v - (1 - \overline{p}_1^{CA}) \le \psi_B^{-1}(v)$, then it continues to the price $\overline{\overline{p}}_1^{CA}$ at which demand decreases from K_1 to $K_1 - 1$. All K_ℓ agents who are active at the price $\overline{p}_\ell^{CA}$ are allocated good ℓ and pay $\overline{p}_\ell^{CA}$. All the other agents who initially bid on good $\ell \in \{0, 1\}$ (and became inactive by the time the price reached $\overline{p}_\ell^{CA}$) are allocated no good and make no payments if the clock auction stopped at the price $\overline{p}_\ell^{CA}$. Otherwise, they are allocated good $-\ell$, for which they pay $2v - 1 - \overline{\overline{p}}_\ell^{CA}$.[28]

Similar to the case of "no bid information" in Ausubel (2004), the only information available to bidders in the clock auction phase is whether the auction price is increasing or has stopped increasing (or always stayed at s_ℓ). We say that agent n bids *sincerely* if: for $v - x_n > s_0$, n bids on good 0 and then becomes inactive at $p_0^{CA} = v - x_n$; for $v - (1 - x_n) > s_1$, n bids on good 1 and then becomes inactive at $p_1^{CA} = v - (1 - x_n)$; and for $v - x_n \le s_0$ and $v - (1 - x_n) \le s_1$, n does not bid on good 0 or good 1.

The two-stage clock auction for cases involving an extremal ironing interval with either $\underline{x}(\omega^*) = 0$ or $\overline{x}(\omega^*) = 1$ is encompassed as a special case of the clock auction we just described since the starting prices for goods 1 and 0 simply become $s_1 = v$ and $s_0 = v$, respectively. In summary, we have the following theorem.

Theorem 3. *The two-stage clock auction makes sincere bidding a weakly dominant strategy for every agent and the equilibrium in weakly dominant strategies implements the allocation rule of the optimal mechanism. If every agent pays an upfront fee of $u^{EIR}(\omega^*)$ to participate,*

[28] Note that if $v - x_0 = \overline{\overline{p}}_0^{CA}$, then $v - (1 - x_0) = 2v - 1 - \overline{\overline{p}}_0^{CA}$ and if $v - (1 - x_1) = \overline{\overline{p}}_1^{CA}$, then $v - x_1 = 2v - 1 - \overline{\overline{p}}_1^{CA}$.

then the two-stage clock auction implements the optimal mechanism in weakly dominant strategies subject to IR.

The two-stage clock auction fails to preserve the privacy of the winners whenever the ascending auction phase ends at a price $\overline{\overline{p}}_{\ell}^{CA}$. This always occurs if $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$ and if good ℓ is over-demanded at s_{ℓ}, as this implies that $\psi_S^{-1}(1-v) = 0$ and $\psi_B^{-1}(v) = 1$. Eliciting the marginal winner's location is then necessary to determine the dominant strategy price for all the agents who obtain the other good. Yet, there is a sense in which the two-stage clock auction maximizes the privacy of all the agents, subject to DIC and implementing the allocation rule of the optimal mechanism. First, agents with types inside the ironing interval never reveal their types. Second, if $D_{\ell}(s_{\ell}) \leq K_{\ell}$, then the types of agents who bid on good ℓ are not revealed either. Only if one of the two pure goods is over-demanded will the privacy of some agents be violated, but this violation is necessary to obtain the dominant strategy prices. Finally, notice that in the first stage of the two-stage clock auction, bidders only submit coarse or simple bids ("good 0", "good 1", "indifferent"). "Finer" bidding via an ascending auction is only required if one of the pure goods is over-demanded in the coarse bidding stage.

6 Conclusions

We derive the optimal selling mechanism for a multi-product seller who has goods at each end of the Hotelling line for sale, and faces buyers with linear transportation costs who are privately informed about their locations, which are independently and identically distributed. Unless the buyers' gross valuation is so small that it is efficient to sell the goods via two independent auctions, the optimal selling mechanism always involves lotteries and random allocation. An implication of this is that the optimal selling mechanism is always ex post inefficient, even if all buyers are served.

While the paper focuses on revenue maximization, one can show that lotteries remain part of the optimal selling mechanism for any designer whose objective consists of a non-trivial convex combination of revenue and social surplus. The reason underlying this robust optimality of lotteries is that they increase the marginal revenue that can be extracted from all buyers. Since the designer can always select a small interval over which randomization occurs and include the midpoint of the Hotelling line (where consuming either good is efficient) in this interval, the loss in social surplus from the inefficient random allocation remains second order even when the weight on revenue is small.[29]

[29] See Online Appendix B for background and details. It might also appear that lotteries are an artefact of

The paper offers many avenues for further research. Natural yet non-trivial generalizations include allowing for heterogeneity in gross valuations across the two locations (thereby introducing vertical differentiation) and across agents, as well as heterogeneity in the agents' type distributions. The analysis could also be generalized by allowing the designer to have a network of goods for sale, with goods being represented as nodes and buyers being distributed along edges of the network. Allowing the buyers to have demand for more than one good would also permit an analysis of optimal bundling. Another aspect that could be explored in future work relates to the observation that our setup gives rise to merger synergies without imposing contractual restrictions or introducing increasing returns to scale.[30] Finally, building on the clock auction implementation, one could investigate when there exists an asymptotically optimal prior-free clock auction.[31]

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the exogeneously specified, suboptimal placement of the two varieties of goods. However, Online Appendix B also shows that this is not the case and a lottery remains part of the optimal selling mechanism even when the revenue-maximizing seller can choose the placements of the two varieties of goods. In that same appendix we also show that the optimality of lotteries does not hinge on the transportation costs being linear.

[30]To see this, consider the case with a single buyer and uniformly distributed locations. As mentioned in Footnote 1, if $v \in \left(\frac{1}{2}, 1\right)$, then two independent sellers will not serve the buyer if its location is sufficiently close to the center of the interval. In contrast, if the two sellers merge and become a multi-product seller, the buyer is always served and the profit of the multi-product seller is larger than the sum of the profits of the standalone sellers. Post merger consumer surplus is smaller but social surplus is larger if $v \leq \frac{5}{6}$.

[31]That is, in the limit as the number of agents and goods grows large, the designer makes the same expected profit per agent as it would if it knew the distribution from which the agents draw their types.

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A Proofs of main results

A.1 Proof of Theorem 1

Proof. Suppose that there exists a saddle point $(\boldsymbol{Q}^*, \omega^*)$ satisfying (6) and (7). First, we show that $\boldsymbol{Q}^*$ satisfies (8) and solves the designer's revenue-maximization problem. To that end, for all $\boldsymbol{Q} \in \mathcal{Q}$ we have

$$\min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x}) = \tilde{R}(\boldsymbol{Q}^*, \omega^*) \geq \tilde{R}(\boldsymbol{Q}, \omega^*) \geq \min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}), \tag{25}$$

where the first equality follows from (7) and the second inequality from (6). Thus, $\boldsymbol{Q}^*$ satisfies (8) as required. Second, we show that if $\boldsymbol{Q'}$ satisfies (8) then $(\boldsymbol{Q'}, \omega^*)$ is a saddle point. Since $\boldsymbol{Q'}$ satisfies (8) we have $\min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q'}, \hat{x}) \geq \min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x})$, while setting $\boldsymbol{Q} = \boldsymbol{Q'}$ in (25) yields $\min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x}) = \tilde{R}(\boldsymbol{Q}^*, \omega^*) \geq \tilde{R}(\boldsymbol{Q'}, \omega^*) \geq \min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q'}, \hat{x})$. Combining these, we have $\tilde{R}(\boldsymbol{Q'}, \omega^*) = \min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q'}, \hat{x})$, which implies that $\omega^* \in \arg\min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q'}, \hat{x})$. Combining this with (8) we also have $\boldsymbol{Q'} \in \arg\max_{\boldsymbol{Q}\in\mathcal{Q}} \tilde{R}(\boldsymbol{Q}, \omega^*)$, and $(\boldsymbol{Q'}, \omega^*)$ is thus a saddle point as required.

It only remains to show that a saddle point exists. Since $\mathcal{Q}$ is compact in the product topology[32], $\tilde{R}(\boldsymbol{Q}, \hat{x})$ is linear in $\boldsymbol{Q}$ for all $\hat{x} \in [0,1]$ and $\tilde{R}(\boldsymbol{Q}, \hat{x})$ is concave in $\hat{x}$ for all $\boldsymbol{Q} \in \mathcal{Q}$[33], by Sion's minimax theorem a solution $\boldsymbol{Q}^*$ to (8) exists and we have

$$\max_{\boldsymbol{Q}\in\mathcal{Q}} \min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}) = \min_{\hat{x}\in[0,1]} \max_{\boldsymbol{Q}\in\mathcal{Q}} \tilde{R}(\boldsymbol{Q}, \hat{x}) = \min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x}). \tag{26}$$

Suppose every $\omega \in \arg\min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x})$ is such that $(\boldsymbol{Q}^*, \omega)$ is not a saddle point (i.e. does not satisfy (6)). This then implies that

$$\max_{\boldsymbol{Q}\in\mathcal{Q}} \tilde{R}(\boldsymbol{Q}, \omega) > \tilde{R}(\boldsymbol{Q}^*, \omega)$$

holds for all $\omega \in \arg\min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x})$, which contradicts (26). Consequently, there exists $\omega^* \in \min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x})$ such that $(\boldsymbol{Q}^*, \omega^*)$ is a saddle point. $\square$

[32] Since $\Delta(\{(0,0),(0,1),(1,0)\})$ is compact in the product topology, $\prod_{x\in[0,1]} \Delta(\{(0,0),(0,1),(1,0)\})$ is also compact in the product topology by Tychonoff's theorem. Since $\mathcal{Q} \subset \prod_{x\in[0,1]} \Delta(\{(0,0),(0,1),(1,0)\})$ and the feasibility constraints (F) and monotonicity condition (M) are weak constraints that are linear in the allocation rule $\boldsymbol{Q}$, it follows that $\mathcal{Q}$ is also compact in the product topology.

[33] Differentiating $\tilde{R}(\boldsymbol{Q}, \hat{x})$ with respect to $\hat{x}$ using the Leibniz integral rule yields $\frac{\partial \tilde{R}(\boldsymbol{Q},\hat{x})}{\partial \hat{x}} = [q_0(\hat{x})\,(v - \psi_S(\hat{x})) + q_1(\hat{x})\,(v - (1 - \psi_S(\hat{x}))) - q_0(\hat{x})\,(v - \psi_B(\hat{x})) - q_1(x)\,(v - (1 - \psi_B(\hat{x})))]\, f(\hat{x})$. Simplifying this expression reveals that $\frac{\partial \tilde{R}(\boldsymbol{Q},\hat{x})}{\partial \hat{x}} = (q_0(\hat{x}) - q_1(\hat{x}))(\psi_B(\hat{x}) - \psi_S(\hat{x}))f(\hat{x}) = q_0(\hat{x}) - q_1(\hat{x})$. Since all $\boldsymbol{Q} \in \mathcal{Q}$ are such that $q_0(\hat{x}) - q_1(\hat{x})$ is decreasing in $\hat{x}$, $\tilde{R}(\boldsymbol{Q}, \cdot)$ is concave in $\hat{x}$ for all $\boldsymbol{Q} \in \mathcal{Q}$.

A.2 Proof of Theorem 2

Proof. Theorem 2 is largely proven in the body of the paper; combining the construction of the functions $\overline{q}_\ell$ with Lemma 6, it only remains to prove the uniqueness claims. To that end, for $\hat{x} \in [\hat{x}_0, \hat{x}_1]$, it is useful to introduce the correspondence

$$\Delta\overline{q}(\hat{x}) := \{\overline{q}_0(\hat{x};\gamma) - \overline{q}_1(\hat{x};\gamma) : \gamma \in [0,1]\},$$

since the critical worst-off type ω^* is then such that $0 \in \Delta\overline{q}(\omega^*)$. Combining the comparative statics from Lemma 5 with continuity and monotonicity of the functions $\overline{\Psi}_0$ and $\overline{\Psi}_1$ and the definitions of the functions $\overline{q}_0$ and $\overline{q}_1$ shows that $\Delta\overline{q}$ is an upper hemicontinuous correspondence with a closed graph (see figures 3, 4, 5 and 6 for some examples). Moreover, $\Delta\overline{q}$ is decreasing (in the sense of set inclusion) on $[\hat{x}_0, \hat{x}_1]$. In fact, $\Delta\overline{q}$ is strictly decreasing on $[\hat{x}_0, \hat{x}_A]$ unless $K_0 = N$, in which case $\Delta\overline{q}(\hat{x}) = 1$ for all $\hat{x} \in [\hat{x}_0, \hat{x}_A)$, and $\Delta\overline{q}$ is strictly decreasing on $[\hat{x}_A, \hat{x}_0]$ unless $K_1 = N$, in which case $\Delta\overline{q}(\hat{x}) = -1$ for all $\hat{x} \in (\hat{x}_A, \hat{x}_1]$. Putting all of this together shows that there is most one $\omega^* \in [\hat{x}_0, \hat{x}_1]$ satisfying $0 \in \Delta\overline{q}(\omega^*)$. Moreover, since the ex post maximizing allocation rules are uniquely defined for all $\hat{x} \in (\hat{x}_0, \hat{x}_A) \cup (\hat{x}_A, \hat{x}_1)$, if $\omega^* \in (\hat{x}_0, \hat{x}_A) \cup (\hat{x}_A, \hat{x}_1)$ then $\overline{q}_0(\omega^*;\gamma) = \overline{q}_1(\omega^*;\gamma)$ holds for all $\gamma \in [0,1]$. Otherwise, if $\omega^* \in \{\hat{x}_0, \hat{x}_A, \hat{x}_1\}$, then by construction $\overline{q}_0(\omega^*;\gamma) - \overline{q}_1(\omega^*;\gamma)$ is strictly monotone in γ and, consequently, there is a unique $\gamma \in [0,1]$ satisfying $\overline{q}_0(\omega^*;\gamma^*) = \overline{q}_1(\omega^*;\gamma^*)$. $\square$

A.3 Proof of Theorem 3

Proof. The theorem statement follows immediately from combining the proof of Proposition 6 with the description of the two-stage clock auction provided in Section 5.2. $\square$

B Proofs of auxiliary results

B.1 Proof of Lemma 1

Proof. Suppose we have an incentive compatible direct mechanism $\langle \boldsymbol{Q}, T \rangle$. We start by showing that $q_1(x) - q_0(x)$ is non-decreasing in x. To see this, notice that the incentive compatibility constraints for the types $x, \hat{x} \in [0,1]$ require that $q_0(x)(v-x) + q_1(x)(v-(1-x)) - t(x) \geq q_0(\hat{x})(v-x) + q_1(\hat{x})(v-(1-x)) - t(\hat{x})$ and $q_0(x)(v-\hat{x}) + q_1(x)(v-(1-\hat{x})) - t(x) \leq q_0(\hat{x})(v-\hat{x}) + q_1(\hat{x})(v-(1-\hat{x})) - t(\hat{x})$. Subtracting the latter inequality from the former inequality then yields $(q_1(x) - q_0(x))(x-\hat{x}) \geq (q_1(\hat{x}) - q_0(\hat{x}))(x-\hat{x})$. Without loss of generality we can assume that $x > \hat{x}$. The last inequality then holds if and only if

$q_1(x) - q_0(x) \geq q_1(\hat{x}) - q_0(\hat{x})$. Since $x, \hat{x} \in [0,1]$ were arbitrarily chosen, this shows that incentive compatible also implies $q_1(x) - q_0(x)$ is non-decreasing in x.

Incentive compatibility also implies that $U(x) = \max_{x\in[0,1]}\{q_0(\hat{x})(v-x) + q_1(\hat{x})(v-1+x) - t(\hat{x})\}$. Applying the envelope theorem we then have that $U(x)$ is differentiable almost everywhere and at any point $x \in [0,1]$ such that $U(x)$ is differentiable its derivative $U'(x)$ satisfies $U'(x) = q_1(x) - q_0(x)$. Since $q_1(x) - q_0(x)$ is non-decreasing in x, this also implies that U is a convex function. Moreover, for any $x, \hat{x} \in [0,1]$ we have $U(x) = U(\hat{x}) + \int_{\hat{x}}^{x}(q_1(y) - q_0(y))\,dy$. Combining this last equation with $U(x) = q_0(x)(v-x) + q_1(x)(v-(1-x)) - t(x)$ and solving for $t(x)$ shows that, for any $x, \hat{x} \in [0,1]$,

$$t(x) = q_0(x)(v-x) + q_1(x)(v-(1-x)) - U(\hat{x}) - \int_{\hat{x}}^{x}(q_1(y) - q_0(y))\,dy. \tag{27}$$

Conversely, if we arbitrarily choose a critical type $\hat{x} \in [0,1]$ and a sufficiently high value for $U(\hat{x})$ (so that individual rationality is satisfied for all types), then using (27) we can construct transfers that implement any allocation rule $\boldsymbol{Q}$ such that $q_1(x) - q_0(x)$ is non-decreasing in x. This completes the proof. $\square$

B.2 Proof of Proposition 1

Proof. Given an incentive compatible direct mechanism $\langle \boldsymbol{Q}, T\rangle$ and using (2), the ex ante expected payment made by each buyer to the seller is given by $\mathbb{E}[t(x)] = \int_0^1 t(x)\,dF(x)$. Using (2) we have, for all $\hat{x} \in [0,1]$,

$$\mathbb{E}[t(x)] = \int_0^1 [q_0(x)(v-x) + q_1(x)(v-(1-x))]\,dF(x) - \int_0^1 \int_{\hat{x}}^{x}(q_1(y) - q_0(y))\,dy\,dF(x) - U(\hat{x}).$$

Applying Fubini's theorem yields $\int_0^1 \int_{\hat{x}}^{x}(q_1(y) - q_0(y))\,dy\,dF(x) = \int_{\hat{x}}^{1}(q_1(y) - q_0(y))(1 - F(y))\,dy - \int_0^{\hat{x}}(q_1(y) - q_0(y))F(y)\,dy$. Substituting this into our expression for $\mathbb{E}[t(x)]$ we have

$$\mathbb{E}[t(x)] = \int_0^{\hat{x}} \left[q_0(x)\left(v - x - \frac{F(x)}{f(x)}\right) + q_1(x)\left(v - (1-x) + \frac{F(x)}{f(x)}\right)\right] dF(x)$$
$$+ \int_{\hat{x}}^{1} \left[q_0(x)\left(v - x + \frac{1-F(x)}{f(x)}\right) + q_1(x)\left(v - (1-x) - \frac{1-F(x)}{f(x)}\right)\right] dF(x) - U(\hat{x}).$$

Introducing the virtual type functions ψ_B and ψ_S as defined in (1), this is equivalent to

$$\mathbb{E}[t(x)] = \int_0^{\hat{x}} \left[q_0(x)\left(v - \psi_S(x)\right) + q_1(x)\left(v - (1 - \psi_S(x))\right)\right] dF(x)$$
$$+ \int_{\hat{x}}^1 \left[q_0(x)\left(v - \psi_B(x)\right) + q_1(x)\left(v - (1 - \psi_B(x))\right)\right] dF(x) - U(\hat{x}).$$

Finally, introducing the virtual type functions Ψ_0 and Ψ_1 as defined in the statement of Proposition 1, the ex ante expected payment made by each buyer becomes $\mathbb{E}[t(x)] = \int_0^1 \left[q_0(x)\Psi_0(x,\hat{x}) + q_1(x)\Psi_1(x,\hat{x})\right] dF(x) - U(\hat{x})$. Summing up over each of the buyers finally yields the expression for $R(\boldsymbol{Q}, T)$ from the proposition statement. $\square$

B.3 Proof of Lemma 2

Proof. As noted in the proof of Lemma 1, under any incentive compatible direct mechanism, the interim payoff function U is a convex with $U'(x) = q_1(x) - q_0(x)$ almost everywhere. This implies that $U(x) \geq U(\omega)$ holds for any $x \in [0,1]$ and $\omega \in \{x \in [0,1] : q_1 - q_0 = 0\} \cup \inf\{x \in [0,1] : q_1(x) - q_0(x) > 0\} \cup \sup\{x \in [0,1] : q_1(x) - q_0(x) < 0\}$. Consequently, we have $\Omega(\boldsymbol{Q}) = \{x \in [0,1] : q_1 - q_0 = 0\} \cup \sup\{x \in [0,1] : q_1(x) - q_0(x) > 0\} \cup \inf\{x \in [0,1] : q_1(x) - q_0(x) < 0\}$; the lemma statement immediately follows. $\square$

B.4 Proof of Lemma 3

Proof. Given any $\omega \in \Omega(\boldsymbol{Q})$ and $\hat{x} \in [0,1]$ and using (4), we have

$$\tilde{R}(\boldsymbol{Q}, \hat{x}) - U(\hat{x}) = \tilde{R}(\boldsymbol{Q}, \omega) - U(\omega) \Rightarrow \tilde{R}(\boldsymbol{Q}, \hat{x}) - \tilde{R}(\boldsymbol{Q}, \omega) = U(\hat{x}) - U(\omega). \tag{28}$$

By assumption we have $U(\hat{x}) \geq U(\omega)$. If $\hat{x} \in \Omega(\boldsymbol{Q})$, then $U(\hat{x}) - U(\omega) = 0$ and (28) implies that $\tilde{R}(\boldsymbol{Q}, \hat{x}) = \tilde{R}(\boldsymbol{Q}, \omega)$. If $\hat{x} \notin \Omega(\boldsymbol{Q})$, then $U(\hat{x}) > U(\omega)$ and (28) then implies that $\tilde{R}(\boldsymbol{Q}, \hat{x}) > \tilde{R}(\boldsymbol{Q}, \omega)$. Combining these cases then together shows that we have $\Omega(\boldsymbol{Q}) = \arg\min_{\hat{x} \in [0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x})$ as required. $\square$

B.5 Proof of Lemma 4

Proof. Broadly speaking, our proof strategy involves performing an "ironing" procedure on the interim allocation rules $q_0(x)$ and $q_1(x)$.

We start by taking the interim allocation rule q_1 for location 1 and computing its increasing "ironed" counterpart, which we denote by $\overline{q}_1$.[34] We then perform a transformation

[34]Specifically, we introduce a function $Q_1(x) := \int_0^x q_1(y)\, dF(y)$, compute its convexification, which we

where we replace q_1 with $\overline{q}_1$ and q_0 with $\tilde{q}_0 := q_0 + \overline{q}_1 - q_1$. By construction we have $\int_0^1 (q_1(x) - \overline{q}_1(x))\, dF(x) = 0$, $\int_0^1 (q_0(x) - \tilde{q}_1(x))\, dF(x)$ and $q_1 - q_0 = \overline{q}_1 - \tilde{q}_0$.

Next, we take the transformed interim allocation rule $\tilde{q}_0$ for location 0 and consider its decreasing "ironed" counterpart, which we denote by $\hat{q}_0$.[35] We now perform a second transformation where we replace $\tilde{q}_0$ with $\hat{q}_0$ and $\overline{q}_1$ with $\hat{q}_1 := \overline{q}_1 + \hat{q}_0 - \tilde{q}_0$. By construction we again have $\int_0^1 (q_1(x) - \hat{q}_1(x))\, dF(x) = 0$, $\int_0^1 (q_0(x) - \hat{q}_0(x))\, dF(x) = 0$ and $q_1 - q_0 = \hat{q}_1 - \hat{q}_0$. Moreover, the transformed allocation rule $\hat{q}_0$ is decreasing by construction. We now argue that the allocation rule $\hat{q}_1$ is increasing. Since $\overline{q}_1$ is increasing by construction, it suffices to check this condition for any $x' \in [0,1]$ such that $\hat{q}_0(x') \neq \tilde{q}_0(x')$. However, $\hat{q}_0$ is necessarily constant at such an $x' \in [0,1]$. Consequently, if $\hat{q}_1$ was strictly decreasing at such an $x' \in [0,1]$ then this would contradict the monotonicity of the original allocation rule (which requires that $q_1 - q_0 = \hat{q}_1 - \hat{q}_0$ is increasing). Thus, $\hat{q}_1$ is increasing as required.

Summarizing, the transformed allocation rule $\hat{\boldsymbol{Q}}$ and the original allocation rule $\boldsymbol{Q}$ both allocate the same expected quantity of each good and satisfy the monotonicity condition, and the transformed allocation rule additionally satisfies strong monotonicity. We therefore have $\hat{\boldsymbol{Q}} \in \mathcal{Q}^{\mathrm{SM}}$ as required provided the transformed allocation rule is feasible. That is, we must check that $\hat{q}_0(x) \geq 0$, $\hat{q}_1(x) \geq 0$ and $\hat{q}_1(x) + \hat{q}_0(x) \leq 1$ hold for each type $x \in [0,1]$.

We first address the non-negativity constraints. We start by showing that $\tilde{q}_0(x) \geq 0$ and $\overline{q}_1(x) \geq 0$ hold for all $x \in [0,1]$ (that is, the first step of our transformation does not lead to a violation of the non-negativity constraints). To that end, we only need to consider types $x \in [0,1]$ such that $q_1(x) \geq \overline{q}_1(x)$. By construction, for any such type there exists another type $x' \in [0,1]$ such that $x' > x$ and $\overline{q}_1(x) \geq q_1(x')$. Since $q_1(x') \geq 0$ holds by assumption this shows that $\overline{q}_1(x) \geq 0$. Moreover, by monotonicity we have $q_1(x') - q_0(x') \geq q_1(x) - q_0(x)$. Rearranging this and using $\overline{q}_1(x) \geq q_1(x')$ and $q_0(x') \geq 0$ yields $q_0(x) \geq q_1(x) + q_0(x') - q_1(x') \geq q_1(x) - \overline{q}_1(x)$. We therefore have $\tilde{q}_0(x) = q_0(x) - (q_1(x) - \overline{q}_1(x)) \geq 0$ as required, and $\tilde{q}_0(x) \geq 0$ and $\overline{q}_1(x) \geq 0$ hold for all $x \in [0,1]$. We now show that $\hat{q}_0(x) \geq 0$ and $\hat{q}_1(x) \geq 0$ hold for all $x \in [0,1]$ (that is, the second step of our transformation does not lead to a violation of the non-negativity constraints). To that end, we only need to consider types $x \in [0,1]$ such that $\tilde{q}_0(x) \geq \overline{q}_0(x)$. By construction, for any such type there exists another type $x' \in [0,1]$ such that $x > x'$ and $\overline{q}_0(x) \geq \tilde{q}_0(x')$. Since $\tilde{q}_0(x') \geq 0$ holds by our previous argument, this shows that $\overline{q}_0(x) \geq 0$. Moreover, by monotonicity we have $\overline{q}_1(x) - \tilde{q}_0(x) \geq \overline{q}_1(x') - \tilde{q}_0(x')$. Rearranging this and using $\overline{q}_0(x) \geq \tilde{q}_0(x')$ and $\overline{q}_1(x') \geq 0$ yields $\overline{q}_1(x) \geq \overline{q}_1(x') + \tilde{q}_0(x) - \tilde{q}_0(x') \geq \tilde{q}_0(x) - \hat{q}_0(x)$. We therefore have $\hat{q}_1(x) =$

denote by $\overline{Q}_1$, and then set $\overline{q}_1(x) = \overline{Q}_1'(x)$; see also Footnote 7.

[35]Specifically, we introduce a function $\tilde{Q}_0(x) := \int_0^x \tilde{q}_0(y)\, dF(y)$, compute its concavification, which we denote by $\hat{Q}_0$, and then set $\hat{q}_0(x) = \hat{Q}_0'(x)$; see also Footnote 7.

$\overline{q}_1(x') - (\tilde{q}_0(x) - \hat{q}_0(x)) \geq 0$ as required and $\hat{q}_0(x) \geq 0$ and $\hat{q}_1(x) \geq 0$ hold for all $x \in [0,1]$.

We now address the unit demand constraints. We start by showing that $\tilde{q}_0(x)+\overline{q}_1(x) \leq 1$ holds for all $x \in [0,1]$ (that is, the first transformation cannot result a violation of the unit demand constraints). To that end, we only need to consider types $x \in [0,1]$ such that $\overline{q}_1(x) - q_1(x) > 0$. By construction, for any such type there exists another type $x' \in [0,1]$ such that $x' < x$ and $\overline{q}_1(x) \leq q_1(x')$. Monotonicity implies that $q_1(x') - q_0(x') \leq q_1(x) - q_0(x)$. Rearranging this inequality and adding $q_1(x')$ to both sides then yields

$$q_0(x) + q_1(x') - q_1(x) + q_1(x') \leq q_0(x') + q_1(x'). \tag{29}$$

Combining (29) with $\overline{q}_1(x) \leq q_1(x')$, as well as the fact that $q_0(x') + q_1(x') \leq 1$ holds by assumption, we have $q_0(x) + \overline{q}_1(x) - q_1(x) + \overline{q}_1(x) \leq 1$. Finally, noting that $\tilde{q}_0(x) = q_0(x) + \overline{q}_1(x') - q_1(x)$, shows that we have $\tilde{q}_0(x) + \overline{q}_1(x) \leq 1$ as required. We now show that $\hat{q}_1(x) + \hat{q}_0(x) \leq 1$ holds for all $x \in [0,1]$ (that is, the second transformation also cannot result in such a violation of the unit demand constraints). To that end, we again only need to consider types $x \in [0,1]$ such that $\hat{q}_0(x) - \tilde{q}_0(x) > 0$. For any such type there exists another type $x' \in [0,1]$ such that $x' > x$ and $\hat{q}_0(x) \leq \tilde{q}_0(x')$. Monotonicity implies that $\overline{q}_1(x) - \tilde{q}_0(x) \leq \overline{q}_1(x') - \tilde{q}_0(x')$. Rearranging this inequality and adding $\tilde{q}_0(x')$ to both sides yields

$$\overline{q}_1(x) + \tilde{q}_0(x') - \tilde{q}_0(x) + \tilde{q}_0(x') \leq \overline{q}_1(x') + \tilde{q}_0(x'). \tag{30}$$

Combining (30) with $\hat{q}_0(x) \leq \tilde{q}_0(x')$, as well as the fact that $\overline{q}_1(x') + \tilde{q}_0(x') \leq 1$ holds by our previous argument, we have $\overline{q}_1(x) + \hat{q}_0(x) - \tilde{q}_0(x) + \hat{q}_0(x) \leq 1$. Finally, noting that $\hat{q}_1(x) = \overline{q}_1(x) + \hat{q}_0(x) - \tilde{q}_0(x)$, shows that $\hat{q}_1(x) + \hat{q}_0(x) \leq 1$ holds for all $x \in [0,1]$ and the transformed allocation rule does not violate the unit demand constraint for any type.

To complete the proof, it only remains to verify the final statement of the lemma. Note that since we have $q_1 - q_0 = \hat{q}_1 - \hat{q}_0$, Lemma 2 then immediately implies that $\Omega(\boldsymbol{Q}) = \Omega(\hat{\boldsymbol{Q}})$. If we then take any $\omega \in \Omega(\boldsymbol{Q})$ and set $U(\omega) = 0$, (ICFOC) immediately implies that the interim expected payoff of each agent is invariant under the transformation that replaces the allocation rule $\boldsymbol{Q}$ with the allocation rule $\hat{\boldsymbol{Q}}$. Moreover, by (2) the change in the payment made by type $x \in [0,1]$ under this transformation is given by

$$\begin{aligned}\hat{t}(x) - t(x) &= (\hat{q}_0(x) - q_0(x))(v - x) + (\hat{q}_1(x) - q_1(x))(v - 1 + x) \\ &= (\hat{q}_0(x) - q_0(x))(2v - 1),\end{aligned}$$

where the second inequality follows from the fact that $\hat{q}_0(x) - q_0(x) = \hat{q}_1(x) - q_1(x)$ holds by construction. The corresponding change in the designer's revenue is then given by

$$N \int_0^1 (\hat{t}(x) - t(x))dx = (2v - 1)N(\hat{q}_0(x) - q_0(x))dx = 0$$

as required. $\square$

B.6 Proof of Lemma 5

Proof. Continuity of $z_0(\hat{x})$ and $z_1(\hat{x})$ in $\hat{x}$ follows immediately from (12) and (13), as well as the fact that ψ_S and ψ_B are continuous functions and F is an absolutely continuous distribution. That $z_0(\hat{x})$ and $z_1(\hat{x})$ are respectively decreasing and increasing in $\hat{x}$ follows directly from (12) and (13) and the fact that ψ_S and ψ_B are increasing functions. That $\underline{x}(\hat{x})$ and $\overline{x}(\hat{x})$ are continuous in $\hat{x}$ follows immediately from (14) and (15), and continuity of ψ_B, ψ_S, z^0 and z^1. That $\underline{x}(\hat{x})$ and $\overline{x}(\hat{x})$ are increasing in $\hat{x}$ follows immediately from (14) and (15), that $z_0(\hat{x})$ and $z_1(\hat{x})$ are respectively decreasing and increasing in $\hat{x}$ and that ψ_B^{-1} and ψ_S^{-1} are increasing functions. $\square$

B.7 Proof of Proposition 2

The following proof utilizes a critical type $\hat{x}_A \in (0, 1)$ that is first introduce in Section 4.2 and is such that $z_0(\hat{x}_A) = z_1(\hat{x}_A)$. As is noted in Footnote 15, such an critical type necessarily exists and is such that $\underline{x}(\hat{x}_A) > 0$ and $\overline{x}(\hat{x}_A) < 1$. By construction, the critical type $\hat{x}_A$ also satisfies $\hat{x}_A = \min_{x\in[0,1]}\{\max\{z_0(\hat{x}), z_1(\hat{x})\}\}$. Moreover, setting $\hat{x} = \hat{x}_A$ in (12) and (13), summing these equations and simplifying reveals that $z_0(\hat{x}_A) = z_1(\hat{x}_A) = v - \frac{1}{2}$.

Proof. By Theorem 1, the optimal selling mechanism involves running two independent auctions if and only if there exists $\hat{x} \in (0, 1)$ with $\underline{x}(\hat{x}) > 0$ and $\overline{x}(\hat{x}) < 1$ satisfying $z_0(\hat{x}), z_1(\hat{x}) \leq 0$. Combining this with $\hat{x}_A = \min_{x\in[0,1]}\{\max\{z_0(\hat{x}), z_1(\hat{x})\}\}$ shows that the optimal mechanism involves running two independent auctions if and only if $z_0(\hat{x}_A) = z_1(\hat{x}_A) \leq 0$. Finally, using $z_0(\hat{x}_A) = z_1(\hat{x}_A) = v - \frac{1}{2}$ shows that the optimal mechanism involves running two independent auctions if and only if $v \leq \frac{1}{2}$. $\square$

B.8 Proof of Lemma 6

Proof. Lemma 6 is largely proven in the body of the paper, and it only remains to show that $\omega^* \notin \{0, 1\}$. To that end, setting $\hat{x} = 0$ we have $\Psi_0(x, 0) = v - \psi_B(x)$ and $\Psi_1(x, 0) = v-(1-\psi_B(x))$ almost everywhere. Since $\Psi_0(x, 0)$ is decreasing in x with $\Psi_0(0, 0) = v+\frac{1}{f(0)} > \Psi_1(0, 0) = v-1-\frac{1}{f(0)}$ and $\Psi_1(x, 0)$ is increasing in x with $\Psi_1(1, 0) = v > \Psi_0(1, 0) = v-1-\frac{1}{f(0)}$, the worst-off type ω under the corresponding pointwise maximizing ex post allocation rule must be such that $\omega \in (0,1)$. Consequently, $\omega \neq 0$ and $\hat{x} = 0$ does not satisfy the saddle point condition from Theorem 1. The argument showing that $\omega^* \neq 1$ is similar. □

B.9 Proof of Lemma 7

Proof. From the proof of Proposition 2, we know that whenever $v > \frac{1}{2}$, there is no critical type $\hat{x}$ such that $z_0(\hat{x}), z_1(\hat{x}) \leq 0$. It therefore suffices to show that any critical type $\hat{x}$ with $z_\ell(\hat{x}) > 0$ and $z_{-\ell}(\hat{x}) < 0$ for some $\ell \in \{0,1\}$ cannot satisfy the saddle point condition from Theorem 1. The statement of the lemma then follows.

Suppose that $\hat{x}$ is such that $z_0(\hat{x}) > 0$ and $z_1(\hat{x}) < 0$. Then given a sufficiently small $\epsilon > 0$ we have $\overline{\Psi}_0(\overline{x}(\hat{x}), \hat{x}) > \overline{\Psi}_0(\overline{x}(\hat{x}) + \epsilon, \hat{x}) > 0$ and $\overline{\Psi}_1(\overline{x}(\hat{x}), \hat{x}) < \overline{\Psi}_1(\overline{x}(\hat{x}) + \epsilon, \hat{x}) < 0$. Consequently, under the ex post allocation rule $\overline{Q}(\cdot, \hat{x})$ that pointwise maximizes the designer's ironed virtual surplus function $\overline{R}(\cdot, \hat{x})$ (and breaks any ties uniformly at random), buyers located at $\overline{x}(\hat{x})$ and $\overline{x}(\hat{x}) + \epsilon$ are only ever allocated a unit of the good at location 0. Moreover, buyers at $\overline{x}(\hat{x})$ have higher priority than buyers at $\overline{x}(\hat{x}) + \epsilon$ for a unit of the good at location 0 and, consequently, the interim allocation probability and the interim expected payoff is higher for buyers at location $\overline{x}(\hat{x})$ than location $\overline{x}(\hat{x}) + \epsilon$ under $\overline{Q}(\cdot, \hat{x})$. Since $\hat{x} \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ and all buyers located in the ironing interval must have the same interim expected payoff, this implies that $\hat{x}$ is not a worst-off type under $\overline{Q}(\cdot, \hat{x})$ and hence we cannot have a saddle point involving the critical type $\hat{x}$. The argument for the case where $\hat{x}$ is such that $z_0(\hat{x}) < 0$ and $z_1(\hat{x}) > 0$ is analogous. □

B.10 Derivation of (18), (19), (20), (21) and (22)

Proof. We begin by considering cases with $K_0 + K_1 \leq N$ that involve scarcity. For these cases, we have weakly fewer goods than agents. Consequently, we can independently compute the pointwise maximizing allocation rules $\overline{Q}_0$ and $\overline{Q}_1$ by allocating units of each good to the agents in a positive assortative fashion (breaking ties that arise in the ironing interval uniformly at random) because (unlike cases where $K_0 + K_1 > N$) under scarcity this procedure cannot result in the designer attempting to allocate two goods to a single agent.

Consider critical types $\hat{x} \in (\hat{x}_0, \hat{x}_1)$ that produce a unique pointwise maximizing ex post allocation rule. We then have $Q_0(i,j,\hat{x}) > 0$ provided the state isn't such that all units of good 0 are allocated to buyers with $x < \underline{x}(\hat{x})$. Consequently, we have $Q_0(i,j,\hat{x}) > 0$ if and only if $i < K_0$. Moreover, we have $Q_0(i,j,\hat{x}) = 1$ if and only if $i + N - i - j < K_0$ (that is, if and only if there is a sufficient supply of good 0 to serve all buyers $x \leq \overline{x}(\hat{x})$ with units from this location). Combining these results, we have $Q_0(i,j,\hat{x}) = \frac{K_0 - i}{N-i-j} \in (0,1)$ if and only if $i < K_0$ and $j < N - K_0$. Putting all of this together yields $\overline{Q}_0(i,j,\hat{x}) = \frac{K_0 - i}{N-i-j}\mathbb{1}(j < N - K_0,\, i < K_0) + \mathbb{1}(j \geq N - K_0)$. A similar argument shows that $\overline{Q}_1(i,j,\hat{x}) = \frac{K_1 - j}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(i \geq N - K_1)$. With these equations at hand, the derivation of the general expressions for $\overline{Q}_0$ and $\overline{Q}_1$ that apply for any $\hat{x} \in [\underline{x}_0, \overline{x}_1]$ is provided in the body of the paper and culminates in (18) and (19).

We now move on to considering cases with $K_0 + K_1 > N$ that involve abundance. Since these cases are such that the designer has more goods than agents, the pointwise maximizing allocation rules $\overline{Q}_0$ and $\overline{Q}_1$ cannot be computed independently. Instead, we can exploit the fact that $Q_0(i,j,\hat{x}) + Q_1(i,j,\hat{x}) = 1$ holds for all feasible states (i,j) and critical types $\hat{x} \in (\hat{x}_0, \hat{x}_1)$.

We start by considering critical types such that $\hat{x} \in (\hat{x}_0, \hat{x}_A)$, which implies that $z_0(\hat{x}) > z_1(\hat{x}) > 0$. In this case the seller strictly prefers to allocate the agents in the ironing interval a unit at location 0 whenever this is feasible. Since the feasibility constraint for good 0 then uniquely pin down $\overline{Q}_0$, and—just as we saw for the scarcity case—we have $\overline{Q}_0(i,j,\hat{x}) = \frac{K_0-i}{N-i-j}\mathbb{1}(i < K_0,\, j < N - K_0) + \mathbb{1}(j \geq N - K_0)$. However, unlike under the case involving scarcity, we now have $\overline{Q}_1(i,j,\hat{x}) = 1 - \overline{Q}_0(i,j,\hat{x})$. A similar argument shows that for critical types such that $\hat{x} \in (\hat{x}_A, \hat{x}_1)$ we have $\overline{Q}_1(i,j,\hat{x}) = \frac{K_1-j}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(i \geq N - K_1)$ and $\overline{Q}_0(i,j,\hat{x}) = 1 - \overline{Q}_1(i,j,\hat{x}) = \frac{N-K_1-i}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(j \geq K_1)$. Finally, we consider the case where $\hat{x} = \hat{x}_A$ (which implies that $z_0(\hat{x}) = z_1(\hat{x}) > 0$) and derive (20). As we argue in the body of the paper, for this case the set of pointwise maximizing ex post allocation rules can be constructed by taking convex combinations of the extremal lottery that allocates units of the good at location 0 to agents in the ironing interval to wherever possible and the extremal lottery that allocates units of the good at location 1 to agents in the ironing interval to wherever possible. We let

$$\begin{aligned}\overline{Q}_0(i,j,\hat{x}_A;1) &= \tfrac{K_0-i}{N-i-j}\mathbb{1}(i < K_0,\, j < N - K_0) + \mathbb{1}(j \geq N - K_0)\\ &= \begin{cases} 0, & i \geq K_0 \\ \min\left\{\frac{K_0-i}{N-i-j}, 1\right\}, & i < K_0 \end{cases}\end{aligned}$$

and

$$\begin{aligned}\overline{Q}_0(i,j,\hat{x}_A;0) &= \tfrac{N-K_1-i}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(j \geq K_1)\\ &= \begin{cases} 0, & i \geq N - K_1 \\ \min\left\{\frac{N-K_1-i}{N-i-j}, 1\right\}, & i < N - K_1 \end{cases}\end{aligned}$$

respectively denote the ex post probabilities that buyers in the ironing interval are allocated a unit of the good at location 0 under these extremal lotteries. Taking $\overline{Q}_0(i, j, \hat{x}_A; \gamma) = \gamma\overline{Q}_0(i, j, \hat{x}_A; 1) + (1 - \gamma)\overline{Q}_1(i, j, \hat{x}_A; 0)$ then yields (20) as required. Similar to what we saw for the cases involving scarcity, with all of these expressions at hand, the derivation of the general expressions for $\overline{Q}_0$ and $\overline{Q}_1$ that apply for any $\hat{x} \in [\underline{x}_0, \overline{x}_1]$ is provided in the body of the paper and culminates in (21) and (22). $\square$

B.11 Proof of Proposition 3

Proof. The derivation of q^* is provided in the body of the paper and the corresponding transfer rule t^* can be computed by setting $\hat{x} = \hat{x}_A$, $U(\hat{x}_A) = 0$ and $q = q^*$ in (2). Under t^* all worst-off types $x \in [\underline{x}(\hat{x}_A), \overline{x}(\hat{x})]$ in the ironing interval pay a lottery price of $v - \frac{1}{2}$ that ensures they receive an interim expected payoff of 0. The price $v - \psi_S^{-1}\left(\frac{1}{2}\right)$ paid by buyers with locations $x < \underline{x}(\hat{x}_A)$ is such that buyers at $x = \underline{x}(\hat{x}_A)$ are indifferent between paying to receiving a unit at location 0 with certainty and paying to enter the lottery. The price $v - \left(1 - \psi_B^{-1}\left(\frac{1}{2}\right)\right)$ paid by buyers with locations $x > \overline{x}(\hat{x}_A)$ is such that buyers at $x = \overline{x}(\hat{x}_A)$ are indifferent between paying to receiving a unit at location 1 with certainty and paying to enter the lottery. $\square$

B.12 Proof of Lemma 8

Proof. By Lemma 5, $\underline{x}(\hat{x})$ and $\overline{x}(\hat{x})$ are continuous and increasing in $\hat{x}$. Moreover, by (14) and (15) we have $F(\underline{x}(0)) = F(\overline{x}(0)) = 0$ and $F(\underline{x}(1)) = F(\overline{x}(1)) = 1$. The lemma statement then immediately follows from continuity and monotonicity of F. $\square$

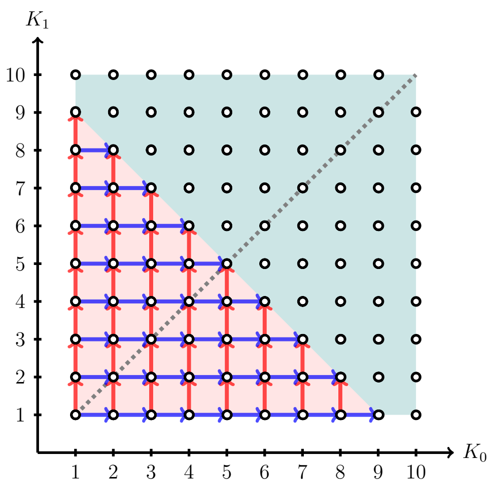
B.13 Proof of Proposition 4

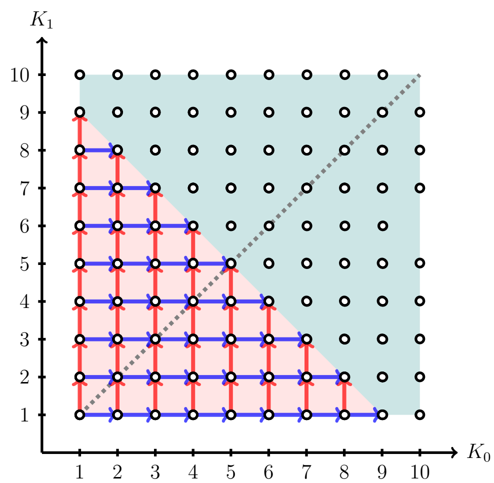
In the following proof we augment our Section 4.2 notation by writing $\overline{q}_\ell(\hat{x}, K_0, K_1; \gamma)$ in order to make explicit the dependence of these interim allocations on the parameters (K_0, K_1).

Proof. We start by proving the comparative statics for the scarcity region and fix any (K_0, K_1) such that $K_0 + K_1 < N$. Since $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$ (which implies that $\hat{x}_0 = 0$ and $\hat{x}_1 = 1$) the pointwise maximizing ex post allocation rules are uniquely defined for all $\hat{x} \in (0, 1)$ and we can drop any dependence of the associated interim allocation rules on the index parameter γ.

We start by showing that $\hat{x}(K_0, K_1) < \hat{x}(K_0 + 1, K_1)$. Fixing the critical type $\hat{x}(K_0, K_1)$, we consider how the allocation rule that pointwise maximizes the designer's ironed virtual objective function $\overline{R}(\cdot, \hat{x}(K_0, K_1))$ varies as the endowment increases from (K_0, K_1) to $(K_0 + 1, K_1)$. In particular, for any feasible state (i, j) such that $i \geq N - K_1$ or $j \geq N - K_0$, increasing the supply of good 0 by one unit will not change the deterministic ex post allocation for buyers in the ironing interval.[36] However, for any feasible state (i, j) such that $i < N - K_1$ and $j < N - K_0$, buyers in the ironing interval are rationed with positive probability because $K_0 + K_1 < N$. Consequently, for any feasible state (i, j) such that $i < N - K_1$ and $j < N - K_0$, increasing the supply of good 0 by one unit ensures that the ex post lottery offered to buyers in the ironing interval includes one additional unit of good 0. Putting all of this together—and using the fact that $\overline{q}_0(\hat{x}(K_0, K_1), K_0, K_1) = \overline{q}_1(\hat{x}(K_0, K_1), K_0, K_1)$ holds by construction—we have $\overline{q}_0(\hat{x}(K_0, K_1), K_0 + 1, K_1) > \overline{q}_1(\hat{x}(K_0, K_1), K_0 + 1, K_1)$. Combining the comparative statics concerning the ironing parameters from Lemma 5 with the continuity and monotonicity of the functions $\overline{\Psi}_0$ and $\overline{\Psi}_1$ and the definitions of the functions $\overline{q}_0$ and $\overline{q}_1$ shows that these interim allocations are decreasing and increasing in $\hat{x}$, respectively. Consequently, combining $\overline{q}_0(\hat{x}(K_0, K_1), K_0 + 1, K_1) > \overline{q}_1(\hat{x}(K_0, K_1), K_0 + 1, K_1)$ with the fact that $\overline{q}_0(\hat{x}(K_0 + 1, K_1), K_0 + 1, K_1) = \overline{q}_1(\hat{x}(K_0 + 1, K_1), K_0 + 1, K_1)$ also holds by construction, shows that $\hat{x}(K_0, K_1) < \hat{x}(K_0 + 1, K_1)$, as required. The argument proving that $\hat{x}(K_0, K_1) > \hat{x}(K_0, K_1 + 1)$ is analogous.

The following figures modify the panels of Figure 7 to include only the information that has been established up to this point of the proof.





We now move onto the abundance region. In this region the balanced markets such that $K_0 + K_1 = N$ play an important role. So we start by studying these markets, which we parameterize by $(K, N - K)$, where $K \in \{1, \ldots, N - 1\}$.

We first show that $\hat{x}(K, N - K)$ is strictly increasing in $K \in \{1, \ldots, N - 1\}$ and, consequently, there is at most one value of K such that $\hat{x}(K, N - K) = \hat{x}_A$. For the proof of

[36] Since we have $K_0 + K_1 + 1 \leq N$, buyers in the ironing interval will be allocated a unit of good 0 (1) before and after the endowment change, regardless of the relative values of the ironing parameters $z_0(\hat{x})$ and $z_1(\hat{x})$ whenever $j \geq N - K_0$ ($i \geq N - K_1$).

this result only, we restrict attention to the case where $N \geq 3$, since this result is vacuous if $N = 2$ (and $K_0 = K_1 = 1$ is then the only balanced market parameterization). The desired result then follows immediately from the comparative statics we just derived concerning the scarcity region. In particular, we have $\hat{x}(K, N-K) < \hat{x}(K, N-K-1)$ and $\hat{x}(K, N-K-1) < \hat{x}(K+1, N-K-1)$ and combining these comparative statics then shows that $\hat{x}(K, N-K) < \hat{x}(K+1, N-K-1)$ as required.

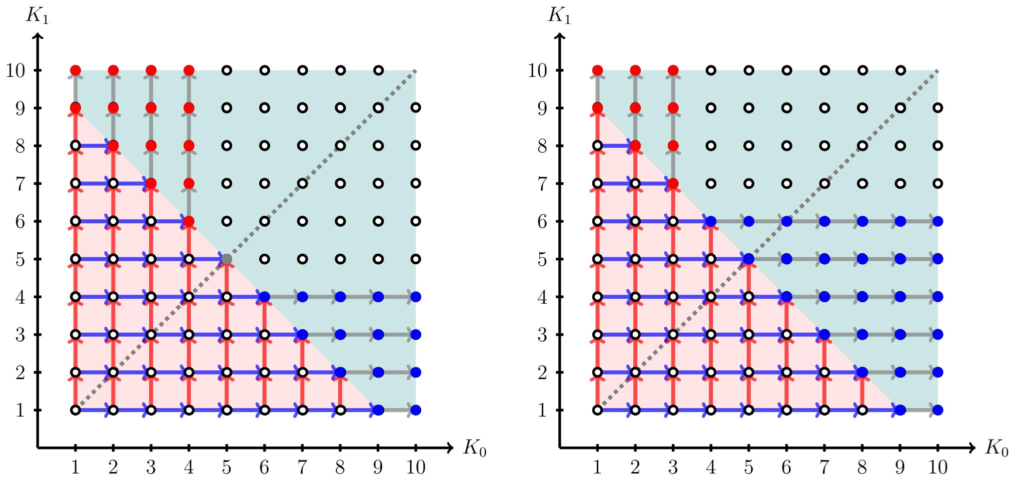
Next, starting from a balanced market we consider the effect of unilaterally increasing the supply of one good. Specifically, we show that whenever $K \in \{1, \ldots, N-1\}$ is such that $\hat{x}(K, N-K) < \hat{x}_A$, we have $\hat{x}(K, K_1) = \hat{x}(K, N-K)$ for all $K_1 \in \{N-K, \ldots, N\}$. Fixing the critical type $\hat{x}(K, N-K)$, we consider how the allocation rule that pointwise maximizes the designer's ironed virtual objective function $\overline{R}(\cdot, \hat{x}(K, N-K))$ varies as the endowment increases from $(K, N-K)$ to (K, K_1). Notice that the critical type $\hat{x}(K, N-K)$ is such that $z_0(\hat{x}(K, N-K)) > z_1(\hat{x}(K, N-K))$ and under the endowment $(K, N-K)$ buyers in the ironing interval are served with probability 1. Consequently, for any feasible state (i, j) such that $i > K_0$ or $j \geq N - K_0$, increasing the supply of good 1 does not change the deterministic ex post allocation for buyers in the ironing interval.[37] Moreover, for any feasible state (i, j) such that $i \leq K_0$ and $j < N - K_0$, increasing the supply of good 1 has no effect on the ex post lottery offered to buyers in the ironing interval. While increasing the supply of good 1 makes it feasible for the seller to include more units of good 1 in the lottery offered to buyers in the ironing interval, this is not consistent with pointwise maximization as $z_0(K, N-K) > z_1(K, N-K)$. Putting all of this together—and using the fact that $\overline{q}_0(\hat{x}(K, N-K), K, N-K) = \overline{q}_1(\hat{x}(K, N-K), K, N-K)$ holds by construction—we have $\overline{q}_0(\hat{x}(K, N-K), K, K_1) = \overline{q}_1(\hat{x}(K, N-K), K, K_1)$.[38] Applying Theorem 2, we then have $\hat{x}(K, K_1) = \hat{x}(K, N-K)$ as required.

We conclude our balanced market analysis by noting that whenever $K \in \{1, \ldots, N-1\}$ is such that $\hat{x}(K, N-K) > \hat{x}_A$ we also have $\hat{x}(K_0, N-K) = \hat{x}(K, N-K)$ for all $K_0 \in \{K, \ldots, N\}$; the proof is analogous to the argument for the case where $\hat{x}(K, N-K) < \hat{x}_A$. The figures at the top of the following page update our modification of the panels of Figure 7 to include the new information concerning the abundance region that has now been established.

As these figures illustrate, at this point, we are left with a "rectangle" of parameterizations (K_0, K_1) in the abundance region that have not already been identified as being such that $\hat{x}(K_0, K_1) < \hat{x}_A$ or $\hat{x}(K_0, K_1) > \hat{x}_A$. Letting (K_0^A, K_1^A) denote the bottom-left corner of this

[37]If $i > K_0$ ($j \geq N - K_0$) all buyers in the ironing interval are allocated a unit of good 1 (0) before and after the endowment change.

[38]Here, the pointwise maximizing ex post allocation rules and the induced interim allocation rules are again uniquely defined so we omit their dependence on the index parameter γ.



"rectangle", one of four cases applies:

(i) If there exists $K \in \{1, \ldots, N-1\}$ with $K_0 = K$, $K_1 = N-K$ and $\hat{x}(K, N-K) = \hat{x}_A$, then we have $K_0^A = K$ and $K_1^A = N-K$.

(ii) If there exists $K \in \{1, \ldots, N-2\}$ with $K_0 = K$, $K_1 = N-K$ and $\hat{x}_A \in (\hat{x}(K, N-K), \hat{x}(K+1, N-K-1))$, then we have $K_0^A = K+1$ and $K_1^A = N-K$.

(iii) If $\hat{x}(1, N-1) > \hat{x}_A$, then we have $K_0^A = 1$ and $K_1^A = N$.

(iv) If $\hat{x}(N-1, 1) < \hat{x}_A$, then we have $K_0^A = N$ and $K_1^A = 1$.

Notice that Panel (a) of Figure 7 provides an example of case (i) and Panel (b) of Figure 7 provides an example of case (ii). An illustration of cases (iii) and (iv) is provided in the figures at the top of the following page.[39]

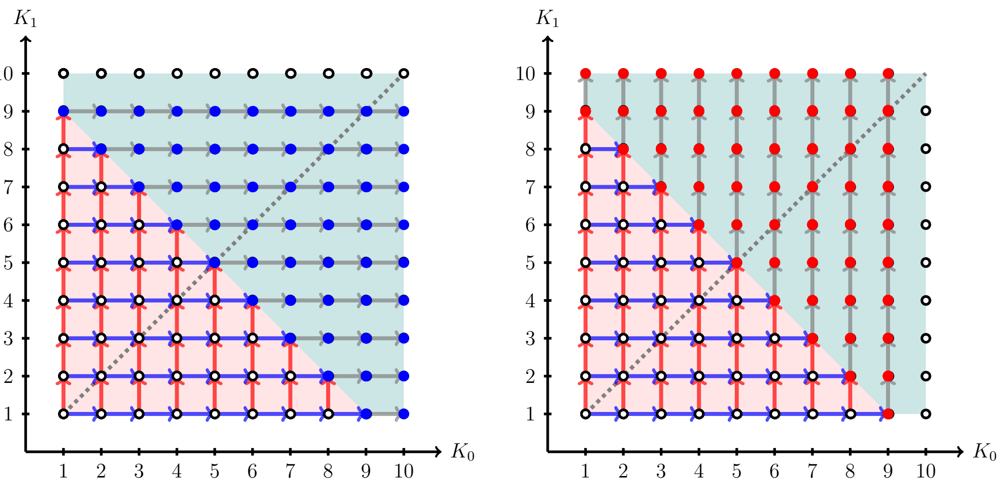
We have the following lemma.

Lemma B.1. *The endowment* (K_0^A, K_1^A) *is such that* $\hat{x}(K_0^A, K_1^A) = \hat{x}_A$.

Proof. For case (i) this holds by construction and the proof for case (iv) is analogous to the proof for case (iii). So we focus cases (ii) and (iii).

Suppose that case (ii) applies and there exists $K_0^A \in \{2, \ldots, N-1\}$ and $K_1^A \in \{2, \ldots, N-1\}$ with $K_0^A + K_1^A = N+1$ such that $\hat{x}_A \in (\hat{x}(K_0^A - 1, K_1^A), \hat{x}(K_0^A, K_1^A - 1))$. In what follows, we refer to an critical type $\check{x}$ as *feasible* under the endowment (K_0, K_1) if it is possible to construct an ex post allocation rule $\boldsymbol{Q}(x_n, \boldsymbol{x}_{-n}, \check{x})$ that pointwise maximizes the ironed virtual surplus function $\overline{R}(\cdot, \check{x})$ on the domain $x_n \in [0, \underline{x}(\check{x})) \cup (\overline{x}(\check{x}), 1]$ and yields an interim allocation of $q_0(x_n, \check{x}) = q_1(x_n, \check{x}) = \frac{1}{2}$ for all $x_n \in [\underline{x}(\check{x}), \overline{x}(\check{x})]$. Intuitively, the proof that

[39] For a given distribution F, cases (iii) or (iv) can only arise if N is sufficiently small.



$\hat{x}(K_0^A, K_1^A) = \hat{x}_A$ then proceeds as follows. First, since $\hat{x}(K_0^A - 1, K_1^A)$ is a feasible critical type under the endowment $(K_0^A - 1, K_1^A)$ and $\hat{x}(K_0^A, K_1^A - 1)$ is a feasible critical type under the endowment $(K_0^A, K_1^A - 1)$, any critical type in the range $[\hat{x}(K_0^A - 1, K_1^A), \hat{x}(K_0^A, K_1^A - 1)]$ must be feasible if we relax the feasibility constraints by taking the join (K_0^A, K_1^A) of these endowments. Consequently, $\hat{x}_A \in [\hat{x}(K_0^A - 1, K_1^A), \hat{x}(K_0^A, K_1^A - 1)]$ is a feasible critical type under the endowment (K_0^A, K_1^A). Moreover, the ex post allocation rule that ensures $\hat{x}_A$ is feasible under the endowment (K_0^A, K_1^A) is also consistent with pointwise maximizing the ironed virtual surplus function $\overline{R}(\cdot, \hat{x}_A)$ subject to the feasibility constraints. Consequently, $\hat{x}_A$ satisfies the saddle point condition and we have $\hat{x}(K_0^A, K_1^A) = \hat{x}_A$.

More formally, suppose the seller's endowment is (K_0^A, K_1^A) and take any $\check{x} \in [\hat{x}(K_0^A - 1, K_1^A), \hat{x}(K_0^A, K_1^A - 1)]$. Consider the ex post allocation rule $\boldsymbol{Q}(x_n, \boldsymbol{x}_{-n}, \check{x}; \gamma)$ that pointwise maximizes the ironed virtual surplus function $\overline{R}(\cdot, \check{x})$ for all $x_n \in [0, \underline{x}(\check{x})) \cup (\overline{x}(\check{x}), 1]$ and serves all types $x_n \in [\underline{x}(\check{x}), \overline{x}(\check{x})]$ with probability 1, allocating them a unit of the good at location 0 wherever possible with probability γ, and allocating them a unit of the good at location 1 wherever possible with probability $1 - \gamma$.[40] Let $q_\ell(\check{x}; \gamma)$ denote the corresponding interim allocation for buyers in the ironing interval. Since $\hat{x}(K_0^A - 1, K_1^A)$ is feasible under the endowment $(K_0^A - 1, K_1^A)$ and $\hat{x}(K_0^A, K_1^A - 1)$ is feasible under the endowment $(K_0^A, K_1^A - 1)$, any critical type $\check{x} \in [\hat{x}(K_0^A - 1, K_1^A), \hat{x}(K_0^A, K_1^A - 1)]$ is feasible under the given endowment (K_0^A, K_1^A). Consequently, for all $\check{x} \in [\hat{x}(K_0^A - 1, K_1^A), \hat{x}(K_0^A, K_1^A - 1)]$, there exists $\gamma \in (0, 1)$ such that $q_0(\check{x}; \gamma) = q_1(\check{x}; \gamma) = \frac{1}{2}$. Moreover, whenever $\check{x} = \hat{x}_A$ the corresponding ex post allocation rule is consistent with pointwise maximizing the ironed virtual surplus function $\overline{R}(\cdot, \hat{x}_A)$ subject to the feasibility constraints. This implies that there exists $\gamma^* \in (0, 1)$ such

[40]Notice that whenever $\gamma \in (0, 1)$, this allocation rule is consistent with pointwise maximizing the designer's ironed objective function subject to the feasibility constraints if and only if $\check{x} = \hat{x}_A$.

that $\overline{q}_0(\hat{x}_A;\gamma^*) = \overline{q}_1(\hat{x}_A;\gamma^*)$ and by Theorem 2 we have $\hat{x}(K_0^A, K_1^A) = \hat{x}_A$ as required.

We now suppose case (iii) applies and we have $K_0^A = 1$ and $K_1^A = N$ with $\hat{x}(1, N-1) > \hat{x}_A$. Notice that whenever $K_1 = N$, $\overline{q}_1(\hat{x};\gamma) = 1$ must hold for any $\hat{x} > \hat{x}_A$ and, consequently, we must have $\hat{x}(1,N) \leq \hat{x}_A$. Assume, seeking a contradiction, that $\hat{x}(1,N) < \hat{x}_A$. Then leveraging the arguments we just introduced for case (ii), we have that any critical type $\check{x} \in [\hat{x}(1,N), \hat{x}(1,N-1)]$ is feasible under the endowment $(1,N)$. Since $\hat{x}_A \in [\hat{x}(1,N), \hat{x}(1,N-1)]$ holds by construction, this implies that the critical type $\hat{x}_A$ is feasible under the endowment $(1,N)$. Moreover, the ex post allocation rule that ensures the feasibility of $\hat{x}_A$ under the endowment $(1,N)$ is also consistent with pointwise maximizing the ironed virtual type function $\overline{R}(\cdot,\hat{x}_A)$ subject to the feasibility constraints and, consequently, $\hat{x}_A$ is a critical worst-off type under the endowment $(1,N)$. Since the critical worst-off type is unique by Theorem 2, this contradicts our initial assumption that $\hat{x}(1,N) < \hat{x}_A$. Consequently, we have $\hat{x}(1,N) = \hat{x}(K_0^A, K_1^A) = \hat{x}_A$ as required. □

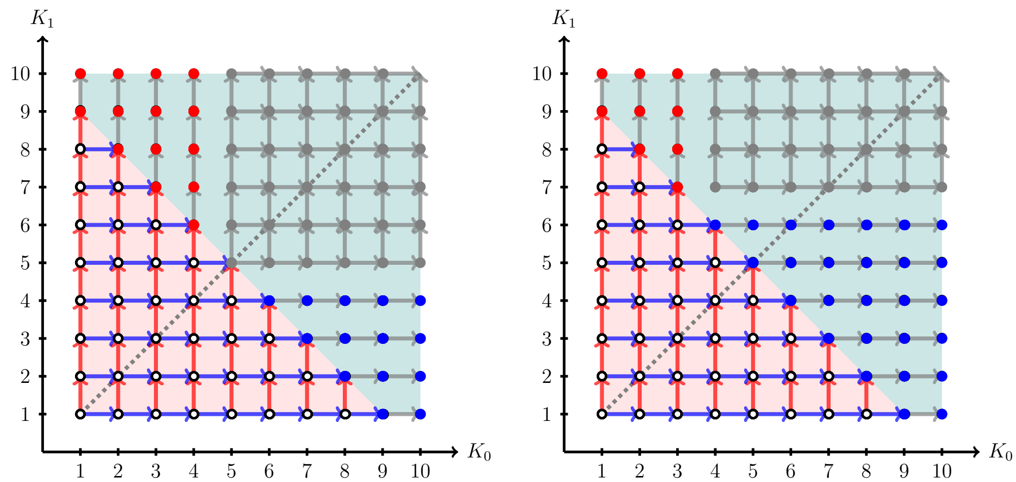
As the next lemma shows, the entire "rectangle" of parameterizations (K_0, K_1) in the abundance region (where the bottom-left corner of this rectangle is given by (K_0^A, K_1^A)) that haven't been identified as being such that $\hat{x}(K_0,K_1) < \hat{x}_A$ or $\hat{x}(K_0,K_1) > \hat{x}_A$ are in fact such that $\hat{x}(K_0,K_1) = \hat{x}_A$.

Lemma B.2. *In the abundance region with $K_0 + K_1 \geq N$, $\hat{x}(K_0,K_1) = \hat{x}_A$ if and only if $K_0 \geq K_0^A$ and $K_1 \geq K_1^A$.*

Proof. We have already shown that for any (K_0,K_1) in the abundance region that does not satisfy $K_0 \geq K_0^A$ and $K_1 \geq K_1^A$, we either have $(K_0,K_1) < \hat{x}_A$ or $(K_0,K_1) > \hat{x}_A$. So it only remains to show that $\hat{x}(K_0,K_1) = \hat{x}_A$ holds whenever $K_0 \geq K_0^A$ and $K_1 \geq K_1^A$. From Lemma B.1 we know that $\hat{x}(K_0^A, K_1^A) = \hat{x}_A$. Utilizing the machinery introduced in the proof of Lemma B.1, we now show that this in turn implies that $\hat{x}(K_0,K_1) = \hat{x}_A$ holds whenever $K_0 \geq K_0^A$ and $K_1 \geq K_1^A$. In particular, since $\hat{x}_A$ is a feasible ironing parameter under the endowment (K_0^A, K_1^A), it is necessarily a feasible ironing parameter under the endowment (K_0,K_1) with $K_0 \geq K_0^A$ and $K_1 \geq K_1^A$ as increasing the seller's endowment can only make the feasibility constraints less tight. Moreover, the ex post allocation rule that ensures the feasibility of $\hat{x}_A$ under the endowment (K_0,K_1) is also consistent with pointwise maximizing the ironed virtual type function $\overline{R}(\cdot,\hat{x}_A)$ subject to the feasibility constraints. Thus, $\hat{x}_A$ is a critical worst-off type under the endowment (K_0,K_1) and we have $\hat{x}(K_0,K_1) = \hat{x}_A$, which concludes the proof. □

Notice that Lemma B.2 immediately implies that $\hat{x}(K_0,K_1) = \hat{x}(K_0+1,K_1)$ if $K_0 \geq K_0^A$ and $\hat{x}(K_0,K_1) = \hat{x}(K_0,K_1+1)$ if $K_1 \geq K_1^A$.

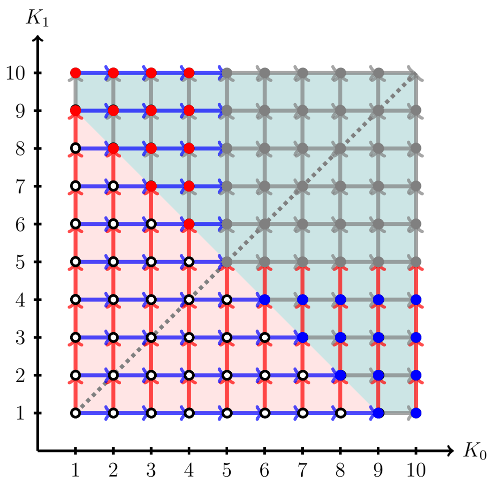
The figures below modify the panels of Figure 7 to include the information that has now been established up to this point. It only remains to show that $\hat{x}(K_0, K_1) < \hat{x}(K_0+1, K_1) \leq \hat{x}_A$ if $K_0 < K_0^A$ and $\hat{x}(K_0, K_1) > \hat{x}(K_0, K_1+1) \geq \hat{x}_A$ if $K_1 < K_1^A$.

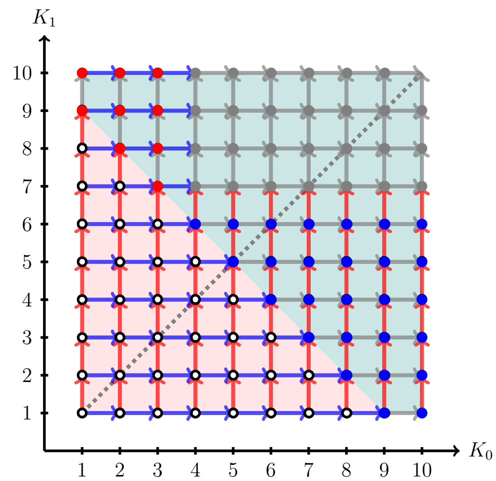


We now conclude the proof by establishing that we have $\hat{x}(K_0, K_1) < \hat{x}(K_0 + 1, K_1) \leq \hat{x}_A$ if $K_0 < K_0^A$. The argument showing that we have $\hat{x}(K_0, K_1) > \hat{x}(K_0, K_1 + 1) \geq \hat{x}_A$ whenever $K_1 < K_1^A$ is analogous. We have already established that $\hat{x}(K_0, K_1) < \hat{x}_A$ and $\hat{x}(K_0+1, K_1) \leq \hat{x}_A$ holds in the abundance region whenever $K_0 < K_0^A$. So it only remains to show that $\hat{x}(K_0, K_1) < \hat{x}(K_0 + 1, K_1)$. Fixing the critical type $\hat{x}(K_0, K_1) < \hat{x}_A$, we consider how the allocation rule that pointwise maximizes the designer's ironed virtual objective function $\overline{R}(\cdot, \hat{x}(K_0, K_1))$ varies as the endowment increases from (K_0, K_1) to $(K_0 + 1, K_1)$. Notice that the critical type $\hat{x}(K_0, K_1)$ is such that $z_0(\hat{x}(K_0, K_1)) > z_1(\hat{x}(K_0, K_1))$ and under the endowment (K_0, K_1) buyers in the ironing interval are served with probability 1. Consequently, for any feasible state (i, j) such that $i > K_0$ or $j \geq N - K_0$, increasing the supply of good 0 by one unit does not change the deterministic ex post allocation for buyers in the ironing interval. However, for any feasible state (i, j) such that $i \leq K_0$ and $j \geq N - K_0$, increasing the supply of good 0 by one unit ensures that the ex post lottery offered to buyers in the ironing interval includes one less unit of good 1 and one additional unit of good 0. Putting all of this together—and using the fact that $\overline{q}_0(\hat{x}(K_0, K_1), K_0, K_1) = \overline{q}_1(\hat{x}(K_0, K_1), K_0, K_1)$ holds by construction—we have $\overline{q}_0(\hat{x}(K_0, K_1), K_0 + 1, K_1) > \overline{q}_1(\hat{x}(K_0, K_1), K_0 + 1, K_1)$. As we noted earlier in the proof, the functions $\overline{q}_0$ and $\overline{q}_1$ are decreasing and increasing in $\hat{x}$, respectively. Consequently, taking $\overline{q}_0(\hat{x}(K_0, K_1), K_0 + 1, K_1) > \overline{q}_1(\hat{x}(K_0, K_1), K_0 + 1, K_1)$ together with the fact that $\overline{q}_0(\hat{x}(K_0 + 1, K_1), K_0 + 1, K_1) = \overline{q}_1(\hat{x}(K_0 + 1, K_1), K_0 + 1, K_1)$ also holds by construction, we have $\hat{x}(K_0, K_1) < \hat{x}(K_0 + 1, K_1)$, as required.

This finally concludes the proof.

For completeness, the figures below modify the panels of Figure 7 to include only the information that can be inferred from the statement of Proposition 4, contingent on knowing the point (K_0^A, K_1^A) (which can of course be determined by computing the critical worst-off types for the set of balanced markets (K_0, K_1) such that $K_0 + K_1 = N$). The value of the critical worst-off type relative to the critical type $\hat{x}_A$ cannot be inferred for any parameterizations in the scarcity region purely on the basis of the information provided in Proposition 4. $\square$





B.14 Proof of Corollary 1

Proof. In light of the results of Proposition 4, it only remains to show that $\hat{x}(K, K) = \hat{x}_S$ for $K \leq \lfloor \frac{N}{2} \rfloor$. Since these cases involve scarcity and $v \geq \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$ (which implies that $z_0(\hat{x}_S) > 0$ and $z_1(\hat{x}_S) > 1$), we have a unique pointwise maximizing ex post allocation rule. Consequently, dropping the γ argument, (18) and (19) become

$$\overline{Q}_0(i, j, \hat{x}_S) = \begin{cases} 0, & i \geq K \\ 1, & j \geq N - K \\ \frac{N-i}{N-i-j}, & i < K,\, j < N - K \end{cases}, \quad \overline{Q}_1(i, j, \hat{x}_S) = \begin{cases} 0, & j \geq K \\ 1, & i \geq N - K \\ \frac{N-j}{N-i-j}, & j < K,\, i < N - K \end{cases}.$$

Notice that, for all $(i, j) \in \{0, 1, \ldots, N-1\}^2$ such that $i + j \leq N - 1$, we have $\overline{Q}_0(i, j, \hat{x}_S) = \overline{Q}_1(j, i, \hat{x}_S)$. Moreover, by the definition of $\hat{x}_S$ we also have $p(i, j, \hat{x}_S) = p(j, i, \hat{x}_S)$. Putting all of this together yields

$$\overline{q}_0(\hat{x}_S) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-j} p(i,j,\hat{x}_S)\overline{Q}_0(i,j,\hat{x}_S) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-j} p(j,i,\hat{x}_S)\overline{Q}_1(j,i,\hat{x}_S) = \overline{q}_1(\hat{x}_S)$$

and by Theorem 2, we then have $\omega^* = \hat{x}_S$ as required. □

B.15 Proof of Corollary 2

Proof. See proof of Lemma B.1 in the proof of Proposition 4. □

B.16 Proof of Proposition 5

Proof. We start by proving the first statement of Proposition 5 and suppose K_0 and K_1 vary with N in such a way that $\frac{K_0(N)}{N} \to 0$ and $\frac{K_1(N)}{N} \to 1$ as $N \to \infty$. Then in the limit as $N \to \infty$, the seller faces the problem of Myerson (1981) at location 1 and the optimal selling mechanism reduces to running a single optimal auction at location 1. Consequently, since the worst-off type under the mechanism is located at 0, we have $\lim_{N\to\infty} \omega^*(K_0, K_1, v) = 0$. The proof of the second statement of Proposition 5 is analogous. We conclude the proof by moving onto the final statement of Proposition 5 and suppose that K_0 and K_1 vary with N in such a way that $\frac{K_0(N)}{N} \to \mu_0$ and $\frac{K_1(N)}{N} \to \mu_1$ and let $\hat{x} = \lim_{N\to\infty} \omega^*(K_0, K_1, v)$. If $\mu_0 \leq F(\underline{x}(\hat{x}))$ and $\mu_1 \leq 1 - F(\overline{x}(\hat{x}))$, then in the limit as $N \to \infty$, no buyers in the ironing interval $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ are served. Consequently, the optimal selling mechanism converges to running two independent auctions as $N \to \infty$. □

B.17 Proof of Lemma 9

For the sake of notational brevity, we drop the dependence of prices and allocations on $\boldsymbol{x}_{-n}$ throughout this proof.

Proof. For the case where the optimal mechanism consists of two independent optimal auctions, i.e. for $v \leq \frac{1}{2}$, the dominant strategy prices are familiar from generalized second-price auctions (with multi-unit supply but single-unit demands), in which case the optimal reserve prices are $v - \psi_S^{-1}(v)$ for good 0 and $v - (1 - \psi_B^{-1}(1-v))$ for good 1. So we are left to derive the dominant strategy prices for $v > \frac{1}{2}$.

We begin with the case where $v > \frac{1}{2}$ and $\overline{Q}_\ell \in [0, 1)$ for all $\ell \in \{0, 1\}$. We first derive the lottery price p_L. We proceed by determining the type for which EIR binds, which pins down the price for the worst-off type (i.e. the lottery price). DIC then pins down the two other prices (i.e. the two pure good prices). The utility from the lottery, excluding payments, as a function of x_n is $\overline{Q}_0(v - x_n) + \overline{Q}_1(v - (1 - x_n))$, which is decreasing in x_n if $\overline{Q}_0 \geq \overline{Q}_1$ and strictly increasing in x_n if $\overline{Q}_1 > \overline{Q}_0$. Consequently, if $\overline{Q}_0 \geq \overline{Q}_1$, then $\overline{x}$ is ex post worst off and obtains a utility of $\overline{Q}_0(v - \overline{x}) + \overline{Q}_1(v - (1 - \overline{x}))$, which is also the lottery price p_L. Similarly, if $\overline{Q}_0 < \overline{Q}_1$, then $\underline{x}$ is ex post worst off and obtains a utility $\overline{Q}_0(v - \underline{x}) + \overline{Q}_1(v - (1 - \underline{x}))$, which is also the lottery price p_L. Combining these cases then yields the expression for p_L from the lemma, as required.

Next, we derive the pure prices p_ℓ when $\overline{Q}_\ell \in [0, 1)$ holds for all $\ell \in \{0, 1\}$. We start with good 0. Notice that if $\overline{Q}_0 = 0$, then we simply have $p_0 = v - x^{-n}_{[K_0]}$, so we suppose that $\overline{Q}_0 \in (0, 1)$. For this case, if agent n reports any type $x_n < \underline{x}$, then n will obtain a unit of good 0 with certainty. Consequently, DIC requires that when agent n's type is $\underline{x}$, agent n is indifferent between reporting $\underline{x}$ and any type arbitrarily close to but less than $\underline{x}$. Therefore, if $\underline{x}$ is ex post worst-off, then $p_0 = s_0$ must hold. If $\overline{x}$ is ex post worst-off, then $v - \underline{x} - p_0 = (\overline{Q}_0 - \overline{Q}_1)(\overline{x} - \underline{x})$ must hold, where the right-hand side is the utility of the type $\underline{x}$ minus the lottery price $p_L = \overline{Q}_0(v - \overline{x}) + \overline{Q}_1(v - (1 - \overline{x}))$. Simplifying, if $\overline{x}$ is ex post worst-off, then $p_0 = s_0 - (\overline{Q}_0 - \overline{Q}_1)(\overline{x} - \underline{x})$ must hold. We now move onto good 1. Notice that if $\overline{Q}_1 = 0$, then we simply have $p_1 = v - (1 - x^{-n}_{(K_1)})$, so we suppose that $\overline{Q}_1 \in (0, 1)$. For this case, if agent n reports any type $x_n > \overline{x}$, then n will obtain a unit of good 1 with certainty. Consequently, DIC requires that when agent n's type is $\overline{x}$, agent n is indifferent between reporting $\overline{x}$ and any type arbitrarily close to but above $\overline{x}$. Consequently, if $\overline{x}$ is ex post worst-off, then $p_1 = s_1$ must hold. If $\underline{x}$ is ex post worst-off, then $v - (1 - \overline{x}) - p_1 = (\overline{Q}_0 - \overline{Q}_1)(\overline{x} - \underline{x})$ must hold, where the right-hand side is the utility of the type $\overline{x}$ minus the lottery price $p_L = \overline{Q}_0(v - \underline{x}) + \overline{Q}_1(v - (1 - \underline{x}))$. Simplifying, if $\underline{x}$ is ex post worst-off, then $p_1 = s_1 - (\overline{Q}_1 - \overline{Q}_0)(\overline{x} - \underline{x})$ must hold. Combining all of these cases (and noting that if $\overline{Q}_0 \in (0, 1)$, then $s_0 > v - x^{-n}_{[K_0]}$ and if $\overline{Q}_1 \in (0, 1)$, then $s_1 > v - (1 - x^{-n}_{(K_1)})$) yields the expressions for p_0 and p_1 from the lemma, as required.

We are now left to consider the case where $v > \frac{1}{2}$ and $\overline{Q}_\ell = 1$ holds for some $\ell \in \{0, 1\}$. First, assume that $K_0 + K_1 < N$. If $\overline{Q}_1 = 1$, which implies that $x^{-n}_{(K_1)} < \underline{x}$, then $p_0 = v - x^{-n}_{[K_0]}$ and $p_1 = \max\{v - (1 - x^{-n}_{(K_1)}), v - (1 - \psi_S^{-1}(1 - v))\}$ because x_n must be larger than both $x^{-n}_{(K_1)}$ and $\psi_S^{-1}(1 - v)$ for n to obtain a unit of good 1. Conversely, if $\overline{Q}_0 = 1$, which implies that $x^{-n}_{[K_0]} > \overline{x}$, then $p_0 = \max\{v - x^{-n}_{[K_0]}, v - \psi_B^{-1}(v)\}$ and $p_1 = v - (1 - x^{-n}_{(K_1)})$ because x_n must be less than both $x^{-n}_{[K_0]}$ and $\psi_B^{-1}(v)$ for n to obtain a unit of good 0.

Second, suppose that $K_0 + K_1 \geq N$, which implies that $x^{-n}_{(K_1)} \leq x^{-n}_{[K_0]}$. We now need to carefully distinguish for which locations $x \in [0, \underline{x})$ and $x \in (\overline{x}, 1]$ the designer prioritizes the good at location 0 and the good at location 1 because units of both goods may be available.

We begin with the case $\overline{Q}_1 = 1$. Suppose that $\omega^* < \hat{x}_A$ (in which case we have $\Psi_0(x, \omega^*) >$

$\Psi_1(x,\omega^*)$ for all $x < \overline{x}$ and $\tilde{x} > \overline{x}$) or $\omega^* = \hat{x}_A$ and $\gamma^* > 0$. Here, $\overline{Q}_1 = 1$ implies that $x^{-n}_{(K_1)} \leq x^{-n}_{[K_0]} < \underline{x}$ because the designer would allocate units of the good 0 to agents in the ironing interval with positive probability if any were available. The dominant strategy prices are $p_0 = v - x^{-n}_{[K_0]}$ and $p_1 = \max\{v - (1 - x^{-n}_{[K_0]}), v - (1 - \psi_S^{-1}(1-v)\}$. Next, suppose that $\omega^* > \hat{x}_A$ or $\omega^* = \hat{x}_A$ and $\gamma^* = 0$. For these cases we have $\Psi_1(x,\omega^*) \geq \Psi_0(x,\omega^*)$ for all $x \in [\underline{x}, \overline{x}]$ and $\tilde{x} \leq \underline{x}$, where we set $\tilde{x} = \underline{x}$ if $\omega^* = \hat{x}_A$ and $\gamma^* = 0$. Here, while $\overline{Q}_1 = 1$ implies that $x^{-n}_{(K_1)} < \underline{x}$, it does not necessarily imply that $x^{-n}_{[K_0]} < \underline{x}$. Three relevant subcases can occur: (i) $\tilde{x} < x^{-n}_{(K_1)} \leq x^{-n}_{[K_0]}$; (ii) $x^{-n}_{(K_1)} \leq \tilde{x} \leq x^{-n}_{[K_0]}$; or (iii) $x^{-n}_{(K_1)} \leq x^{-n}_{[K_0]} < \tilde{x}$. In subcase (i) the dominant strategy prices are $p_0 = v - x^{-n}_{(K_1)}$ and $p_1 = v - (1 - x^{-n}_{(K_1)})$ and in subcase (iii) they are $p_0 = v - x^{-n}_{[K_0]}$ and $p_1 = \max\{v - (1 - x^{-n}_{[K_0]}), v - (1 - \psi_S^{-1}(1-v)\}$, which is the same as the case with $\omega^* < \hat{x}_A$ or $\omega^* = \hat{x}_A$ and $\gamma^* > 0$. For subcase (ii) the dominant strategy prices are $p_0 = v - \tilde{x}$ and $p_1 = v - (1 - \tilde{x})$.

To wrap up, we need to consider the case $\overline{Q}_0 = 1$, which mirrors the $\overline{Q}_1 = 1$ case. Suppose that $\omega^* > \hat{x}_A$ (in which case we have $\Psi_1(x,\omega^*) > \Psi_0(x,\omega^*)$ for all $x > \underline{x}$ and $\tilde{x} < \underline{x}$) or $\omega^* = \hat{x}_A$ and $\gamma^* < 1$. Here, $\overline{Q}_0 = 1$ implies that $\overline{x} < x^{-n}_{(K_1)} \leq x^{-n}_{[K_0]}$ because the designer would allocate units of good 1 to agents in the ironing interval with positive probability if any were available. The dominant strategy prices are $p_0 = \max\{v - x^{-n}_{(K_1)}, v - \psi_B^{-1}(v)\}$ and $p_1 = v - (1 - x^{-n}_{(K_1)})$. Next, suppose that $\omega^* < \hat{x}_A$ or $\omega^* = \hat{x}_A$ and $\gamma^* = 1$. For these cases we have $\Psi_0(x,\omega^*) \geq \Psi_1(x,\omega^*)$ for all $x \in [\underline{x}, \overline{x}]$ and $\tilde{x} \geq \overline{x}$, where we set $\tilde{x} = \overline{x}$ if $\omega^* = \hat{x}_A$ and $\gamma^* = 1$. Here, while $\overline{Q}_0 = 1$ implies that $x^{-n}_{[K_0]} > \overline{x}$, it does not necessarily imply that $x^{-n}_{(K_1)} > \overline{x}$. Three relevant subcases can occur: (i) $\tilde{x} < x^{-n}_{(K_1)} \leq x^{-n}_{[K_0]}$; (ii) $x^{-n}_{(K_1)} \leq \tilde{x} \leq x^{-n}_{[K_0]}$; or (iii) $x^{-n}_{(K_1)} \leq x^{-n}_{[K_0]} < \tilde{x}$. In subcase (i) the dominant strategy prices are $p_0 = \max\{v - x^{-n}_{(K_1)}, v - \psi_B^{-1}(v)\}$ and $p_1 = v - (1 - x^{-n}_{(K_1)})$—which is the same as the case with $\omega^* > \hat{x}_A$ or $\omega^* = \hat{x}_A$ and $\gamma^* < 1$—and in subcase (iii) they are $p_0 = v - x^{-n}_{[K_0]}$ and $p_1 = v - (1 - x^{-n}_{[K_0]})$. For subcase (ii) the dominant strategy prices are $p_0 = v - \tilde{x}$ and $p_1 = v - (1 - \tilde{x})$.

Putting all of this together yields the prices stated in the lemma, as required. □

B.18 Proof of Proposition 6

Proof. The second statement is proven in the body of the paper following the proposition itself, so we confine attention to the first statement. By (EIR), we have $U(x_n, \boldsymbol{x}_{-n}) \geq 0$ for all $x_n \in [0,1]$ and all $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$, including for $x_n = \omega^*$. Taking expectations, the interim expected payoff under EIR, denoted $u^{EIR}(\omega^*) := \mathbb{E}_{\boldsymbol{x}_{-n}}[U(\omega^*, \boldsymbol{x}_{-n})]$, is thus non-negative, meaning that EIR (unsurprisingly) implies IR. To prove the result, we need to show that if the optimal mechanism involves a lottery and $K_0 < N$ or $K_1 < N$, then $u^{EIR}(\omega^*) > 0$

(by the payoff equivalence theorem this then proves the result for the case where $v > \frac{1}{2}$ two independent auctions are not optimal). To see that this holds, observe that $U(\omega^*, \boldsymbol{x}_{-n}) > 0$ holds for a set of type profiles $\boldsymbol{x}_{-n}$ with positive measure. Indeed, we have $U(\omega^*, \boldsymbol{x}_{-n}) > 0$ whenever either the K_0-lowest element of $\boldsymbol{x}_{-n}$ is smaller than $\underline{x}(\omega^*)$ or its K_1-highest element is larger than $\overline{x}(\omega^*)$. Thus, $u^{EIR}(\omega^*) > 0$. When two independent auctions are optimal, then $u^{EIR}(\omega^*) = 0$, implying that there is no difference in expected revenue. Similarly, in a monopoly pricing problem with $K_0 = K_1 = N$, an agent's ex post allocation and DIC payment is independent of the other agents' reports, implying that $u^{EIR}(\omega^*) = 0$ and that there is no difference in expected revenue. $\square$

B Robustness of lotteries

In this appendix, we show that lotteries remain part of the optimal mechanism if the designer maximizes a convex combination of revenue and social surplus, if goods are optimally placed rather than exogenously placed at 0 and 1 and if transportation costs are not linear. For simplicity, all extensions assume that $v \geq 1 + \max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$ and $K_0 + K_1 \geq N$ so that we have full market coverage under the optimal selling mechanism. The final two extensions further assume that $N = K_0 = K_1 = 1$ and $F(x) = x$.

Preliminaries: Full market coverage. To simplify the analysis, we assume that $v \geq 1+\max\left\{\frac{1}{f(0)}, \frac{1}{f(1)}\right\}$ and $K_0+K_1 \geq N$ so that we have full market coverage under the optimal selling mechanism. (For the setting with quadratic transportation costs, we will assume $v > 3$.) Let $q(x) := q_0(x)$ and $q_1(x) = 1 - q(x)$ (and $Q(\boldsymbol{x}) = Q_0(\boldsymbol{x})$ and $Q_1(\boldsymbol{x}) = 1 - Q(\boldsymbol{x})$). The monotonicity constraint implied by incentive compatibility then reduces to the requirement that q be decreasing, and expected revenue then becomes

$$R(Q,T) = N\left[\int_0^1 q(x)\Psi(x,\hat{x})\,dF(x) + \hat{x} + v - 1 - U(\hat{x})\right], \tag{31}$$

where $\hat{x} \in [0,1]$ is an arbitrarily chosen critical type and the virtual type function $\Psi(x,\hat{x}) := \psi_0(x,\hat{x}) - \psi_1(x,\hat{x}) = (1 - 2\psi_S(x))\mathbb{1}(x \leq \hat{x}) + (1 - 2\psi_B(x))\mathbb{1}(x > \hat{x})$ captures net revenue gain from allocating an agent a unit of good 0 rather than a unit of good 1. Accordingly, the *ironed* virtual type function associated with Ψ is given by

$$ol\Psi(x;\hat{x}) = \begin{cases} 1 - 2\psi_S(x), & x \in [0, \underline{x}(\hat{x})) \\ z, & x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})] \\ 1 - 2\psi_B(x), & x \in (\overline{x}(\hat{x}), 1] \end{cases}.$$

Ramsey objective. We now provide a sketch of the arguments for why lotteries remain part of the optimal mechanism for a designer who maximizes a weighted sum of revenue and social surplus, provided the weight on revenue is greater than 0. To that end, under full market coverage expected social surplus $SS(Q,T)$ under any direct incentive compatible mechanism (Q,T) is given by $SS(Q,T) = N\int_0^1 q(x)(v-x) + (1-q(x))(v-(1-x))dF(x) = N\left(v - 1 + \mathbb{E}[x] + \int_0^1 q(x)(1-2x)dF(x)\right)$. Expected revenue $R(Q,T)$ is computed in (31). For $\alpha \in [0,1]$, the designer's problem is to then maximize over (Q,T) the Ramsey objective

$$W_\alpha(Q,T) := \alpha R(Q,T) + (1-\alpha)SS(Q,T).$$

Letting $\psi_S^\alpha(x) := x + \alpha\frac{F(x)}{f(x)}$, $\psi_B^\alpha(x) := x - \alpha\frac{1-F(x)}{f(x)}$ and $\Psi^\alpha(x,\hat{x}) := (1-2\psi_S^\alpha(x))\mathbb{1}(x \leq \hat{x}) + (1-2\psi_B^\alpha(x))\mathbb{1}(x > \hat{x})$, we have

$$W_\alpha(Q,T) = N\left(v - 1 + (1-\alpha)\mathbb{E}[x] + \alpha(\hat{x} - U(\hat{x})) + \int_0^1 q(x)\Psi^\alpha(x,\hat{x})dF(x)\right).$$

All the preceding analysis then carries over to this generalization, with $\Psi(x,\hat{x})$ replaced by $\Psi^\alpha(x,\hat{x})$. Observe in particular that for any $\alpha > 0$, $\Psi^\alpha(x,\hat{x})$ increases at $x = \hat{x}$, implying that there is a need for ironing for any $\alpha > 0$. Notice also that $\psi_S^\alpha(x)$ increases in α and $\psi_B^\alpha(x)$ decreases in α and, consequently, $1 - 2\psi_S^\alpha(x)$ decreases and $1 - 2\psi_B^\alpha(x)$ increases in α. This in turn implies that, as α decreases, the ironing interval $[\underline{x}^\alpha(\hat{x}), \overline{x}^\alpha(\hat{x})]$ shrinks in a set inclusion sense. So although the lottery interval shrinks in a set-inclusion sense as the designer places less weight on revenue, there is still a lottery involving a positive measure of types under the optimal selling mechanism whenever $\alpha > 0$.

Optimal placement of the goods. Consider the case with $N = K_0 = K_1 = 1$, $F(x) = x$ and $v \geq 2$ and suppose the seller can place one product at location $a \in [0,1]$ and one at location $b \in [0,1]$. Without loss, assume $b \geq a$. If the seller does not use lotteries, it is optimal to place the products so as to minimize the buyer's expected transportation costs and then to compensate for the reduction in costs by increasing the prices. Thus, the optimal locations under posted prices are $a_{PP} = 1/4$ and $b_{PP} = 3/4$ and the optimal prices for the two products are $v - 1/4$, which is also the seller's profit.

Now assume the seller places the two goods at $a_L = 3/16$ and $b_L = 13/16$. A fifty-fifty lottery for types $x \in [a_L, b_L]$ generates a utility of $v - (b_L - a_L)/2 = v - 5/16$, which is also the optimal lottery price. To cover the fully market, the seller can charge the price $v - 3/16$ for each of the two pure goods. This implies that the buyer participates in the lottery for all $x \in [3/8, 5/8]$. Accordingly, the seller's expected profit is $\frac{3}{4}\left(v - \frac{3}{16}\right) + \frac{1}{4}\left(v - \frac{5}{16}\right) = v - \frac{7}{32}$. This is larger than its profit with transportation cost minimizing placements of $1/4$ and $3/4$ and posted prices $v - 1/4$, and shows that for this case the optimal selling mechanism must continue to involve a lottery when the locations are optimally placed.

Non-linear transportation costs. We are now going to show that the optimality of lotteries also doesn't depend on the assumption that buyers' transportation costs are linear by studying a model with quadratic transportation costs. As we will see, the behavior of this model is, perhaps surprisingly, similar to that with linear transportation costs. The only essential change is that the allocation rule in the ironing interval is no longer a constant fifty-fifty lottery. Rather, the probability of obtaining good 1 is increasing in the type.

Specifically, we now assume quadratic transportation costs, uniformly distributed types, $N = 1 = K_0 = K_1$ and that $v \geq 3$ (which ensures optimality of full market coverage). To apply standard results such as single-crossing without relabelling types, we now let $q_0(x) = 1 - q_1(x)$. We then have

$$V(q_1, x) = (1 - q_1)(v - x^2) + q_1(v - (1 - x)^2) = q_1(2x - 1) + v - x^2.$$

Incentive compatibility requires that $q_1(x)(2x-1)+v-x^2-t(x) \geq q_1(\hat{x})(2x-1)+v-x^2-t(\hat{x})$ and $q_1(x)(2\hat{x} - 1) + v - \hat{x}^2 - t(x) \leq q_1(\hat{x})(2\hat{x} - 1) + v - \hat{x}^2 - t(\hat{x})$. Subtracting the latter inequality from the former yields $q_1(x)(2x-1) - q_1(x)(2\hat{x}-1) \geq q_1(\hat{x})(2x-1) - q_1(\hat{x})(2\hat{x}-1)$. Rearranging, we have $2q_1(x)(x - \hat{x}) \geq 2q_1(\hat{x})(x - \hat{x})$. This inequality is satisfied if and only if q_1 is (weakly) increasing. Notice also that $\frac{V(q_1,x)}{\partial q_1 \partial x} = 2 > 0$. Consequently, the Spence-Mirrlees single crossing property holds and q_1 can be implemented using an incentive compatible direct mechanism if and only if q_1 is increasing. Let $U(x, \hat{x}) = q_1(\hat{x})(2x - 1) + v - x^2 - t(\hat{x})$ and $U(x) = U(x, x)$. Applying the envelope theorem, we have

$$U(x) = U(\hat{x}) + \int_{\hat{x}}^{x} (2q_1(y) - 2y)\, dy, \qquad (32)$$

where $\hat{x} \in [0, 1]$ is an arbitrarily chosen critical type. By definition we also have $U(x) = q_1(x)(2x - 1) + v - x^2 - t(x)$. Combining this with (32) and solving for $t(x)$ then yields

$t(x) = q_1(x)(2x-1) + v - x^2 - U(\hat{x}) - \int_{\hat{x}}^{x}(2q_1(y) - 2y)\,dy$. The designer's revenue under any direct mechanism $\langle q_1, t\rangle$ is therefore

$$\begin{aligned} R(q_1, t) &= \int_0^1 \left(q_1(x)(2x-1) + v - x^2 - U(\hat{x}) - \int_{\hat{x}}^{x}(2q_1(y) - 2y)\,dy\right) dx \\ &= \int_0^1 \left(q_1(x)(2x-1) - 2\int_{\hat{x}}^{x} q_1(y)\,dy\right) dx + v - \hat{x}^2 - U(\hat{x}) \end{aligned}$$

Using $\int_0^1 \int_{\hat{x}}^{x} q_1(y)\,dy\,dx = \int_{\hat{x}}^{1} q_1(y)(1-y)\,dy - \int_0^{\hat{x}} q_1(y)y\,dy$, the designer's revenue becomes $R(q_1, t) = \int_{\hat{x}}^{1}(2x - 1 - 2(1-x))\,q_1(x)\,dx + \int_0^{\hat{x}}(2x - 1 + 2x)q_1(x)\,dx + v - \hat{x}^2 - U(\hat{x}) = \int_{\hat{x}}^{1}(4x-3)q_1(x)\,dx + \int_0^{\hat{x}}(4x-1)q_1(x)\,dx + v - \hat{x}^2 - U(\hat{x})$. Introducing the virtual type function $\Psi(x, \hat{x}) = (4x-1)\mathbb{1}(x < \hat{x}) + (4x-3)\mathbb{1}(x \geq \hat{x})$ we can rewrite this as $R(q_1, t) = \int_0^1 \Psi(x, \hat{x})q_1(x)\,dx + v - \hat{x}^2 - U(\hat{x})$. Once again we have a non-regular problem and we iron the virtual type function. For any $\hat{x} \in (0, 1)$. We have

$$\overline{\Psi}(x, \hat{x}) = \begin{cases} 4x - 1, & x \in [0, \underline{x}(\hat{x})) \\ z(\hat{x}), & x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})] \\ 4x - 3, & x \in (\overline{x}(\hat{x}), 1] \end{cases},$$

where $\underline{x}(\hat{x}) = \max\left\{\frac{1 - z(\hat{x})}{4}, 0\right\}$, $\overline{x}(\hat{x}) = \min\left\{\frac{3 - z(\hat{x})}{4}, 1\right\}$ and

$$z(\hat{x}) = \begin{cases} 4\sqrt{\hat{x}} - 3, & \hat{x} \in \left[0, \frac{1}{4}\right) \\ 4\hat{x} - 2, & \hat{x} \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ 3 - 4\sqrt{1 - \hat{x}}, & \hat{x} \in \left(\frac{3}{4}, 1\right] \end{cases}.$$

The saddle point theorem still applies to this problem, and we can use it to show that $\omega^* = \frac{1}{2}$. In particular, if we set $\hat{x} < \frac{1}{2}$ so that $z(\hat{x}) > 0$ and pointwise maximize the ironed virtual surplus function, then we have a worst-off type of $\omega = \frac{3}{4} \neq \hat{x}$. So setting $\hat{x} < \frac{1}{2}$ cannot satisfy the saddle point condition. Similarly, if we set $\hat{x} > \frac{1}{2}$ then we have $z(\hat{x}) < 0$ which yields a worst-off type of $\omega = \frac{1}{4} \neq \hat{x}$ under pointwise maximization of the designer's ironed virtual surplus function. So we must have a critical worst-off type of $\omega^* = \frac{1}{2}$. Setting $\hat{x} = \frac{1}{2}$ in our expression for the ironed virtual type function we have $\overline{\Psi}\left(x, \frac{1}{2}\right) = (4x-1)\mathbb{1}(x \in \left[0, \frac{1}{4}\right)) + (4x-3)(x \in \left(\frac{3}{4}, 1\right])$. Since $z\left(\frac{1}{2}\right) = 0$, any allocation rule

$$\overline{q}\,(x) = q_\ell(x)\mathbb{1}(x \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right]) + \mathbb{1}(x \in \left(\tfrac{3}{4}, 1\right]),$$

with $q_\ell : \left[\frac{1}{4}, \frac{3}{4}\right] \to [0, 1]$ increasing pointwise maximizes the designer's ironed virtual surplus function. The pointwise maximizing allocation rule that makes all types in the ironing interval worst-off satisfies $U'(x) = 2q_\ell(x) - 2x = 0$, which yields $q_\ell(x) = x$. Clearly, this allocation rule satisfies the saddle point condition since $\omega^* = \frac{1}{2}$ is then a worst-off type.

In summary, this analysis shows that the optimality of lotteries does not reply on the assumption of linear transportation costs and that, moreover, our saddle point and ironing machinery apply beyond the case of linear transportation costs.